Chapter 4
Consensus I

Theory of Distributed Systems
Summer Term 2024

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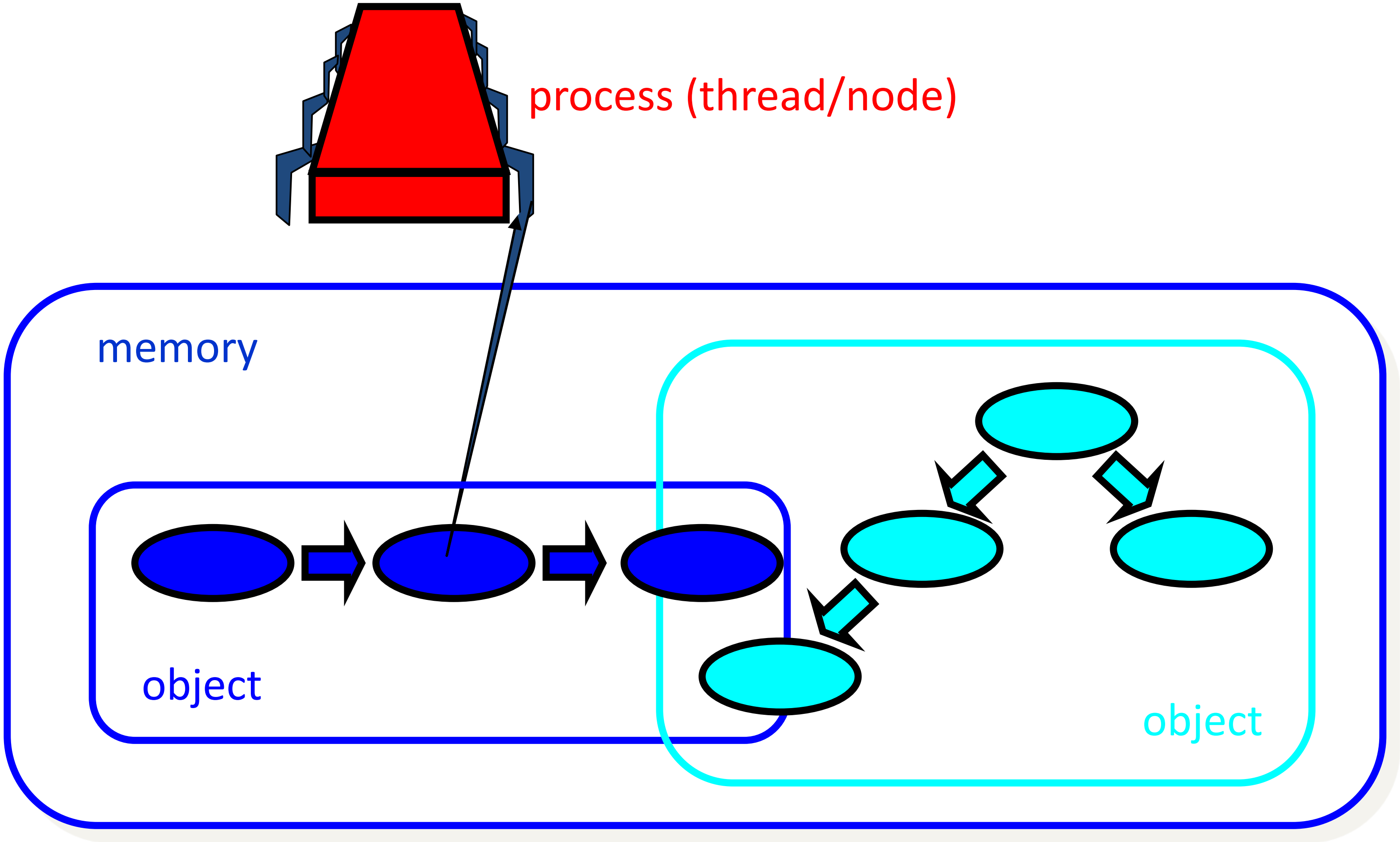
Overview

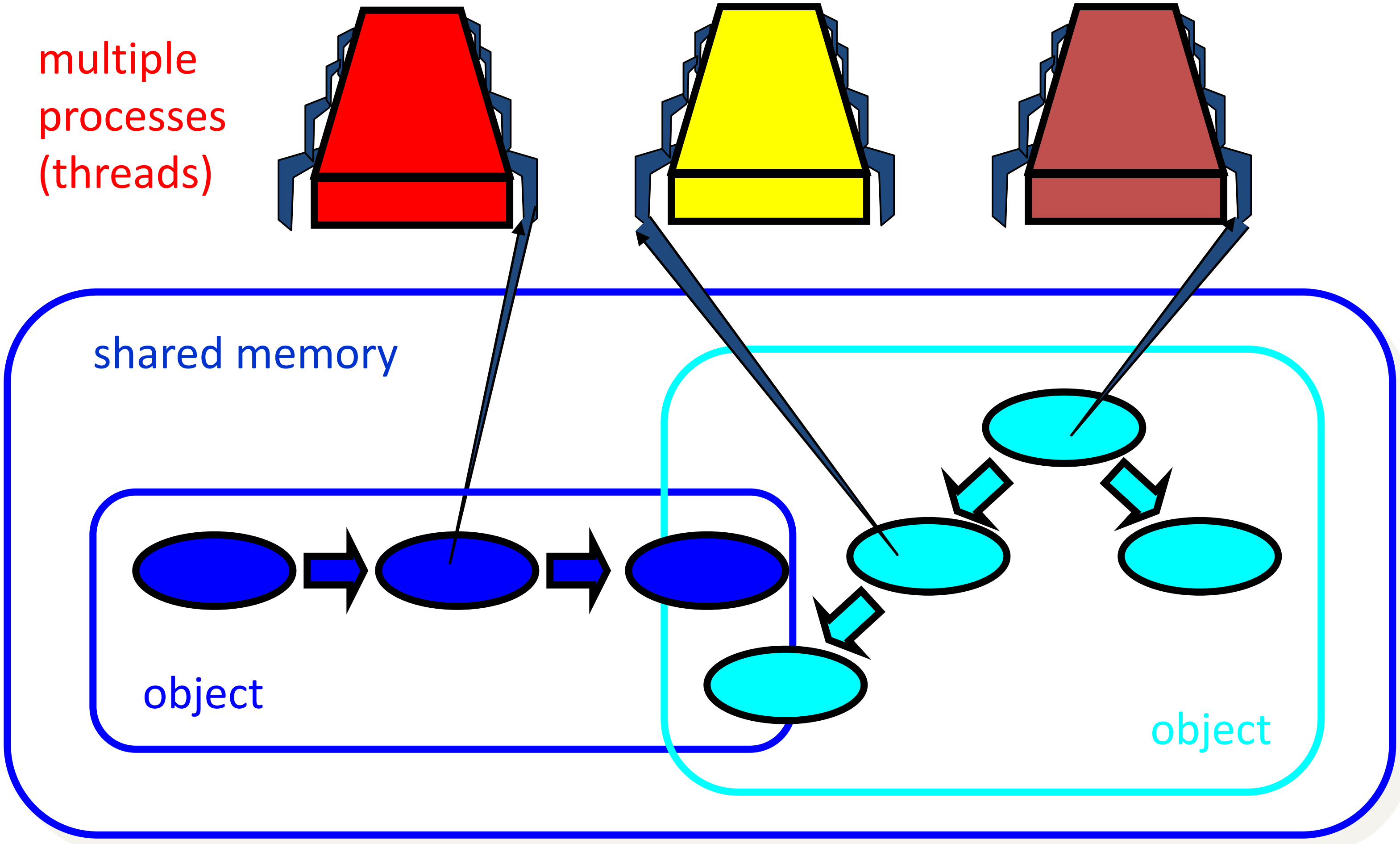
- Introduction
- Consensus #1: Shared Memory
- Consensus #2: Wait-free Shared Memory
- Consensus #3: Read-Modify-Write Shared Memory
- Consensus #4: Synchronous Systems
- Consensus #5: Byzantine Failures
- Consensus #6: A Simple Algorithm for Byzantine Agreement
- Consensus #7: The Queen Algorithm
- Consensus #8: The King Algorithm
- Consensus #9: Byzantine Agreement Using Authentication
- Consensus #10: A Randomized Algorithm
- Shared Coin

Most slides by R. Wattenhofer (ETHZ), based on slides by M. Herlihy (Brown Univ.)

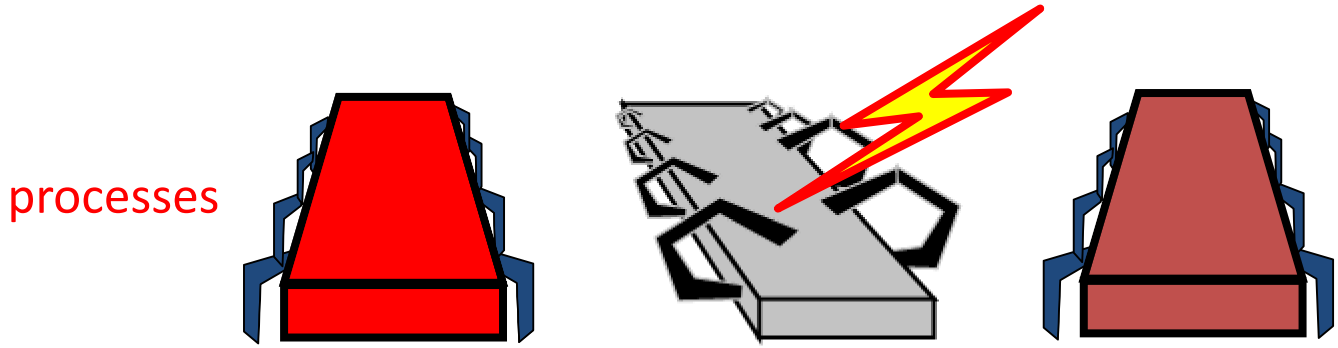
Sequential Computation



Concurrent Computation



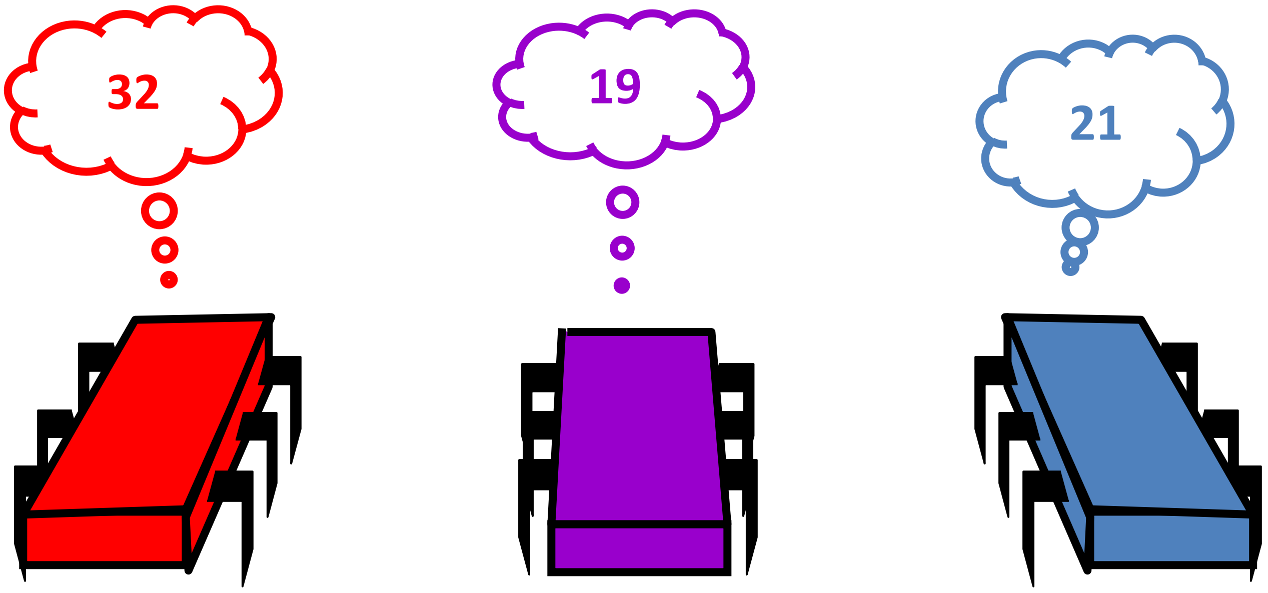
Fault Tolerance & Asynchrony



- Why fault-tolerance?
  - Even if processes do not die, there are “near-death experiences”
- Sudden unpredictable delays:
  - Cache misses (short)
  - Page faults (long)
  - Scheduling quantum used up (really long)

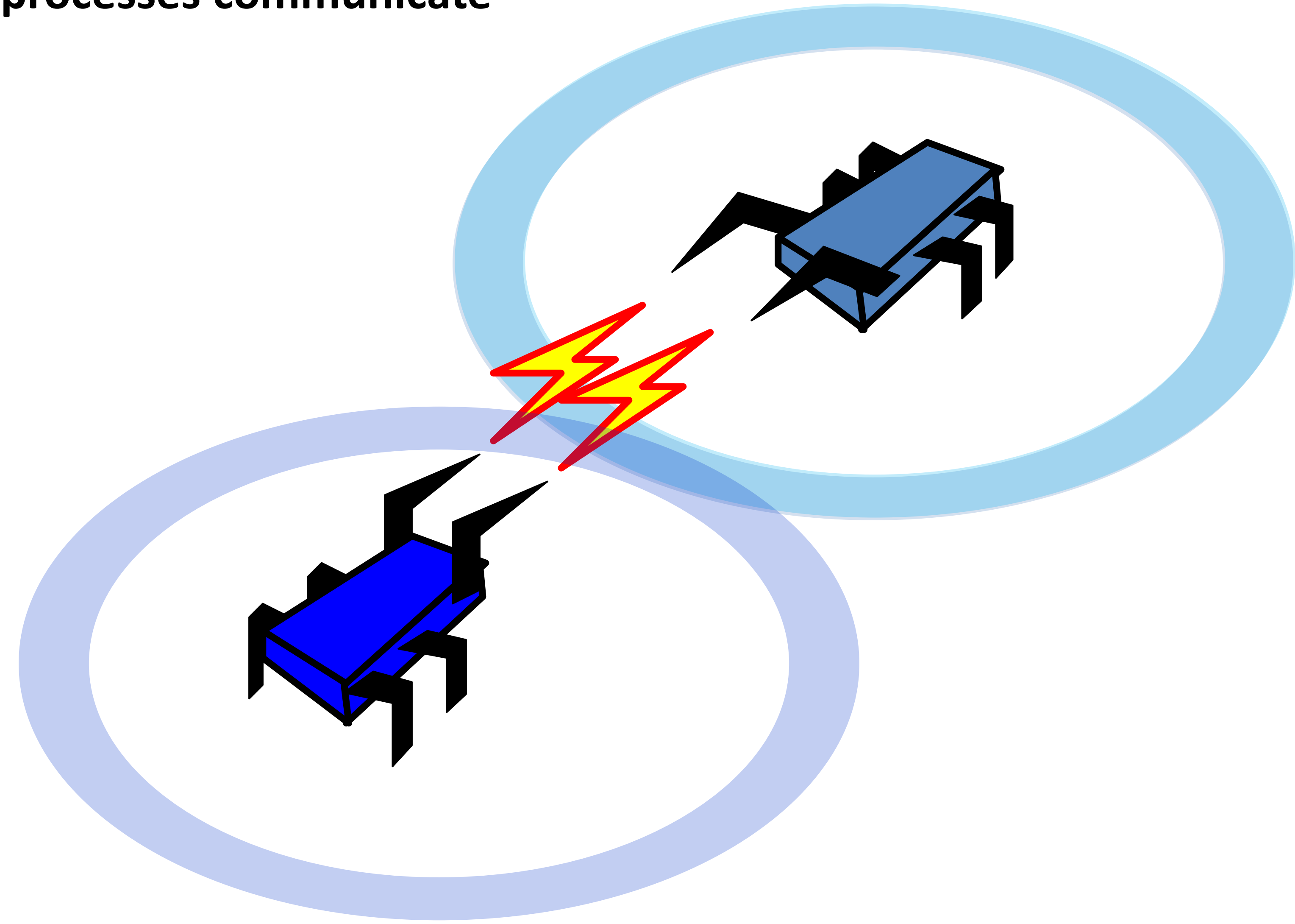
Consensus

Each thread/process has a private input



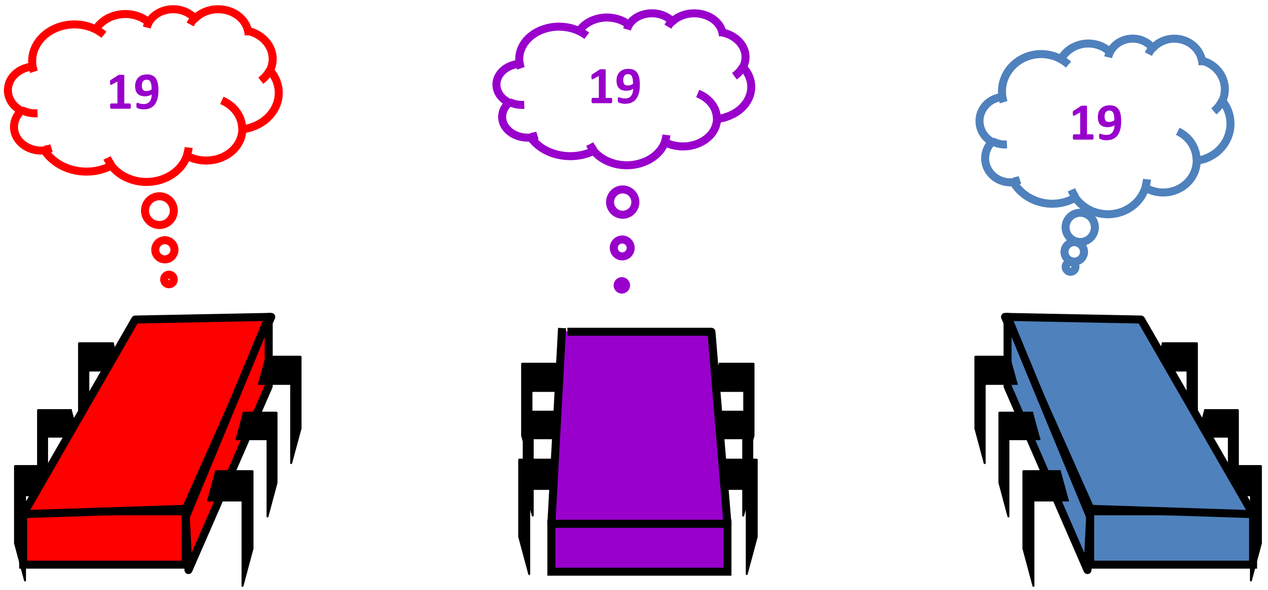
Consensus

The processes communicate



Consensus

They agree on some process's input



Consensus More Formally

Setting:

- n processes/threads/nodes $v_1, v_2, \dots, v_n$
- Each process has an input $x_1, x_2, \dots, x_n \in \mathcal{D}$
- Each (non-failing) process computes an output $y_1, y_2, \dots, y_n \in \mathcal{D}$

binary consensus: $\mathcal{D} = \{0, 1\}$

Agreement:

The outputs of all non-failing processes are equal.

Validity:

If all inputs are equal to x, all outputs are equal to x.

Termination:

All non-failing processes terminate after a finite number of steps.

Remarks

- Validity might sometimes depend on the (failure) model

Two Generals:

- The two generals (coordinated attack) problem is a variant of binary consensus with 2 processes.
- *Model:*
  - Communication is synchronous, messages can be lost
- Validity:
  - If no messages are lost, and both nodes have the same input x, x needs to be the output of both nodes
- We have seen that the problem cannot be solved in this setting.

Consensus is Important

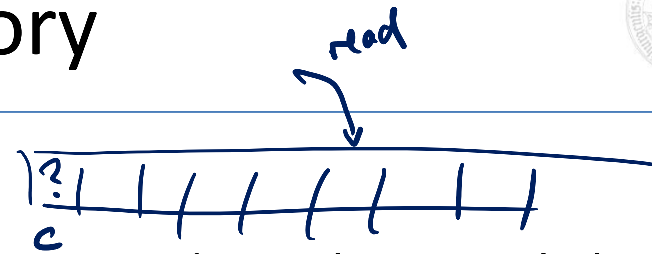
- With consensus, you can implement anything you can imagine…

- Examples:
  - With consensus you can decide on a leader,
  - implement mutual exclusion,
  - or solve the two generals problem
  - and much more…

- We will see that in some models, consensus is possible, in some other models, it is not

- The goal is to learn whether for a given model consensus is possible or not … and prove it!

Consensus #1: Shared Memory



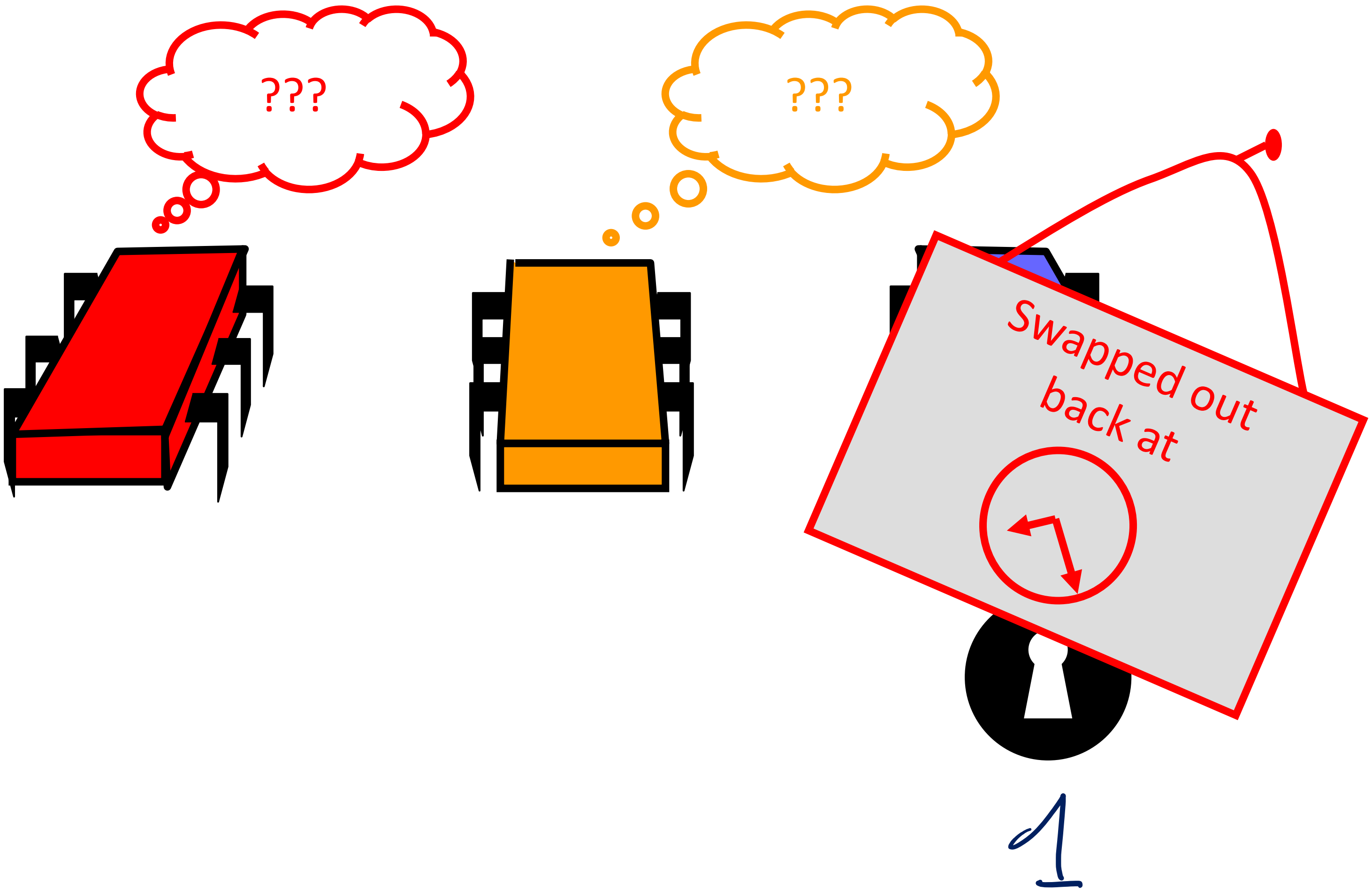
- $n > 1$ processors
- Shared memory is memory that may be accessed simultaneously by multiple threads/processes.
- Processors can atomically *read* from or *write* to (not both) a shared memory cell

Protocol:

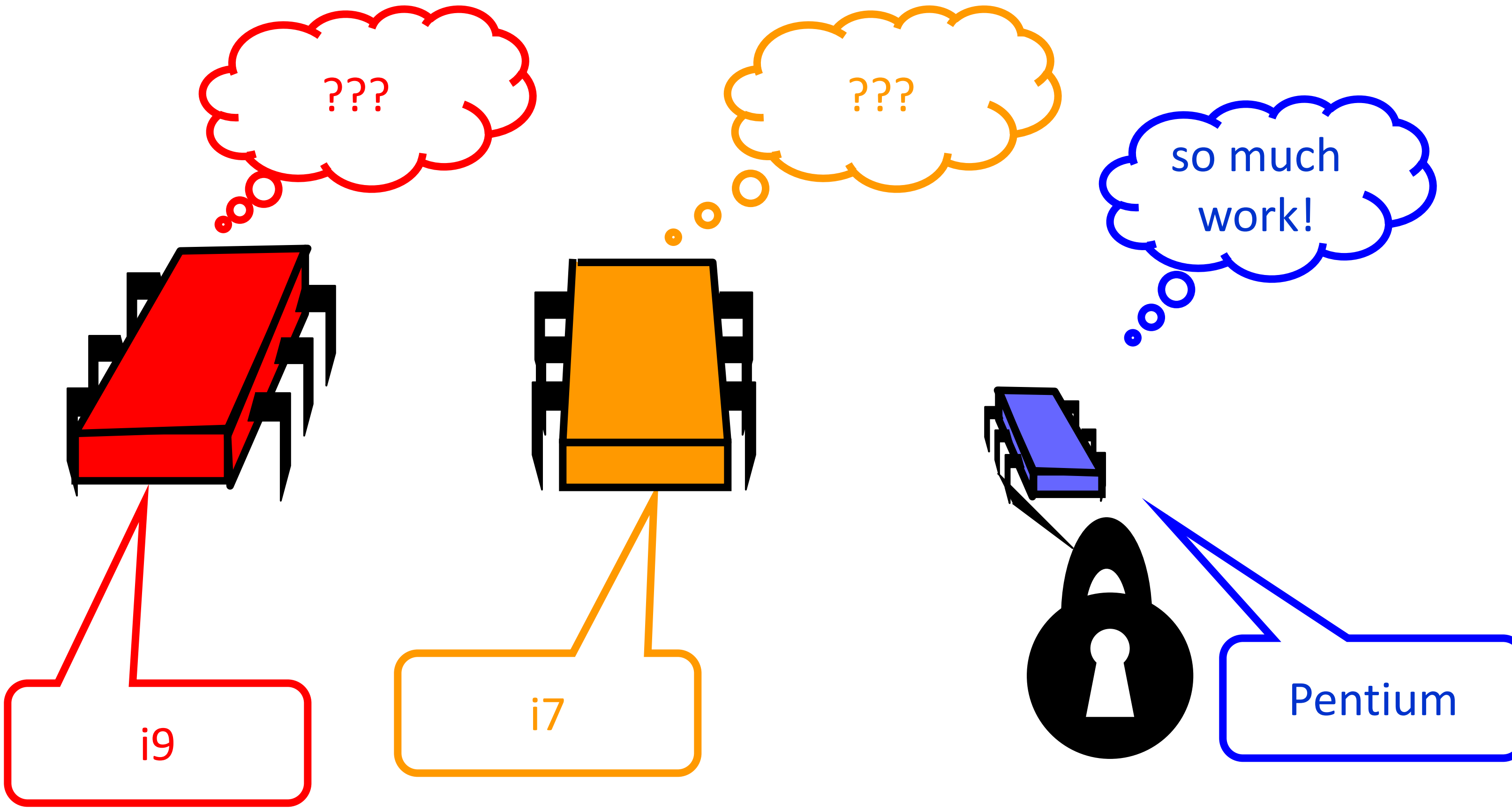
- There is a designated memory cell c.
- Initially c is in a special state "?"
- Processor 1 writes its value x_1 into c, then decides on x_1.
- A processor $j \neq 1$ reads c until j reads something else than "?", and then decides on that.

- Problems with this approach?

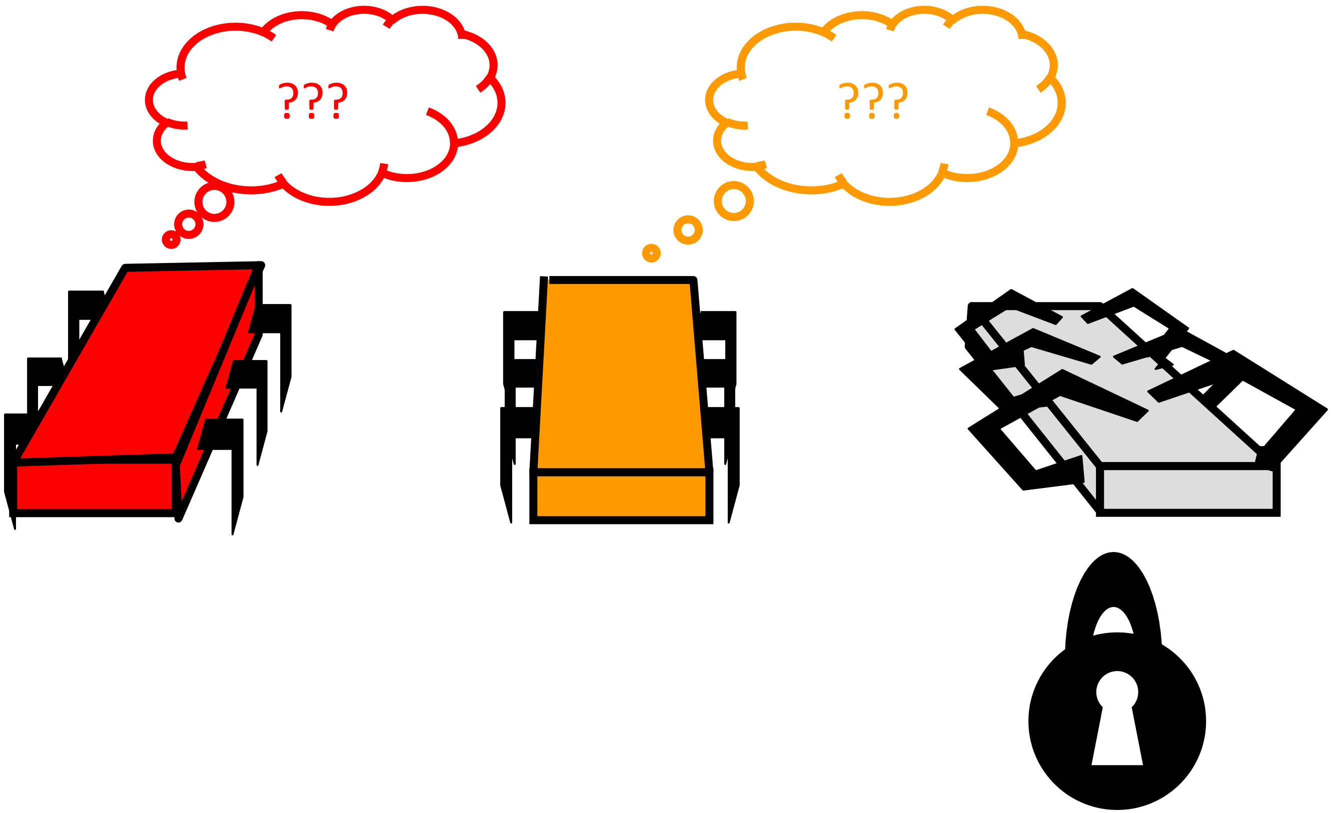
Unexpected Delay



Heterogeneous Architectures

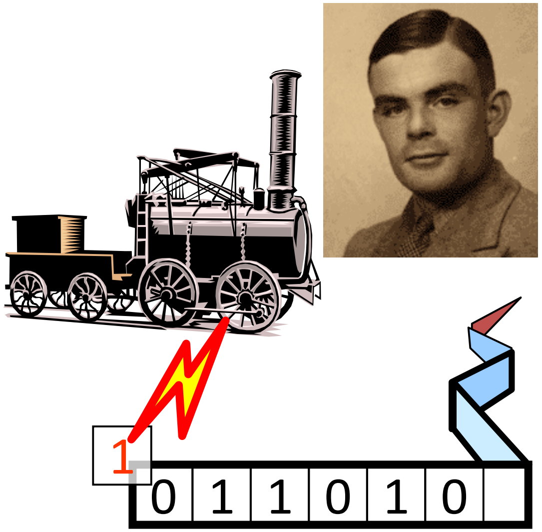


Fault-Tolerance

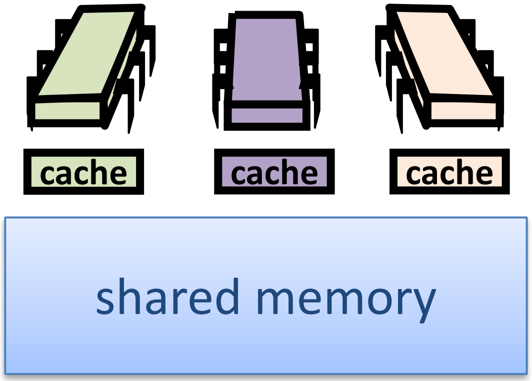


Computability

- Definition of computability
  - Computable usually means Turing-computable, i.e., the given problem can be solved using a Turing machine
  - Strong mathematical model!



- Shared-memory computability
  - Model of asynchronous concurrent computation
  - Computable means it is wait-free computable on a multiprocessor
  - Wait-free…?



Consensus #2: Wait-free Shared Memory

- $n > 1$ processors
- Processors can atomically *read* to or *write* from (not both) a shared memory cell
- Processors might crash (stop… or become very slow…)

Wait-free implementation:

- Every process completes in a finite number of steps
- Implies that locks cannot be used → The thread holding the lock may crash and no other thread can make progress
- We assume that we have wait-free atomic registers (i.e., reads and/or writes to same register do not overlap)

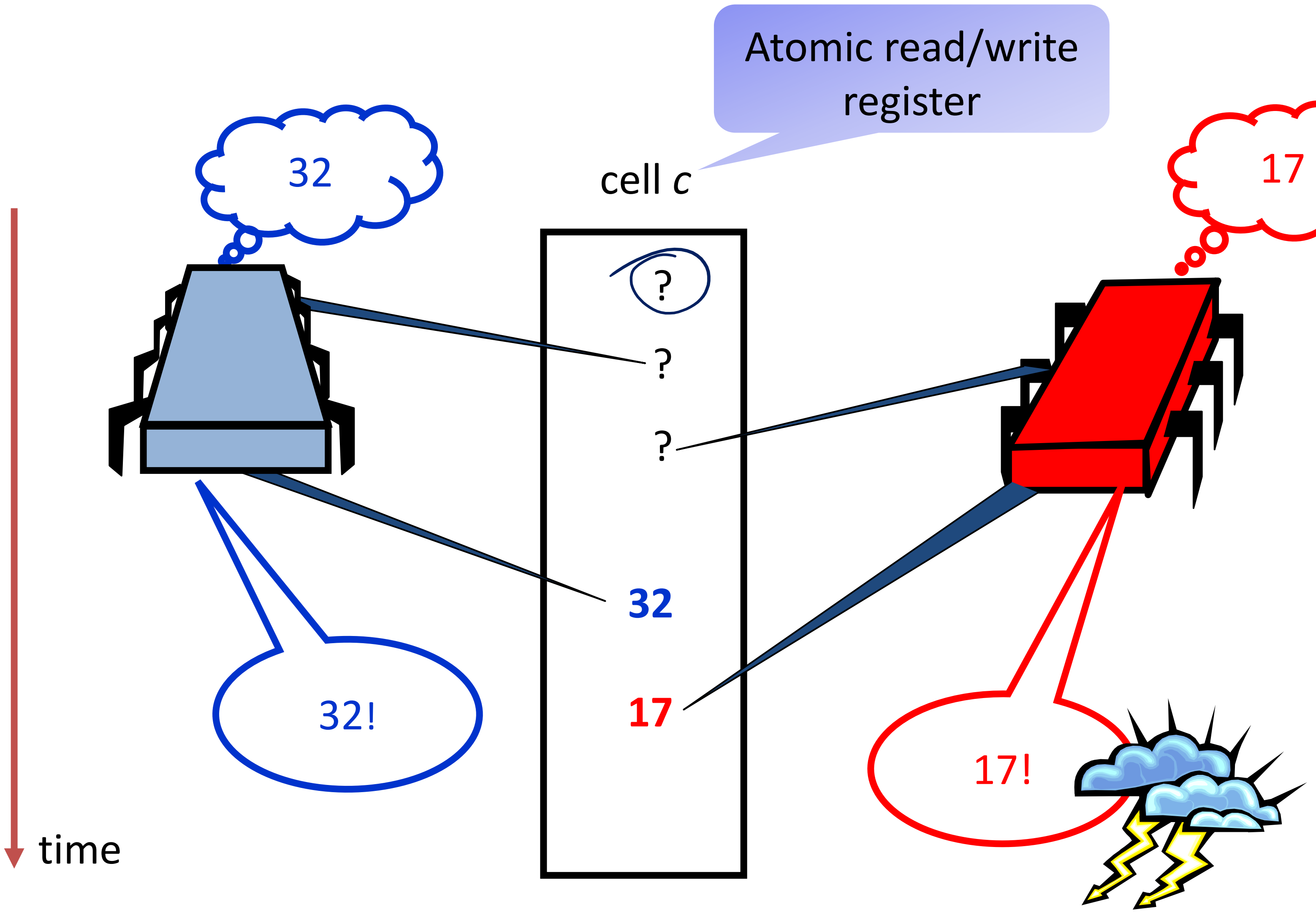
A Wait-Free Algorithm

- There is a cell c, initially $c =$“?”
- Every processor i does the following:

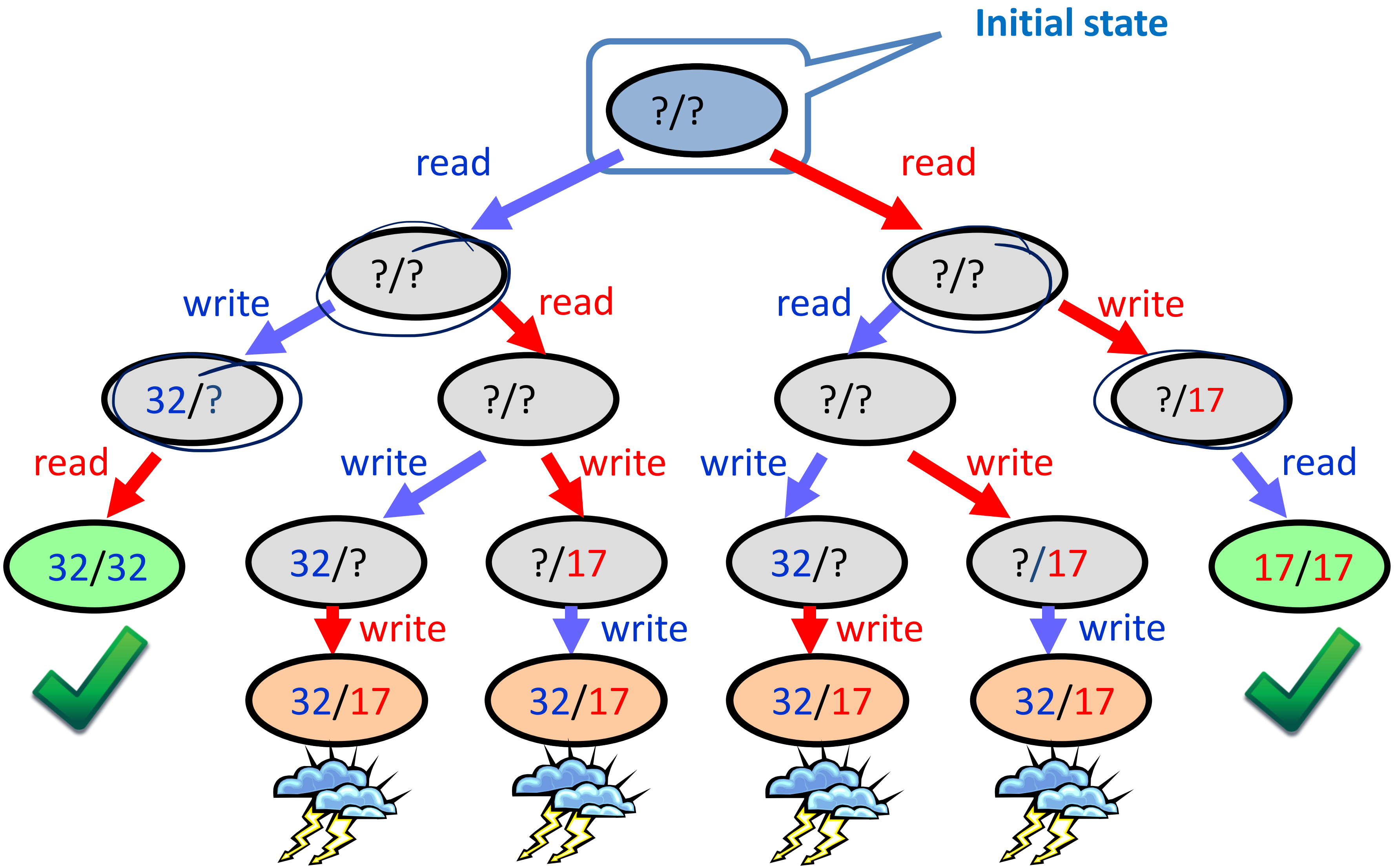
```
r = read(c);
if (r == “?”) then
    write(c, x_i); decide x_i;
else
    decide r;
```

- Is this algorithm correct…?

An Execution



Execution Tree

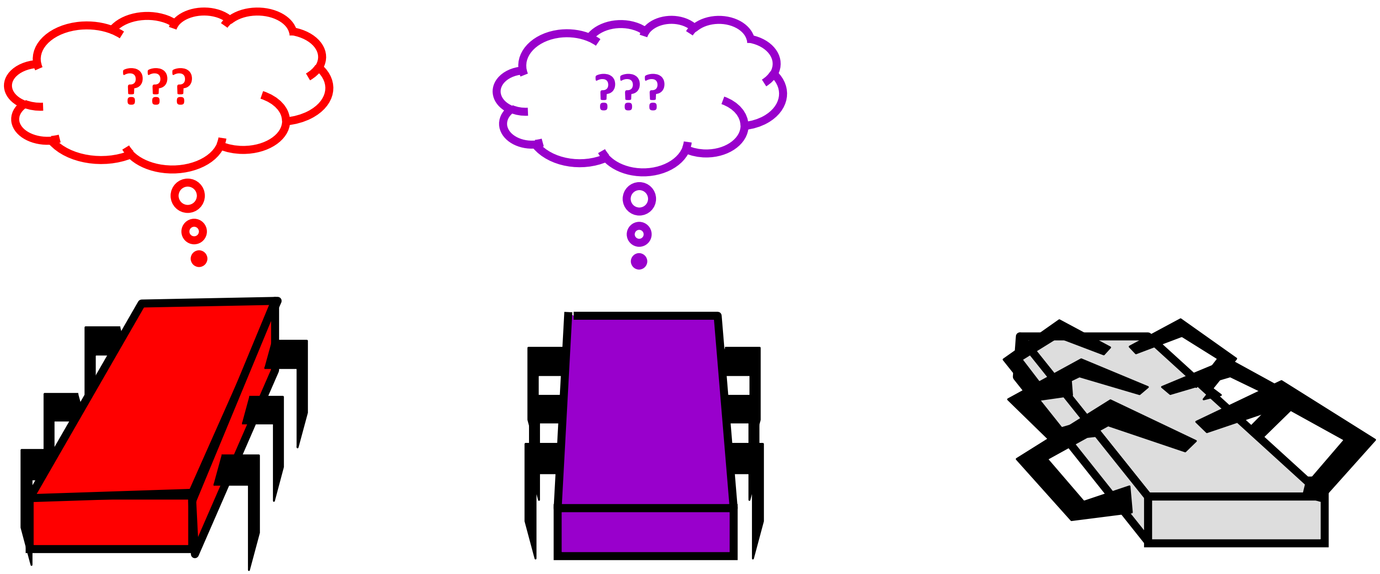


Impossibility

[FLP]: Fischer, Lynch, Paterson, 1985

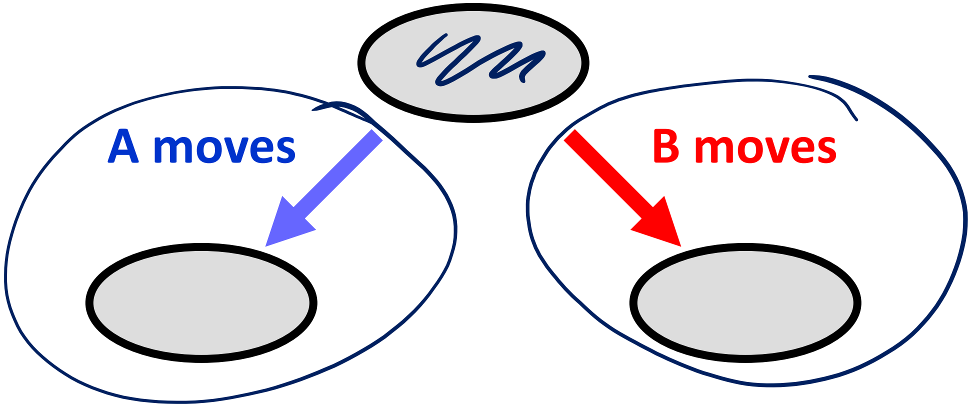
Theorem

There is no deterministic asynchronous wait-free consensus algorithm using read/write atomic registers.

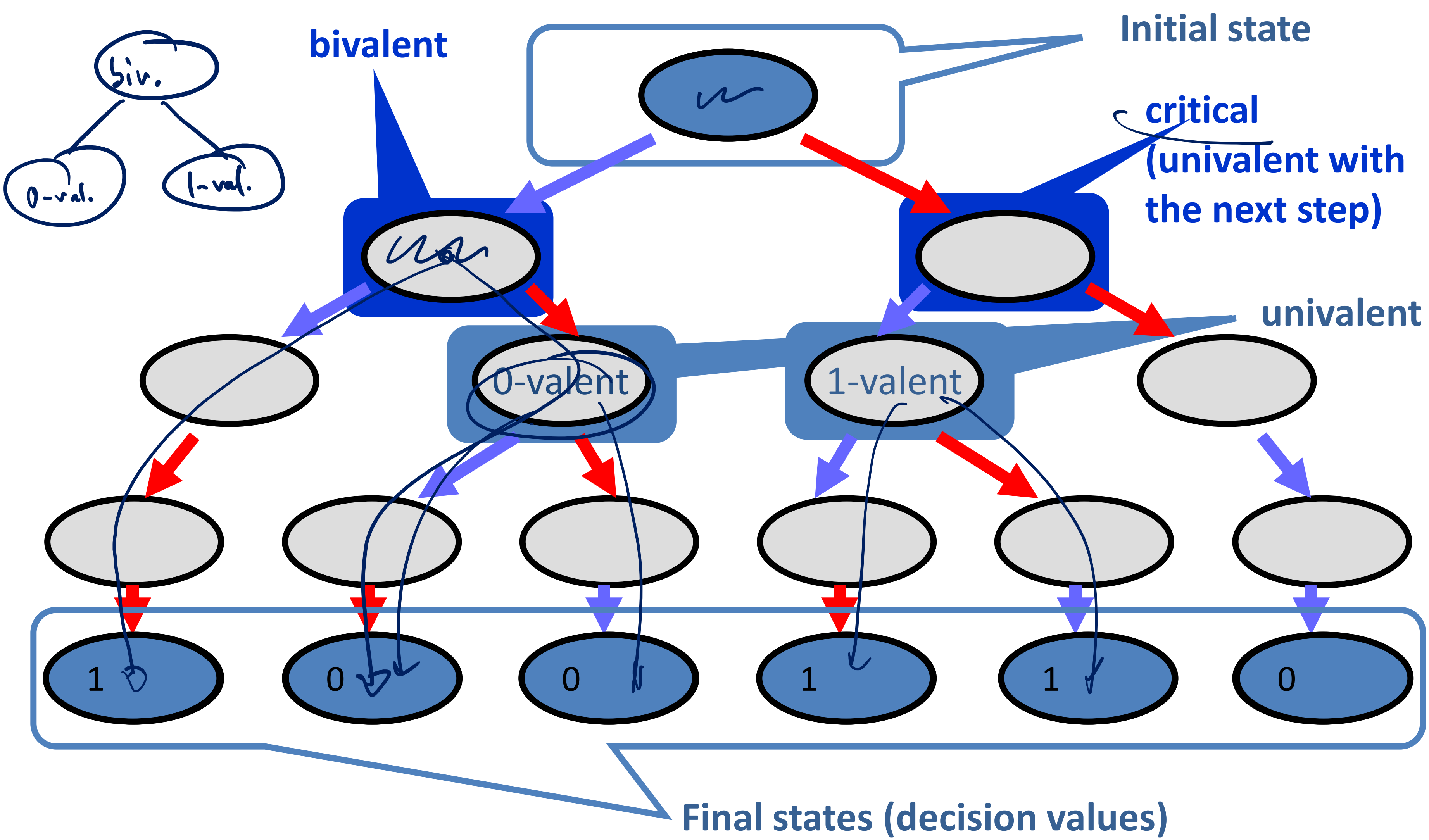


Proof

- Make it simple
  - There are only **two processes** A and B and the **input is binary**
- Assume that there is a protocol
- In this protocol, either A or B “moves” in each step
- Moving means
  - Register read
  - Register write



Execution Tree



Bivalent vs. Univalent

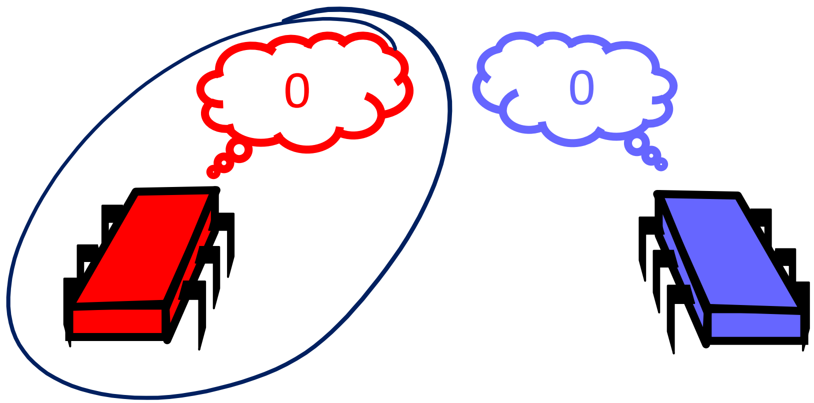
- Wait-free computation is a tree
- Bivalent system states
  - Outcome is not fixed
- Univalent states
  - Outcome is fixed
  - Maybe not “known” yet
  - 1-valent and 0-valent states

Claim:

- Some initial system state is bivalent
- Hence, the outcome is not always fixed from the start

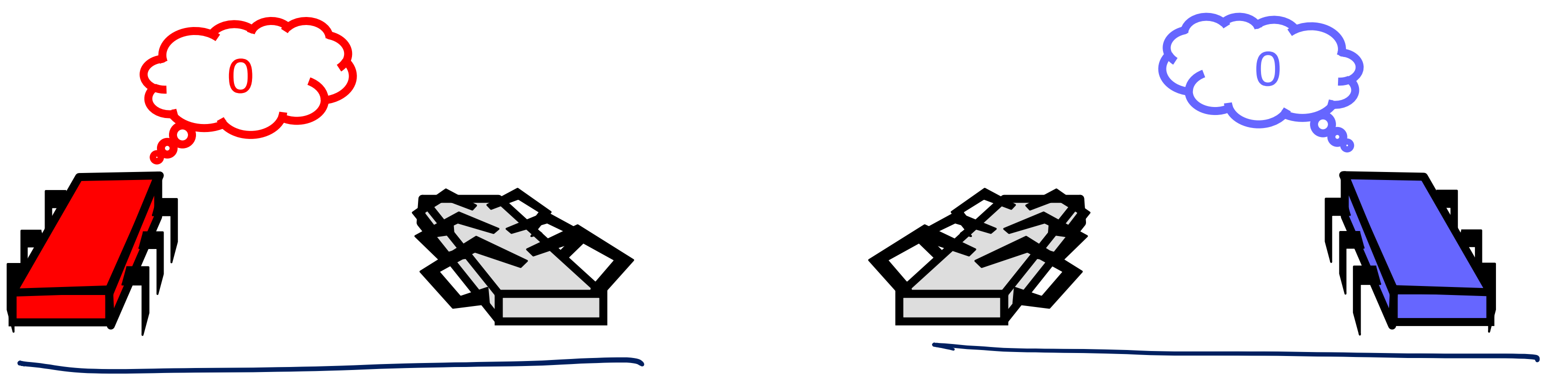
Proof of Claim: A 0-Valent Initial State

- All executions lead to the decision 0



Similarly, the decision is always 1 if both threads start with 1!

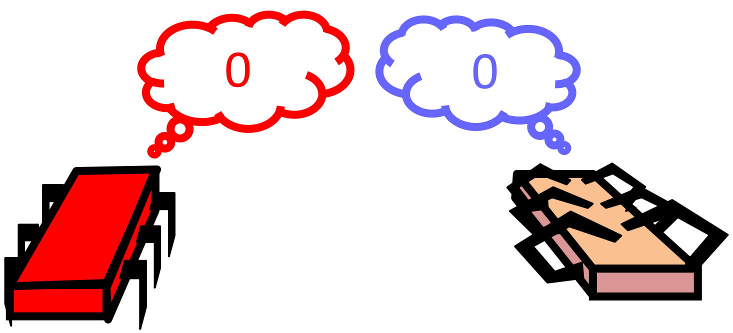
- Solo executions also lead to the decision 0



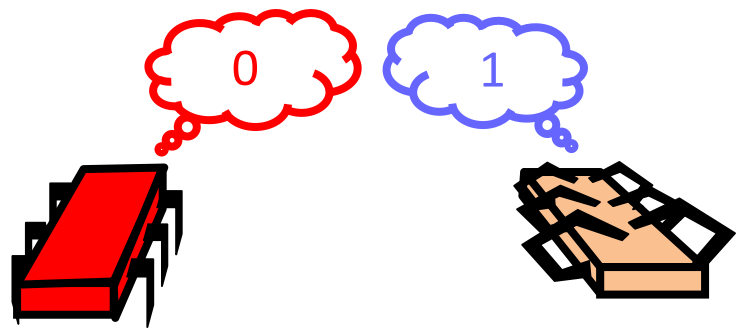
Proof of Claim: Indistinguishable Situations

- Situations are indistinguishable to red process

  $\Longrightarrow$ The outcome must be the same



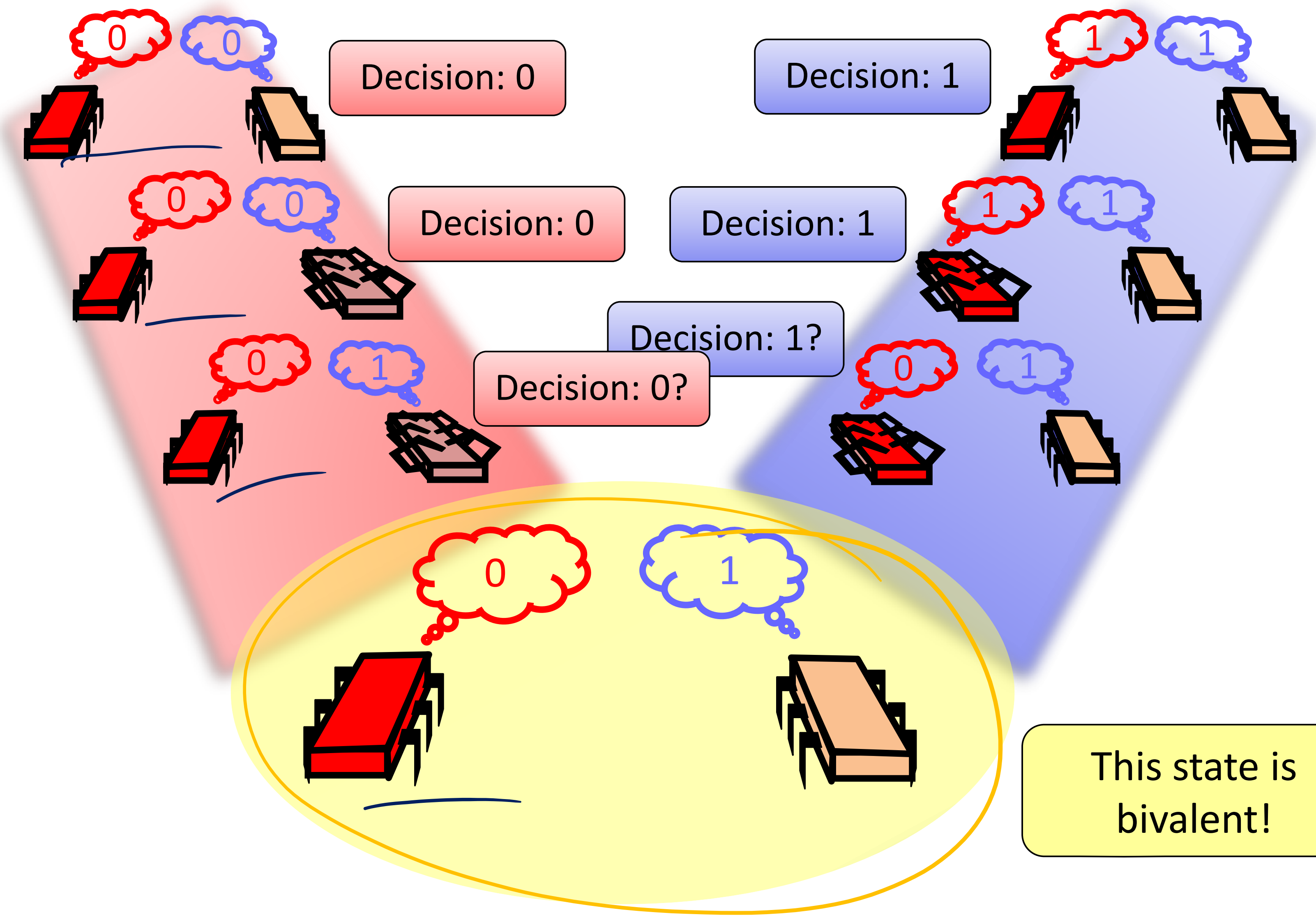
The decision is 0!



The decision is 0!

Similarly, the decision is 1 if the red thread crashed!

Proof of Claim: A Bivalent Initial State

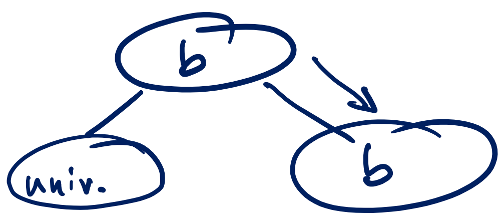


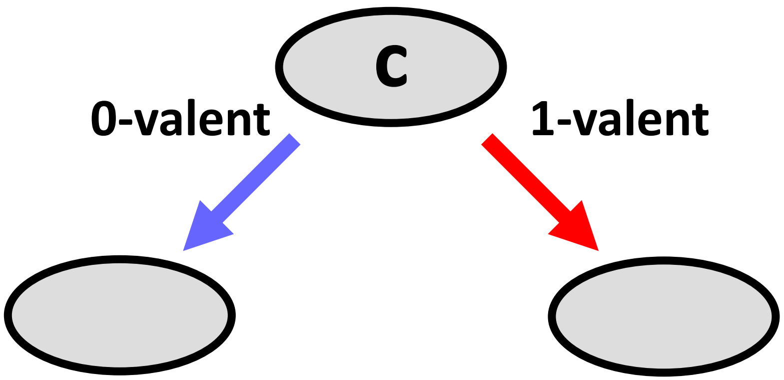
This state is bivalent!

Critical States

- Starting from a bivalent initial state
- The protocol must reach a critical state
  - Otherwise we could stay bivalent forever
  - And the protocol is not wait-free

A bivalent state is critical if all children states are univalent

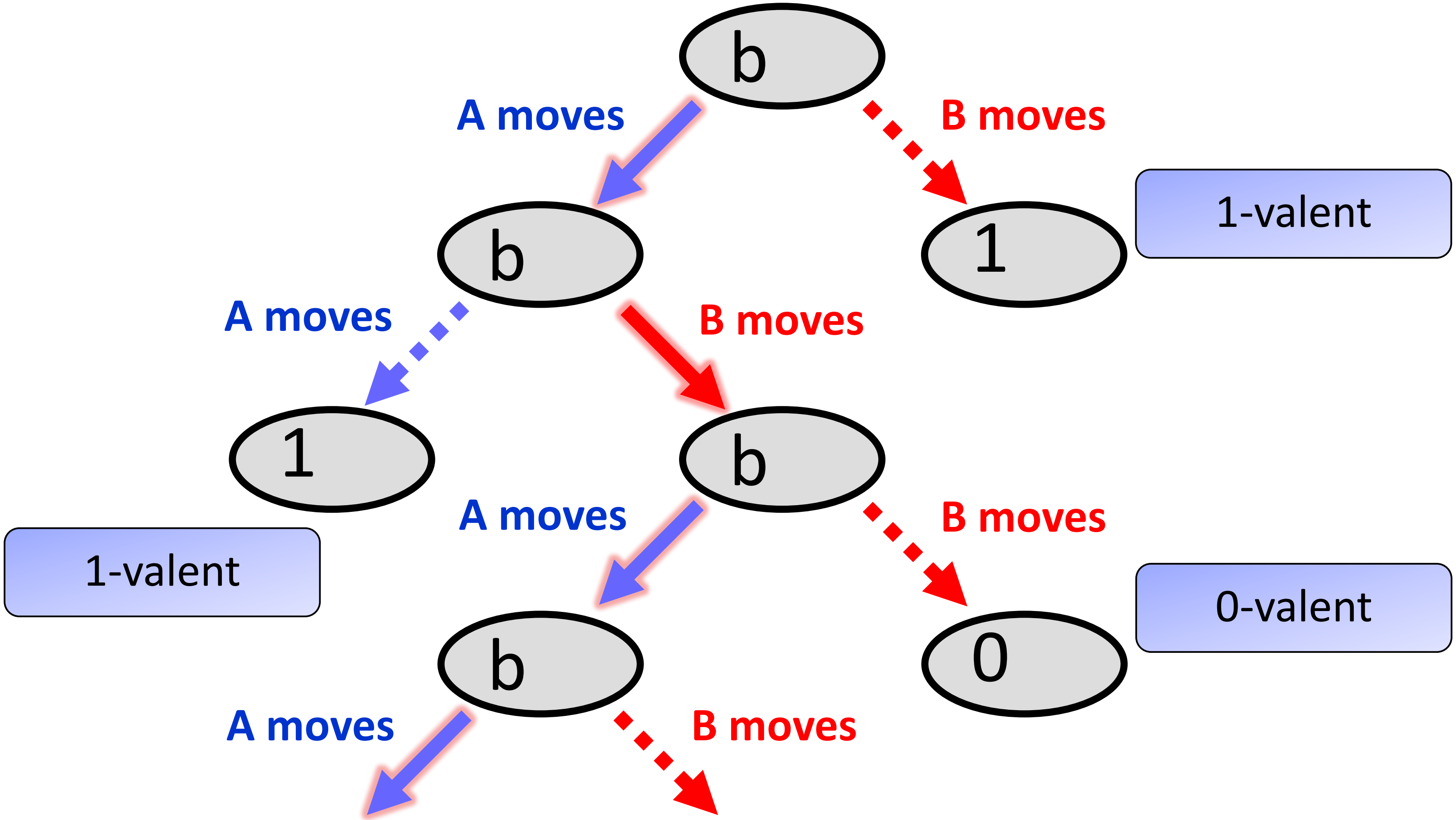




- The goal is now to show that the system can always remain bivalent

Reaching a Critical State

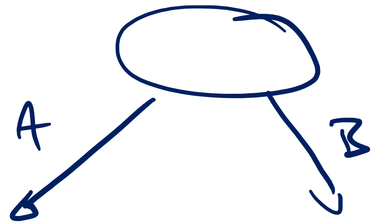
- The system can remain bivalent forever if there is always an action that prevents the system from reaching a critical state:



Model Dependency

- So far, everything was memory-independent!

- True for
  - Registers
  - Message-passing
  - Carrier pigeons
  - Any kind of asynchronous computation



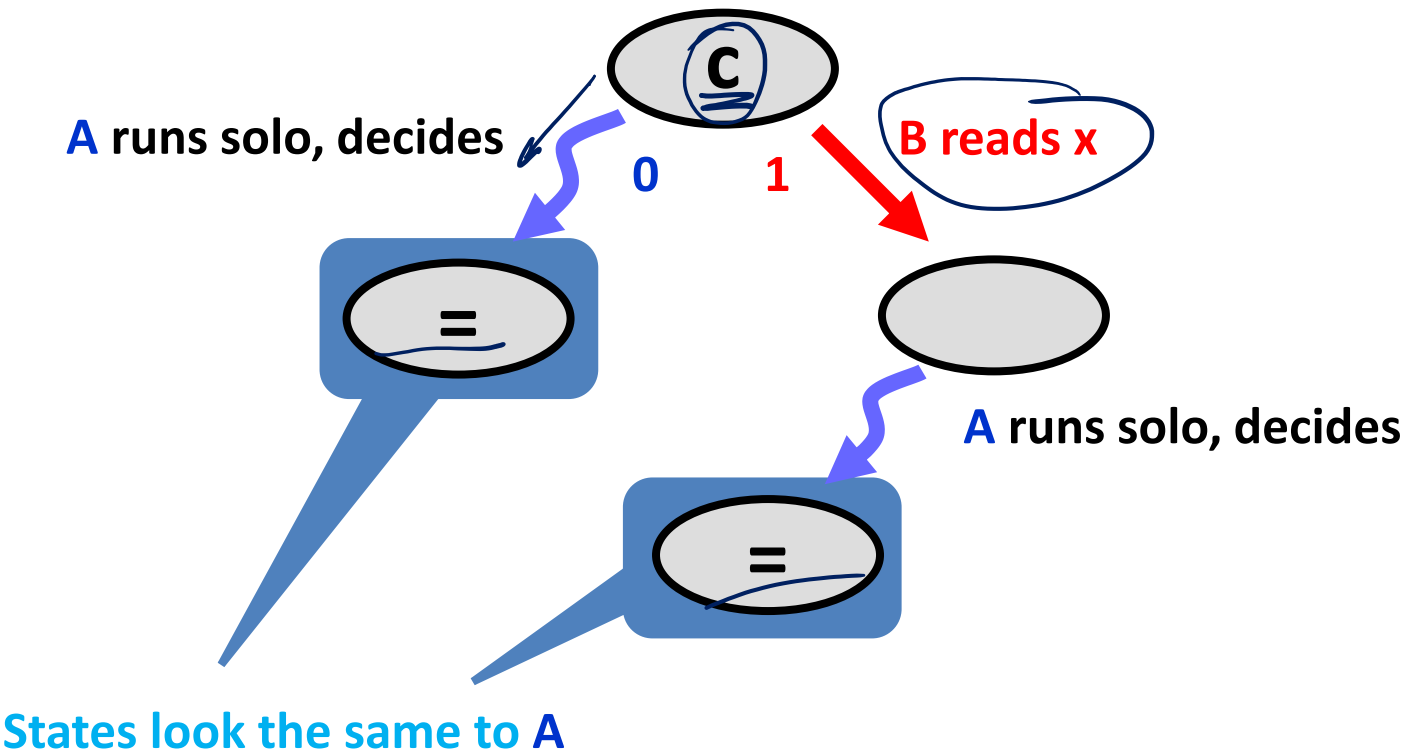
Steps with Shared Read/Write Registers

- Processes/Threads
  - Perform reads and/or writes
  - To the same or different registers
  - Possible interactions?

Possible Interactions

	x.read()	y.read()	x.write()	y.write()
x.read()	?	?	?	?
y.read()	?	?	?	?
x.write()	?	?	?	?
y.write()	?	?	?	?

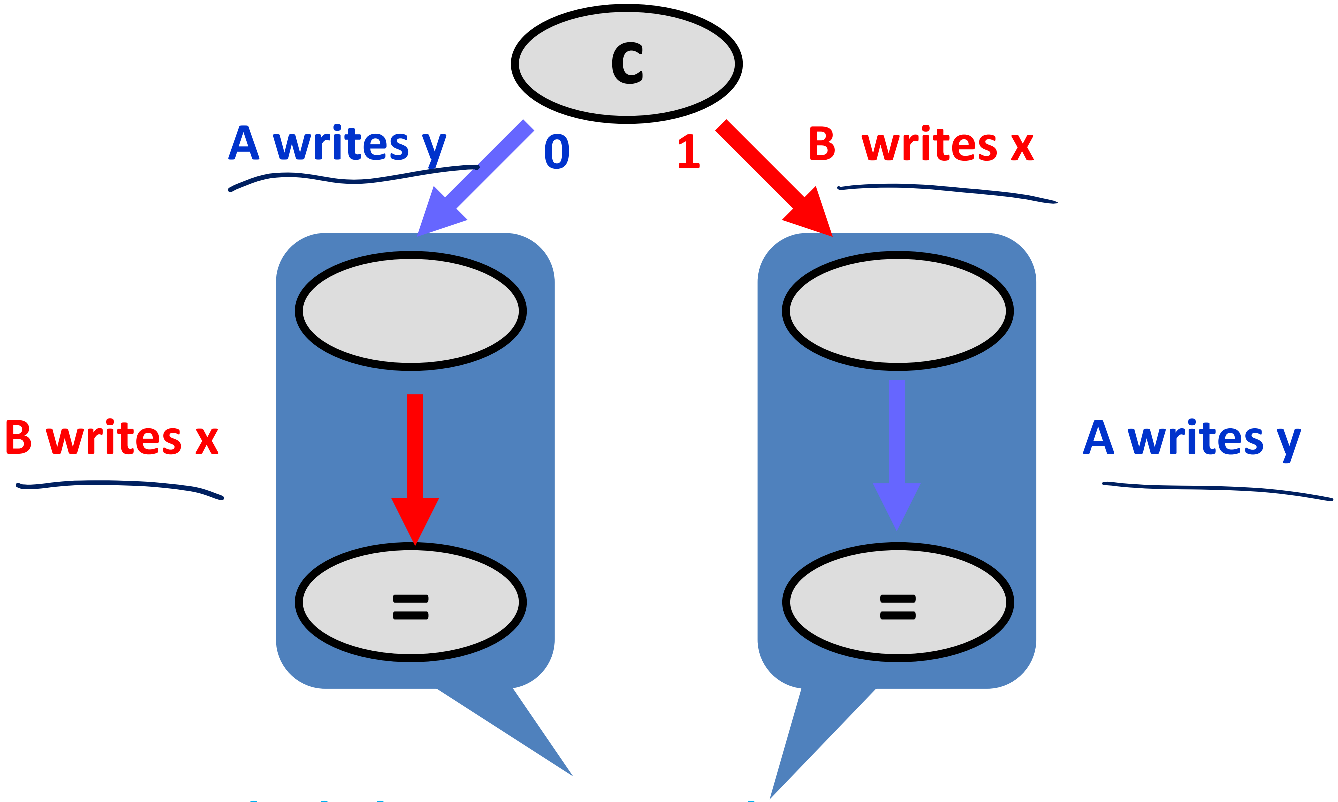
Reading Registers



Possible Interactions

	x.read()	y.read()	x.write()	y.write()
x.read()	no	no	no	no
y.read()	no	no	no	no
x.write()	no	no	?	?
y.write()	no	no	?	?

Writing Distinct Registers

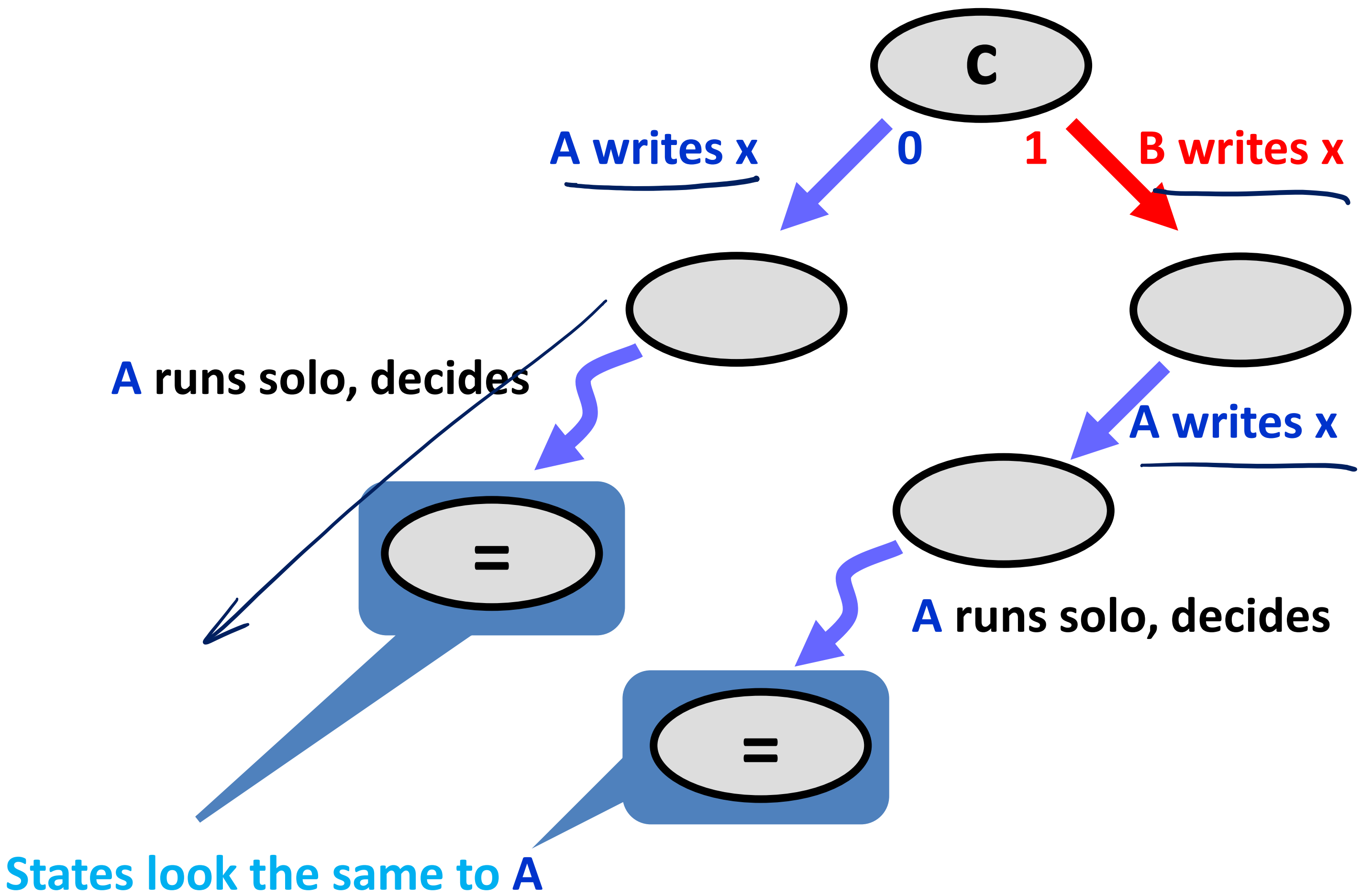


States look the same to A and B

Possible Interactions

	x.read()	y.read()	x.write()	y.write()
x.read()	no	no	no	no
y.read()	no	no	no	no
x.write()	no	no	?	no
y.write()	no	no	no	?

Writing Same Registers

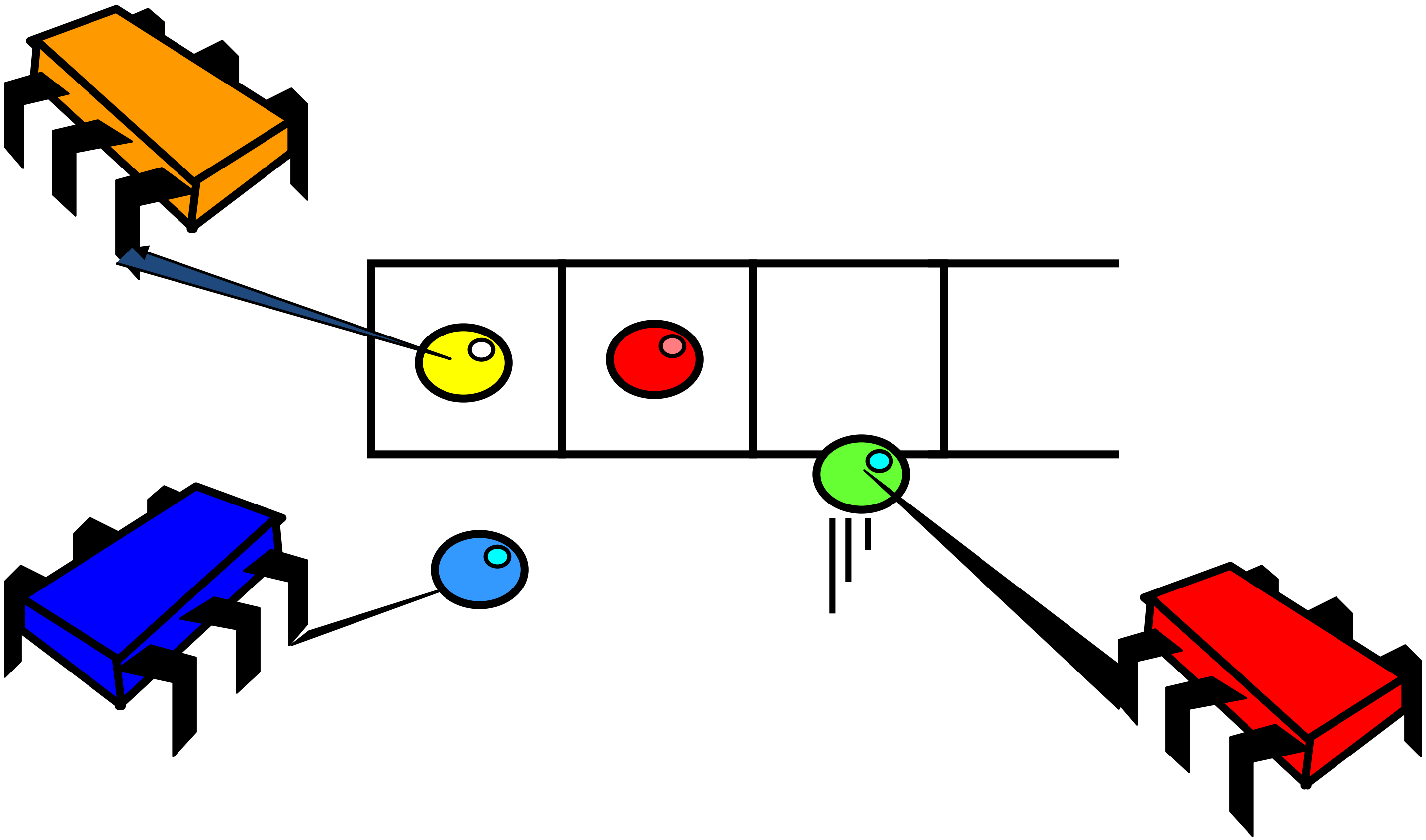


This Concludes the Proof ☺

	x.read()	y.read()	x.write()	y.write()
x.read()	no	no	no	no
y.read()	no	no	no	no
x.write()	no	no	no	no
y.write()	no	no	no	no

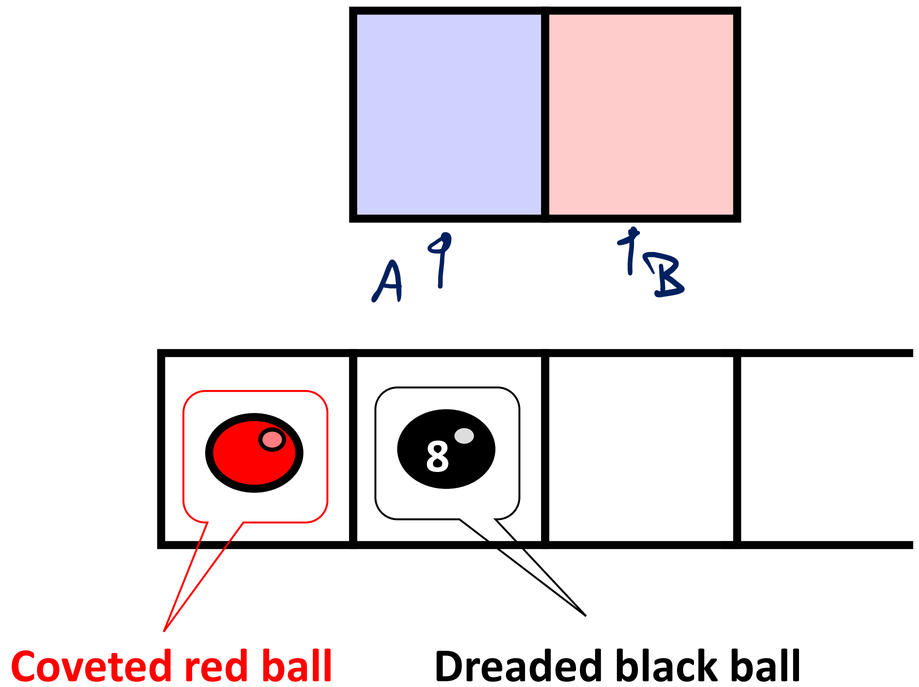
Consensus in Distributed Systems?

- We want to build a concurrent FIFO Queue with multiple dequeuers



A Consensus Protocol

- Assume we have such a FIFO queue and a 2-element array

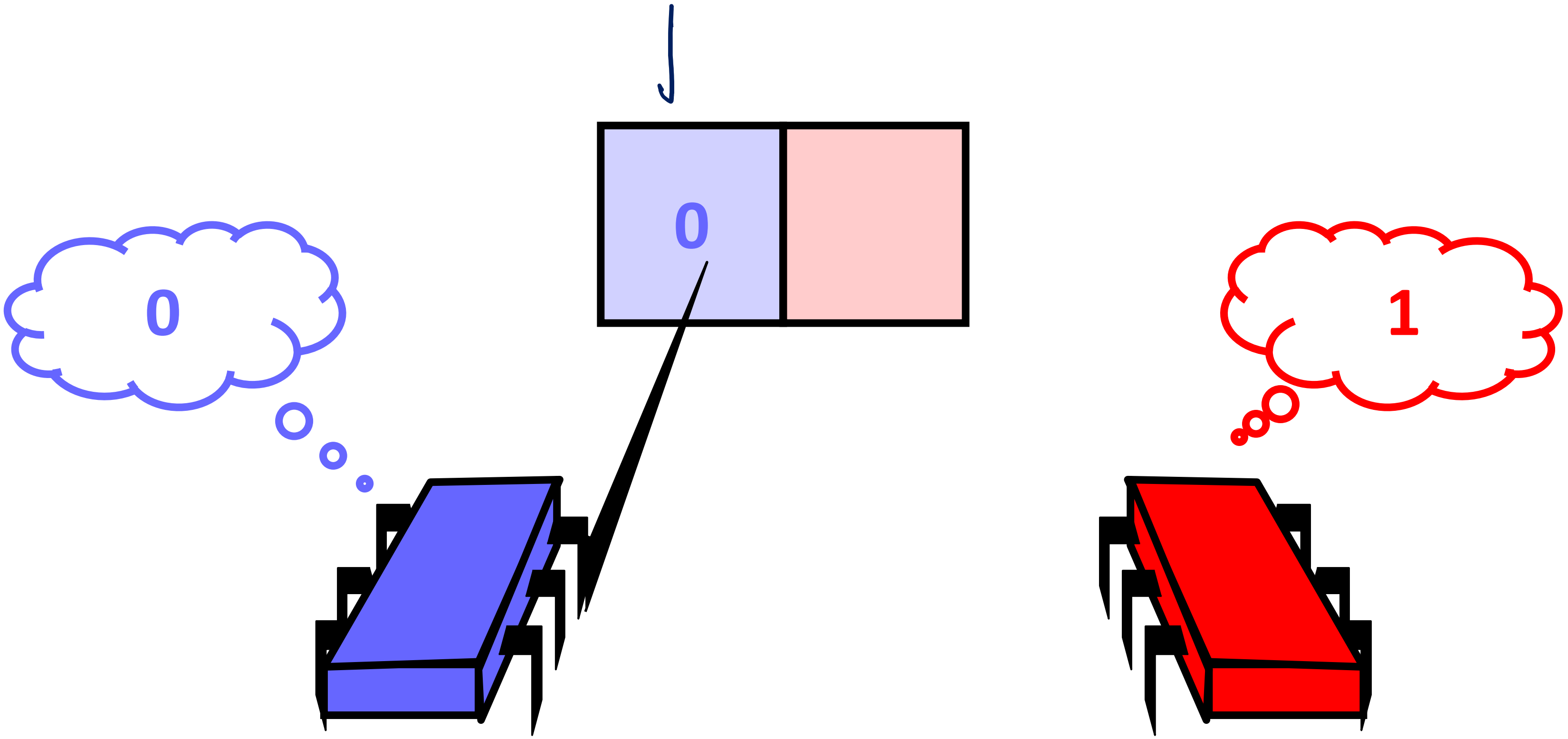


2-element array

FIFO Queue with red and black balls

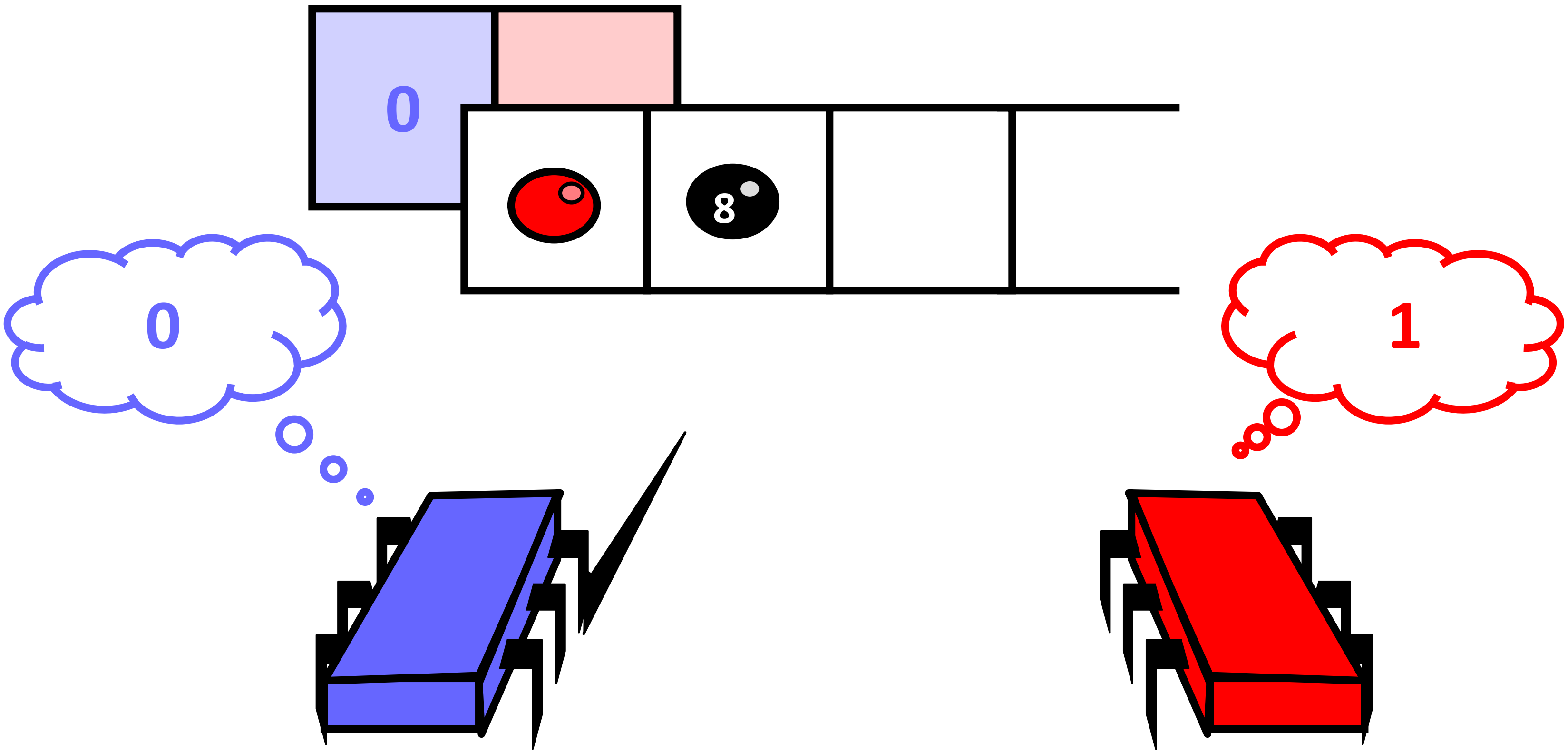
A Consensus Protocol

- Process i writes its value into the array at position i

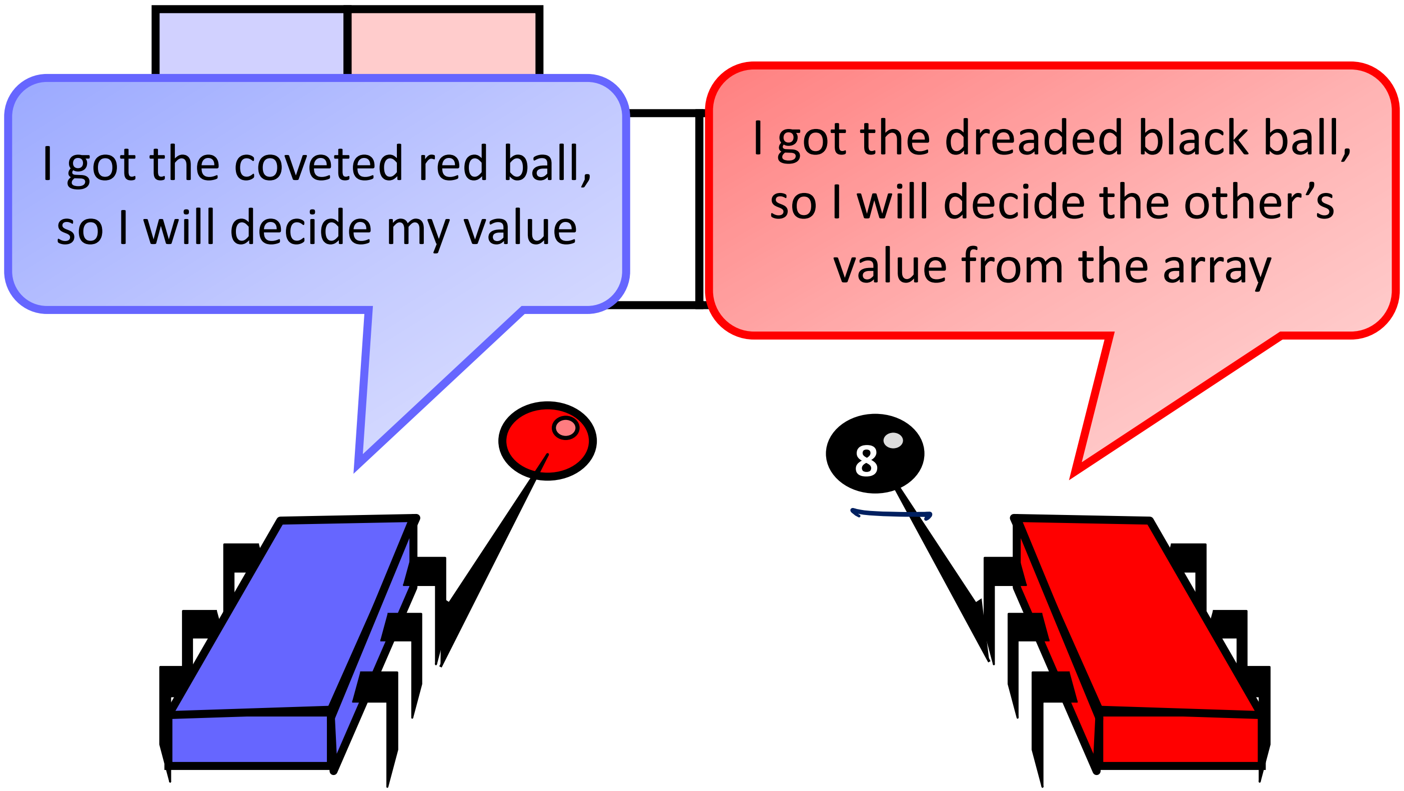


A Consensus Protocol

- Then, the thread takes the next element from the queue



A Consensus Protocol



A Consensus Protocol

Why does this work?

- If one thread gets the red ball, then the other gets the black ball
- Winner can take its own value
- Loser can find winner's value in array
  - Because processes write array before dequeuing from queue

Implication

- We can solve 2-thread consensus using only
  - A two-dequeuer queue
  - Atomic registers

Implications

- Assume there exists
  - A queue implementation from atomic registers
- Given
  - A consensus protocol from queue and registers
- Substitution yields
  - A wait-free consensus protocol from atomic registers

contradiction

Corollary

- It is impossible to implement a two-dequeuer wait-free FIFO queue with read/write shared memory.
- This was a proof by reduction;
  important beyond NP-completeness…

Consensus #3: Read-Modify-Write Memory

- $n > 1$ processes (processors/nodes/threads)
- Wait-free implementation
- Processors can read *and* write a shared memory cell in one atomic step: the value written can depend on the value read
- We call this a read-modify-write (RMW) register
- Can we solve consensus using a RMW register...?

Consensus Protocol Using a RMW Register

- There is a cell c, initially $c =$"?"
- Every processor i does the following

RMW(c)

```
if (c == "?")  then
    write(c, x_i); decide x_i
else
    decide c;
```

atomic step

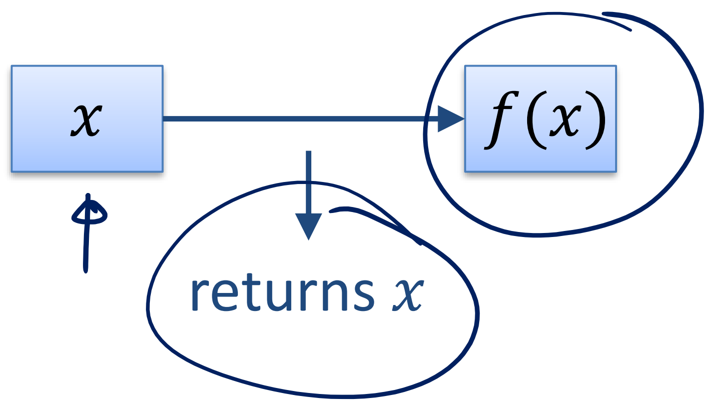
a = read(c)
if a == "?" then
write
atomic

Discussion

- Protocol works correctly
  - One processor accesses c first; this processor will determine decision
- Protocol is wait-free
- RMW is quite a strong primitive
  - Can we achieve the same with a weaker primitive?

Read-Modify-Write More Formally

- Method takes 2 arguments:
  - Cell $\boldsymbol{c}$
  - Function $\boldsymbol{f}$
- Method call:
  - Replaces value $\boldsymbol{x}$ of cell $\boldsymbol{c}$ with $\boldsymbol{f(x)}$
  - Returns value $\boldsymbol{x}$ of cell $\boldsymbol{c}$



Read-Modify-Write

```
public class RMW {
  private int value;

  public synchronized int rmw(function f) {
    int prior  = this.value;
    this.value = f(this.value);
    return prior;
  }

}
```

Return prior value

Apply function

Read-Modify-Write: Read

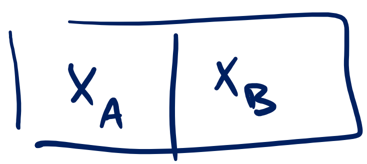
```
public class RMW {
  private int value;

  public synchronized int read() {
    int prior  = this.value;
    this.value = this.value;
    return prior;
  }

}
```

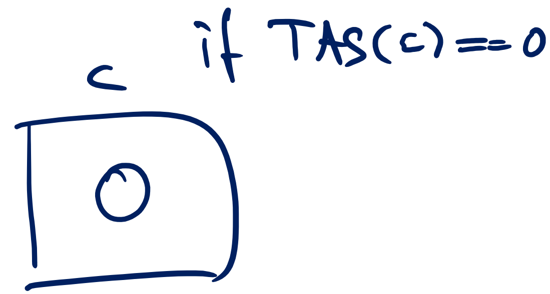
Identify function

Read-Modify-Write: Test&Set

```java
public class RMW {
  private int value;

  public synchronized int TAS() {
    int prior  = this.value;
    this.value = 1;
    return prior;
  }

}
```

Constant function





Read-Modify-Write: Fetch&Inc

```java
public class RMW {
  private int value;

  public synchronized int FAI() {
    int prior  = this.value;
    this.value = this.value+1;
    return prior;
  }

}
```

Increment function

Read-Modify-Write: Fetch&Add

```
public class RMW {
  private int value;

  public synchronized int FAA(int x) {
    int prior  = this.value;
    this.value = this.value+x;
    return prior;
  }

}
```

Addition function

Read-Modify-Write: Swap

```
public class RMW {
  private int value;

  public synchronized int swap(int x) {
    int prior  = this.value;
    this.value = x;
    return prior;
  }

}
```

Set to x

Read-Modify-Write: Compare&Swap

```
public class RMW {
  private int value;

  public synchronized int CAS(int old, int new) {
    int prior  = this.value;
    if(this.value == old)
      this.value = new;
    return prior;
  }

}
```

"Complex" function

Definition of Consensus Number

- An object has **consensus number $\boldsymbol{n}$**
  - If it can be used
    - Together with atomic read/write registers
  - To implement n-process consensus, but not $(n + 1)$-process consensus

- Example: Atomic read/write registers have consensus number 1
  - Works with 1 process
  - We have shown impossibility with 2

Consensus Number Theorem

Theorem

If you can implement $\boldsymbol{X}$ **from** $\boldsymbol{Y}$ and $\boldsymbol{X}$ has consensus number $\boldsymbol{c}$, then $\boldsymbol{Y}$ has consensus number **at least** $\boldsymbol{c}$.

Consensus Number Theorem

Theorem

If you can implement $\boldsymbol{X}$ **from** $\boldsymbol{Y}$ and
$\boldsymbol{X}$ has consensus number $\boldsymbol{c}$, then
$\boldsymbol{Y}$ has consensus number **at least** $\boldsymbol{c}$.

- Consensus numbers are a useful way of measuring synchronization power
- An alternative formulation:
  - If X has consensus number c
  - And Y has consensus number $d < c$
  - Then there is no way to construct a wait-free implementation of X by Y
- This theorem is very useful
  - Unforeseen practical implications!

Theorem

- A RMW is *non-trivial* if there exists a value v such that $v \neq f(v)$
  - Test&Set, Fetch&Inc, Fetch&Add, Swap, Compare&Swap, general RMW…
  - But not read

Theorem

Any non-trivial RMW object has consensus number at least 2.

- Implies no wait-free implementation of RMW registers from read/write registers
- Hardware RMW instructions not just a convenience

Proof

- A two-process consensus protocol using any non-trivial RMW object:

```
public class RMWConsensusFor2 implements Consensus{
  private RMW r;

  public Object decide() {
    int i  = Thread.myIndex();
    if(r.rmw(f) == v)
      return this.announce[i];
    else
      return this.announce[1-i];
  }

}
```

Initialized to v (handwritten: initial value v)

Am I first? (handwritten: this.announce[i] = x_i)

Yes, return my input

No, return other's input

Interfering RMW

- Let F be a set of functions such that for all f_i and f_j, either
  - They commute: $f_i(f_j(x))=f_j(f_i(x))$
  - They overwrite: $f_i(f_j(x))=f_i(x)$

$f_i(x)$ = new value of cell
(not return value of f_i)

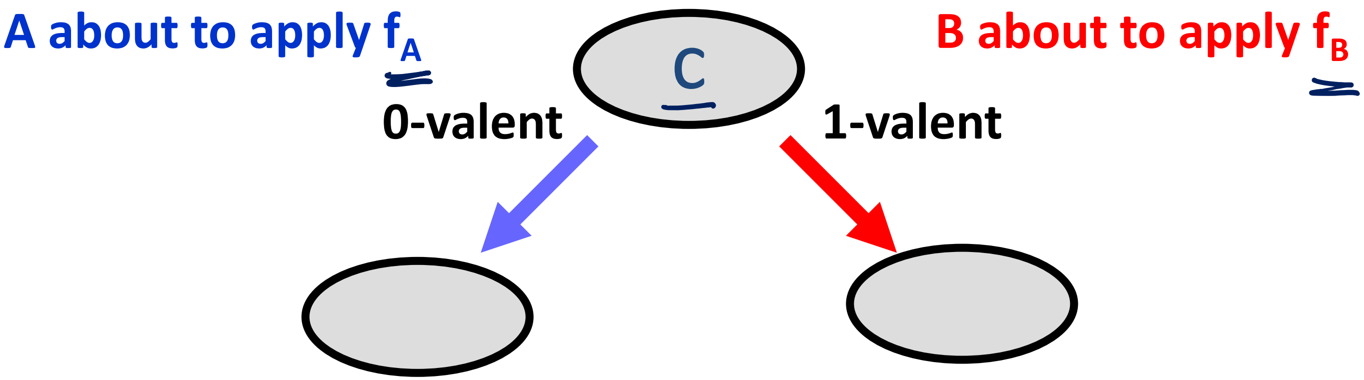
Claim: Any such non-trivial RMW object has consensus number **exactly 2**

Examples:

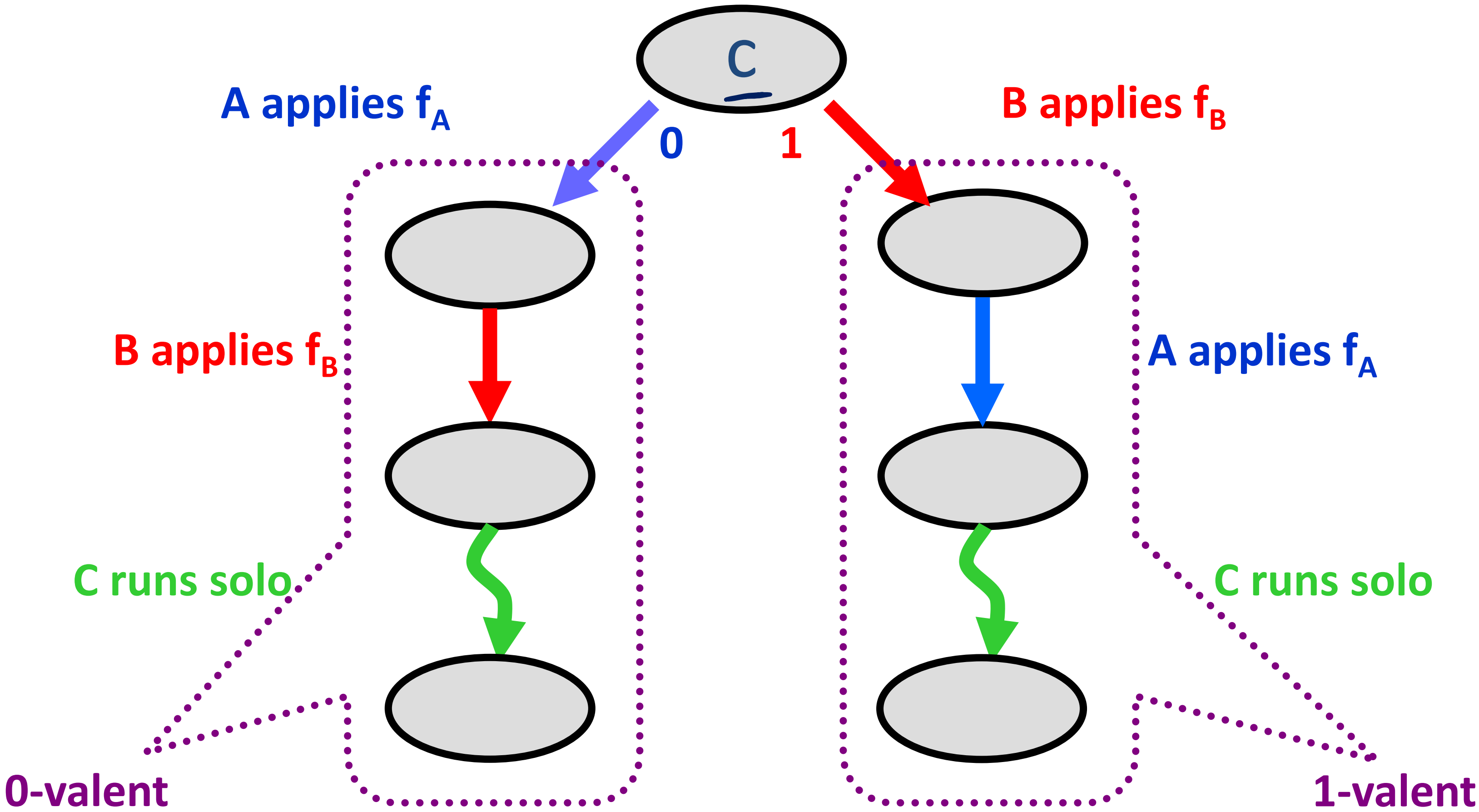
- Overwrite
  - Test&Set , Swap
- Commute
  - Fetch&Inc, Fetch&Add

Proof

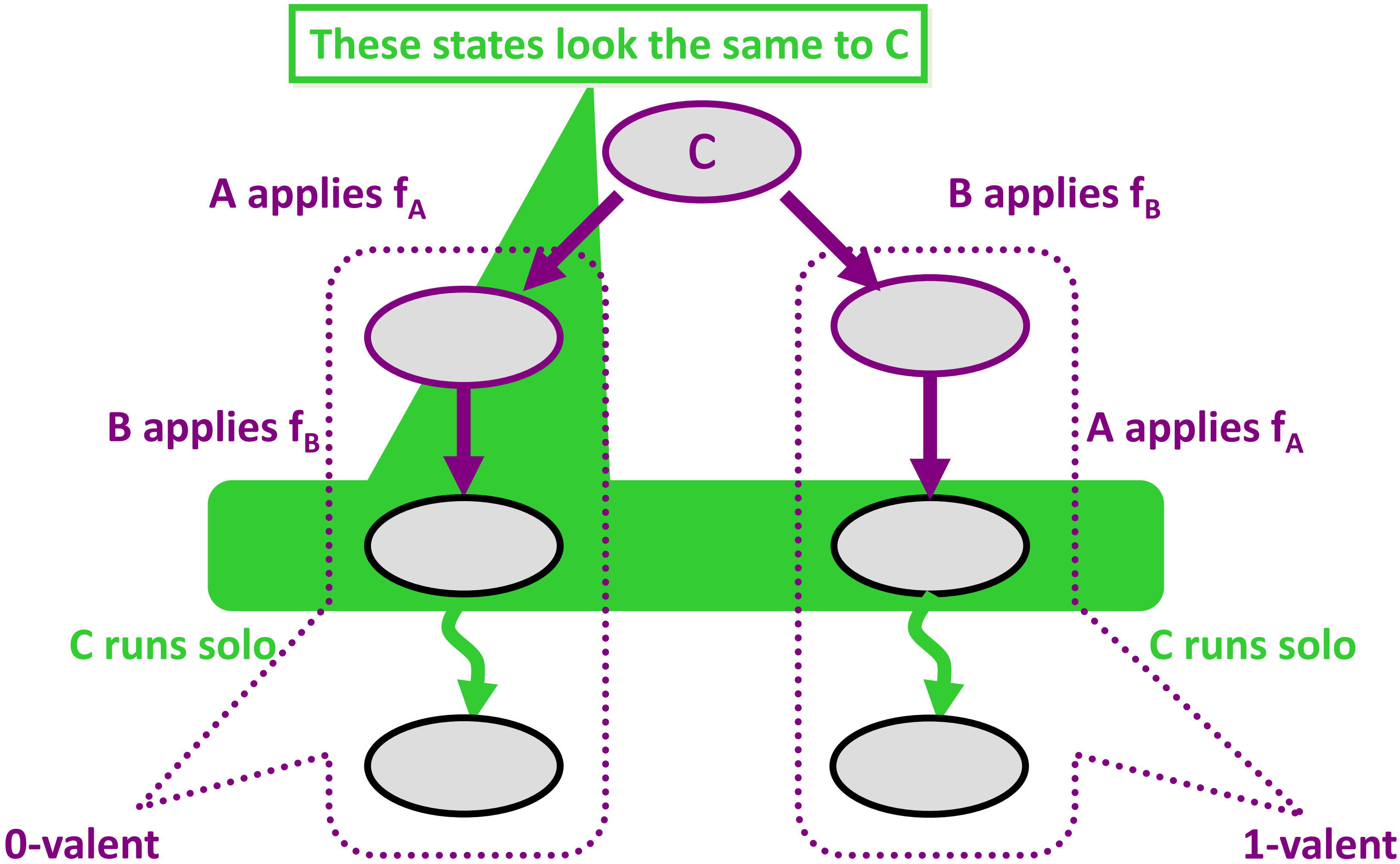
- There are three threads, A, B, and C
- Consider a critical state *c*:



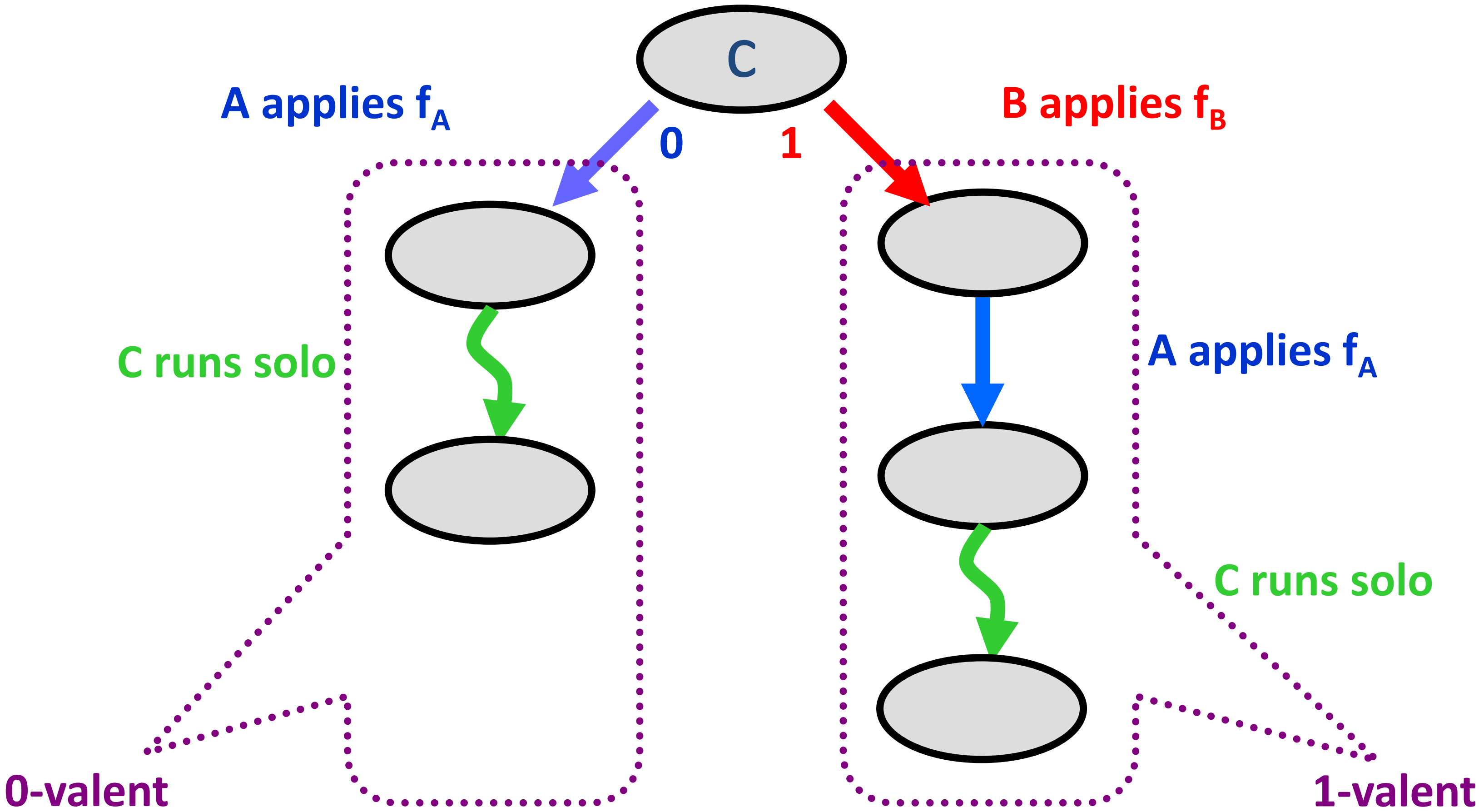
Proof: Maybe the Functions Commute



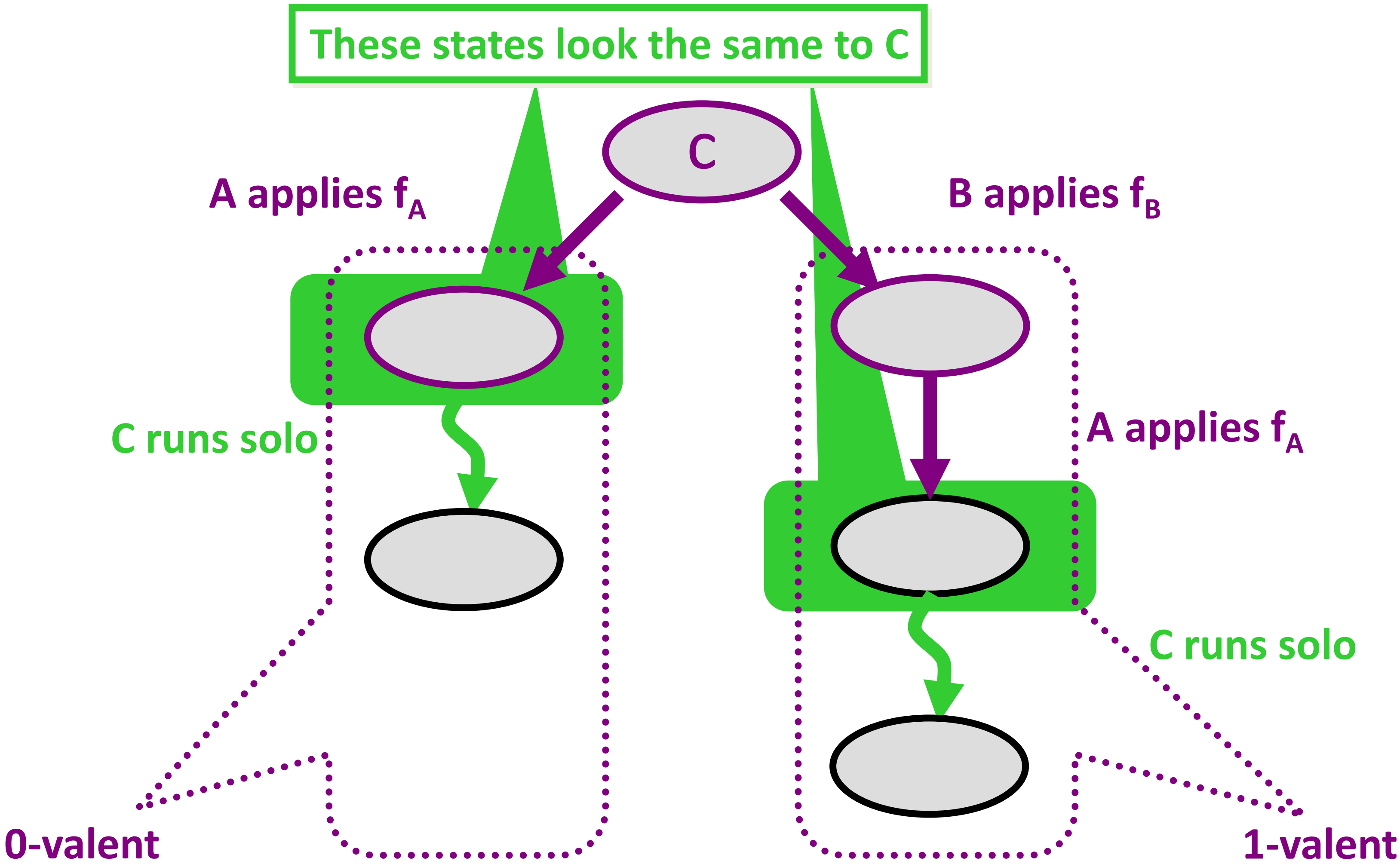
Proof: Maybe the Functions Commute



Proof: Maybe the Functions Overwrite



Proof: Maybe the Functions Overwrite



Impact

- Many early machines used these “weak” RMW instructions
  - Test&Set (IBM 360)
  - Fetch&Add (NYU Ultracomputer)
  - Swap
- We now understand their limitations

Consensus with Compare & Swap

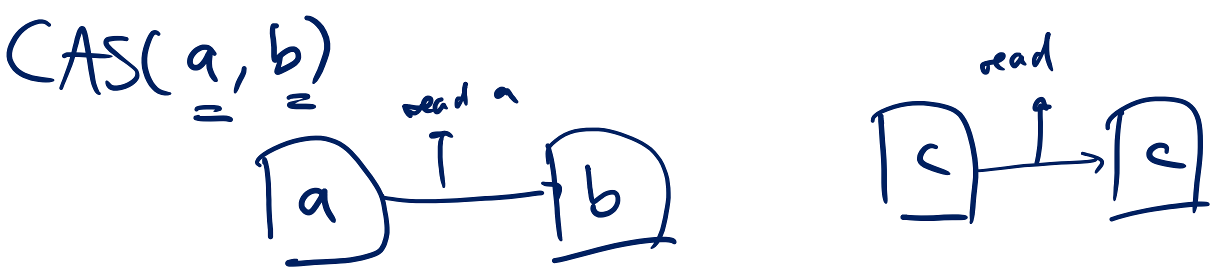
```
public class RMWConsensus implements Consensus {
  private RMW r;

  public Object decide() {
    int i = Thread.myIndex();
    int j = r.CAS(-1,i);
    if(j == -1)
      return this.announce[i];
    else
      return this.announce[j];
  }

}
```

Initialized to -1

Am I first?

Yes, return my input

No, return other's input

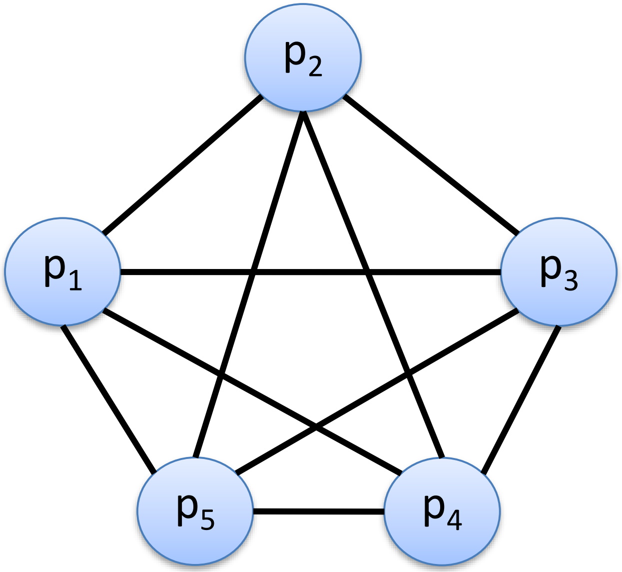


The Consensus Hierarchy

1	2	…	∞
• Read/Write Registers	• Test&Set • Fetch&Inc • Fetch&Add • Swap		• CAS • LL/SC

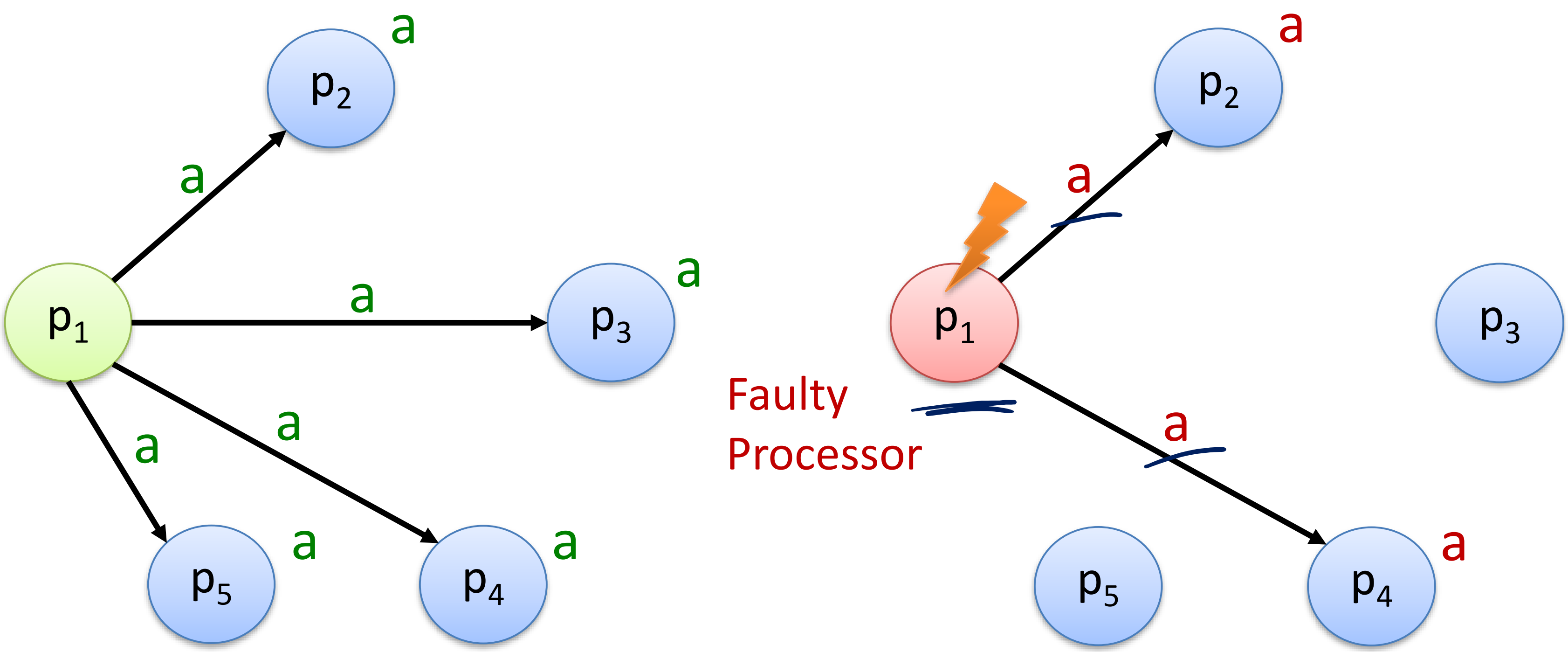
Consensus #4: Synchronous Systems

- One can sometimes tell if a processor had crashed
  - Timeouts
  - Broken TCP connections
- Can one solve consensus at least in synchronous systems?
- Model
  - All communication occurs in synchronous rounds
  - Complete communication graph

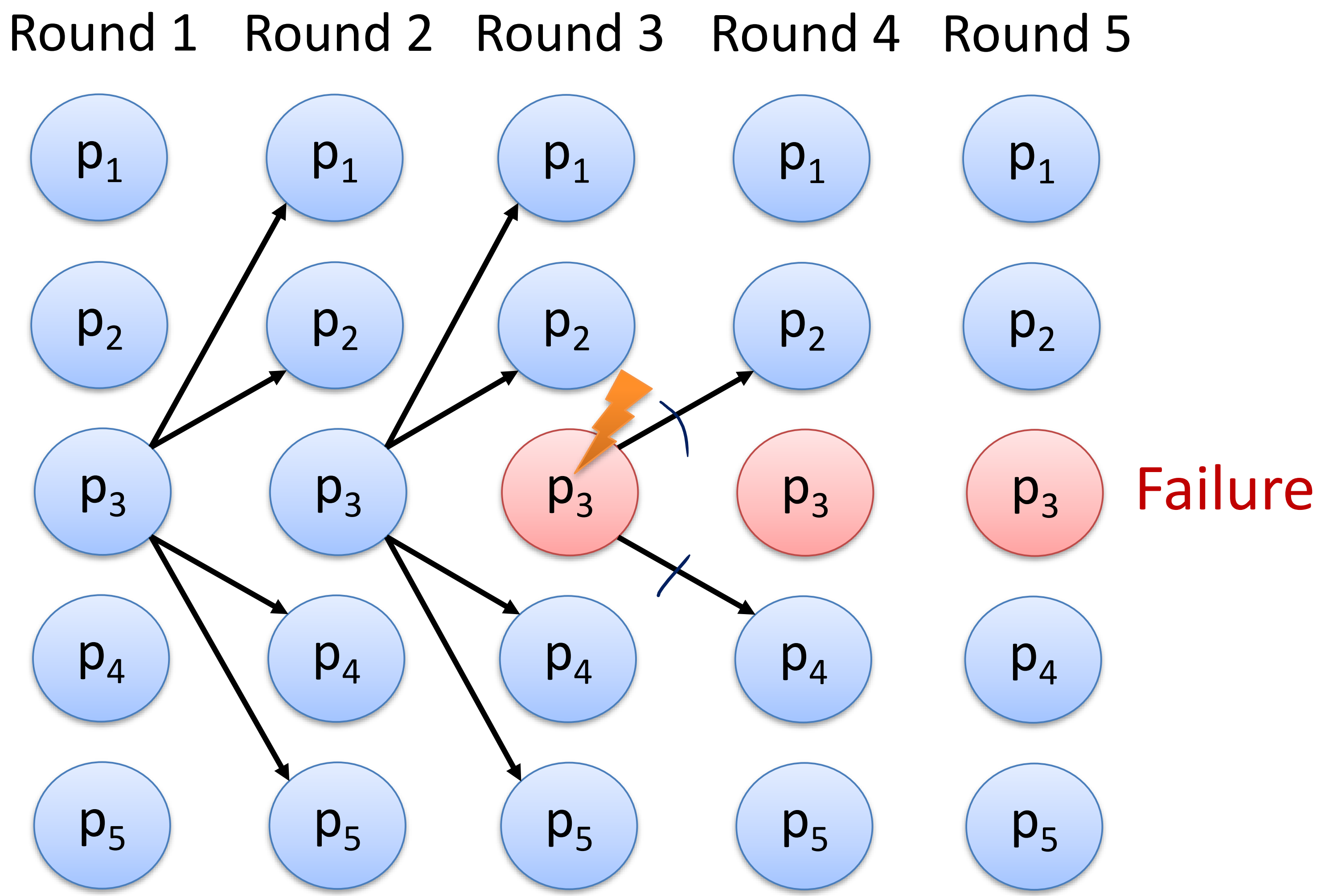


Crash Failures

- Broadcast: Send a message to all nodes in one round
  - At the end of the round everybody receives the message a
  - Every process can broadcast a value in each round
- Crash Failures: A broadcast can fail if a process crashes
  - Some of the messages may be lost, i.e., they are never received

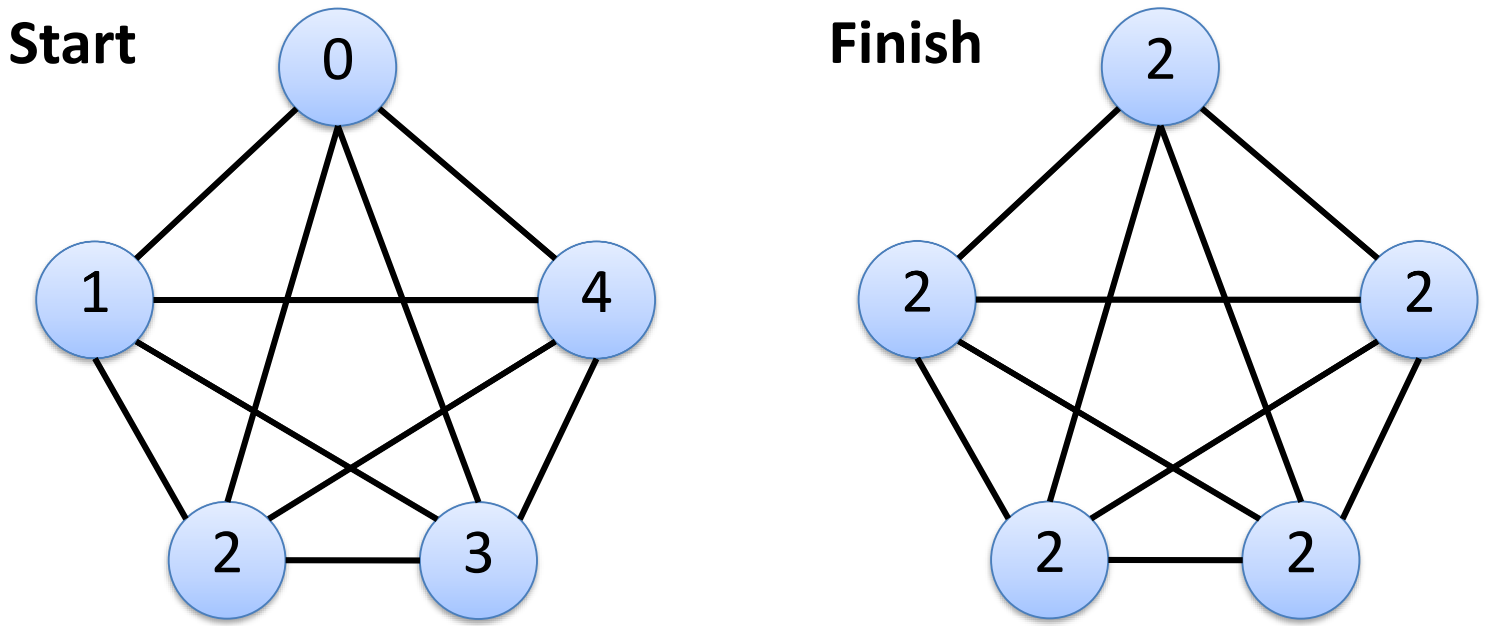


Process disappears after failure



Consensus Repetition

- **Input:** everybody has an initial value
- **Agreement:** everybody must decide on the same value



- **Validity conditon**: If everybody starts with the same value, everybody must decide on that value

A Simple Consensus Algorithm

Each process:

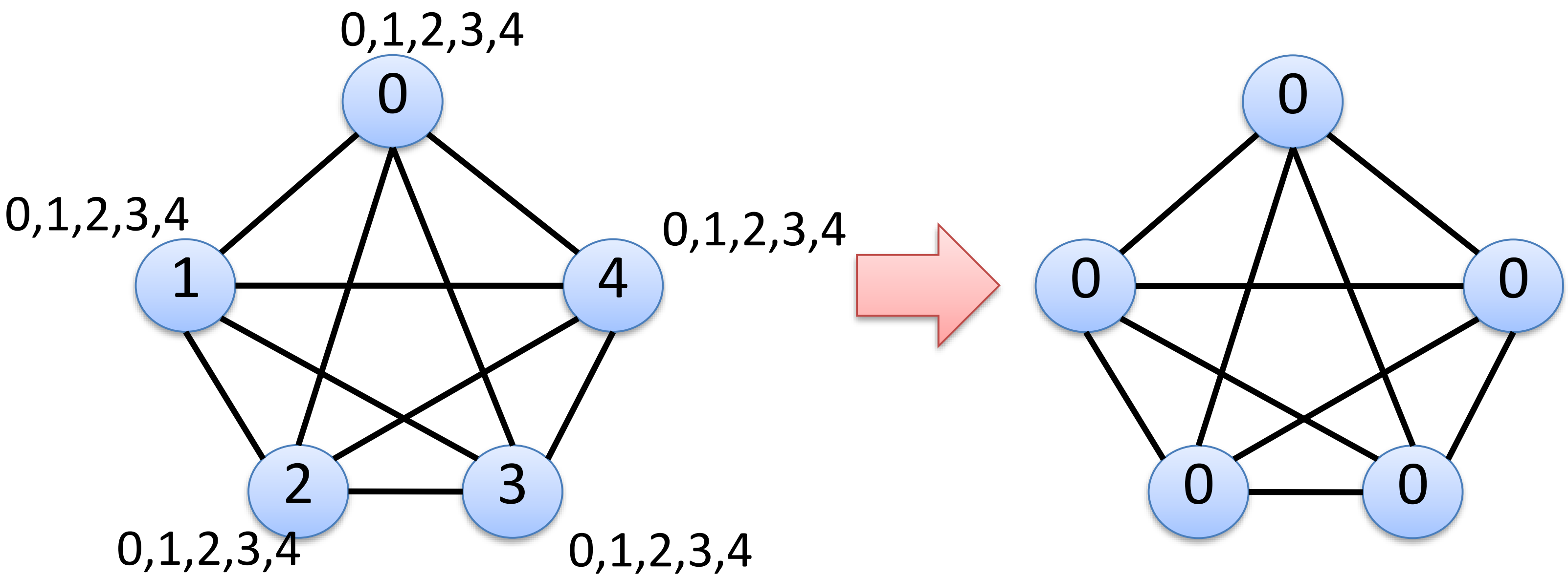
1. Broadcast own value
2. Decide on the minimum of all received values

Including the own value

Note than only one round is needed!

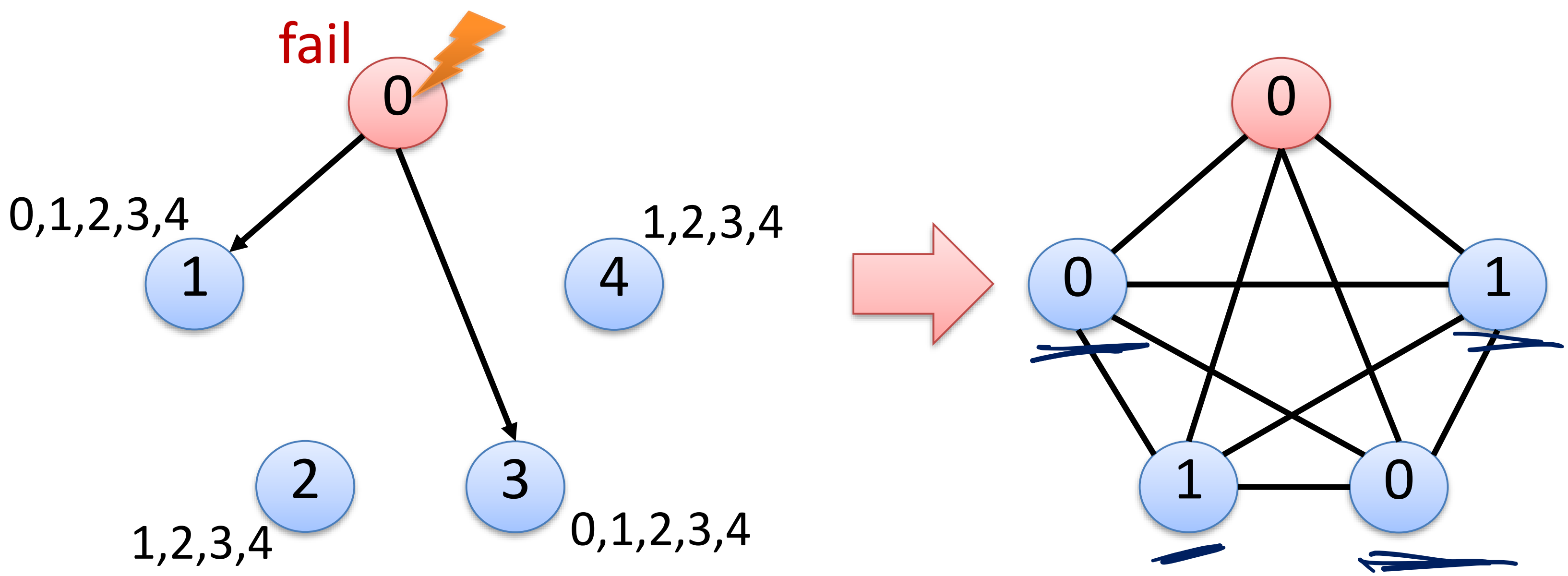
Execution Without Failures

- Broadcast values and decide on minimum → Consensus!
- Validity condition is satisfied: If everybody starts with the same initial value, everybody sticks to that value (minimum)



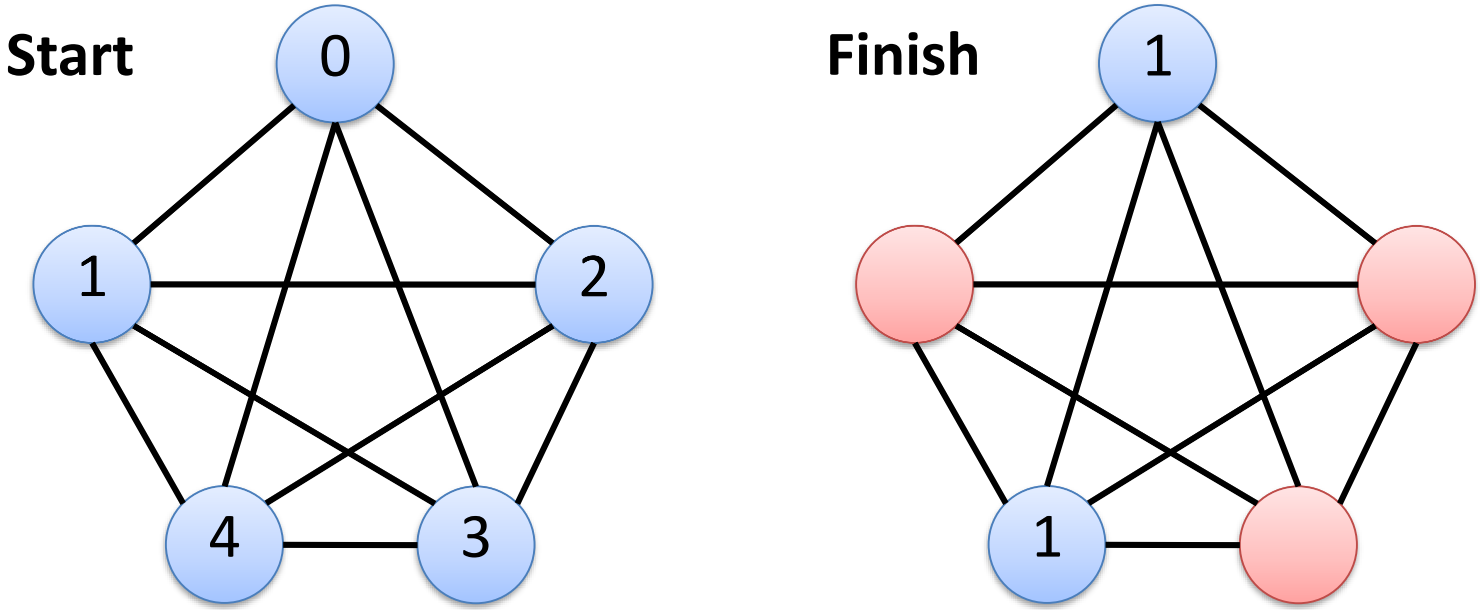
Execution With Failures

- The failed processor doesn't broadcast its value to all processors
- Decide on minimum → No consensus!



f-Resilient Consensus Algorithm

- If an algorithm solves consensus for f failed processes, we say it is an f-resilient consensus algorithm
- Example: The input and output of a 3-resilient consensus alg.



- **Refined validity condition**:
  All processes decide on a value that is available initially

An f-Resilient Consensus Algorithm

Each process:

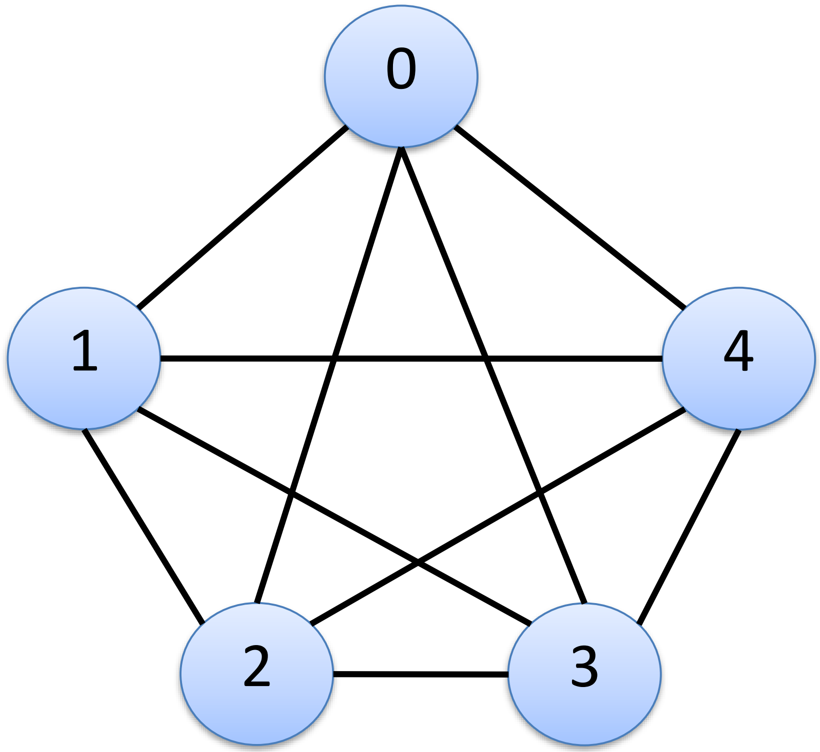
Round 1:
Broadcast own value

Round 2 to round $f + 1$:
Broadcast the minimum of the received values unless it has been sent before

End of round $f + 1$:
Decide on the minimum value received

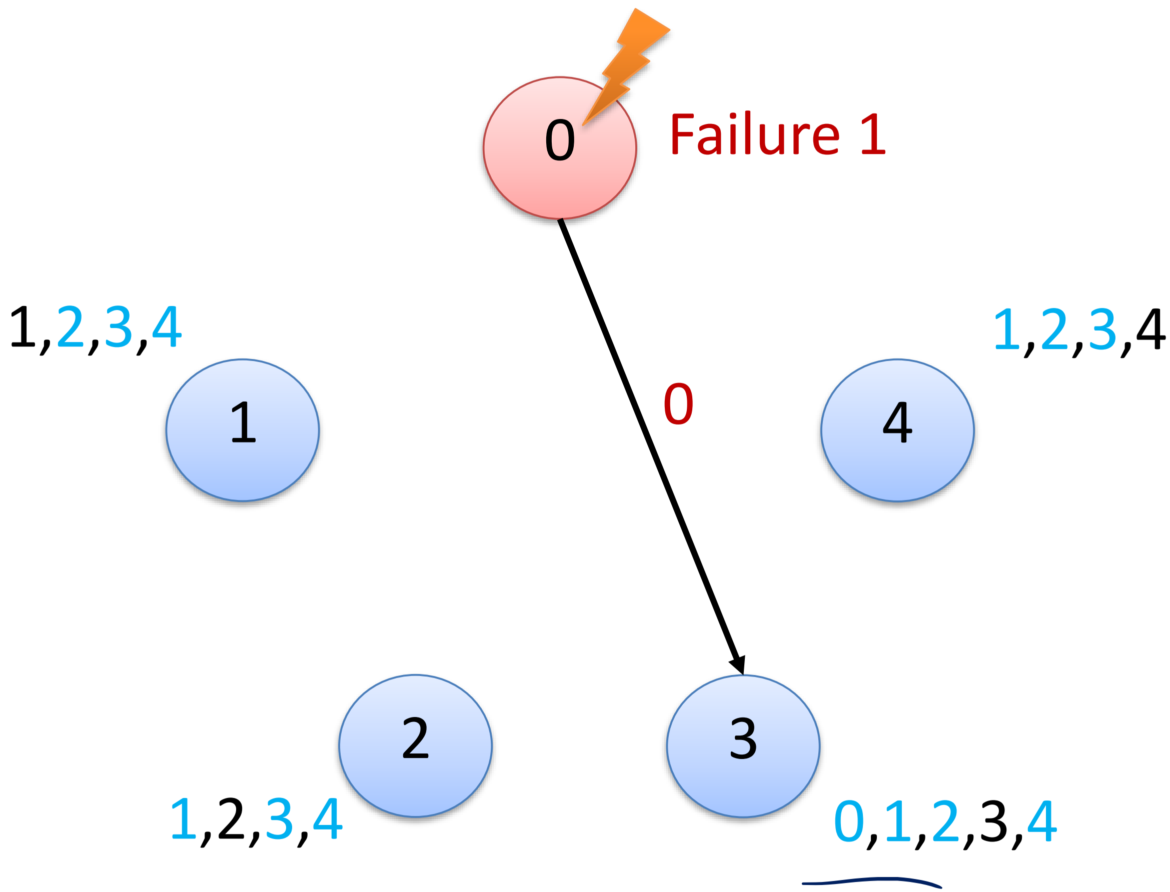
An f-Resilient Consensus Algorithm

- Example: $f = 2$ failures, $f + 1 = 3$ rounds needed



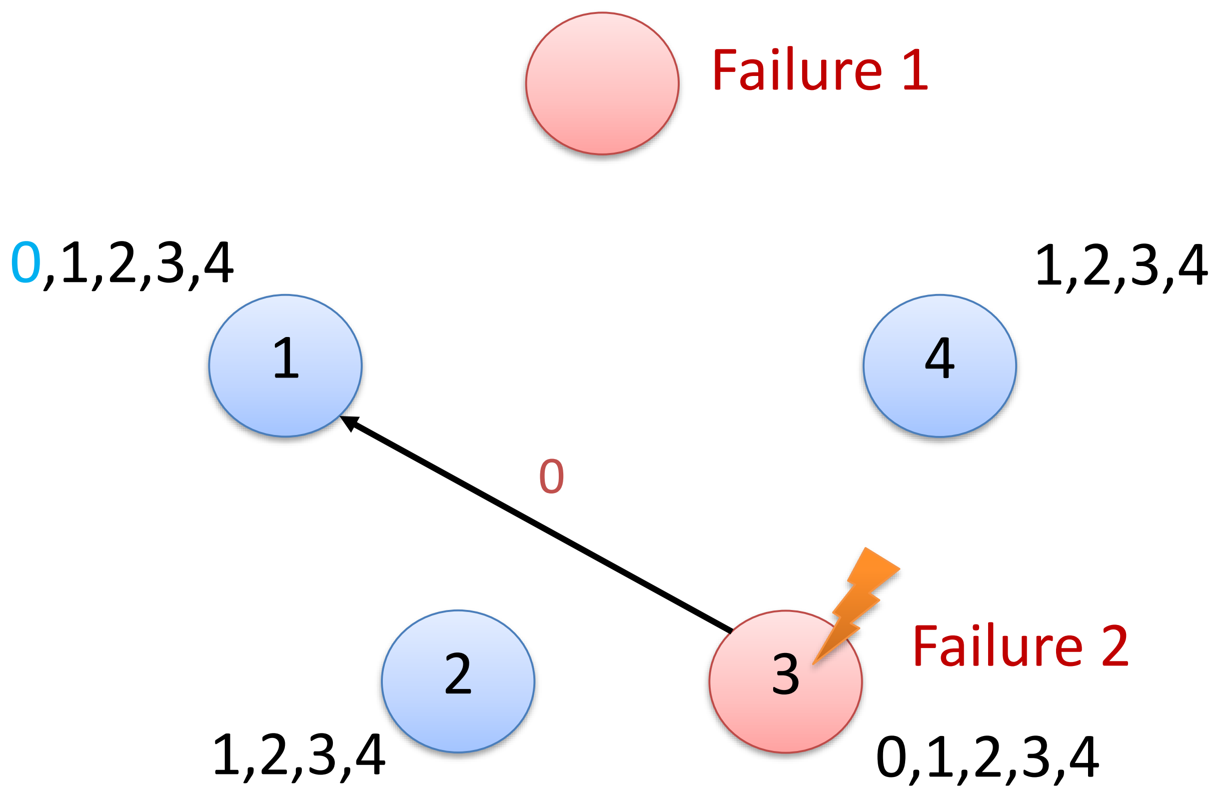
An f-Resilient Consensus Algorithm

- Round 1: Broadcast all values to everybody



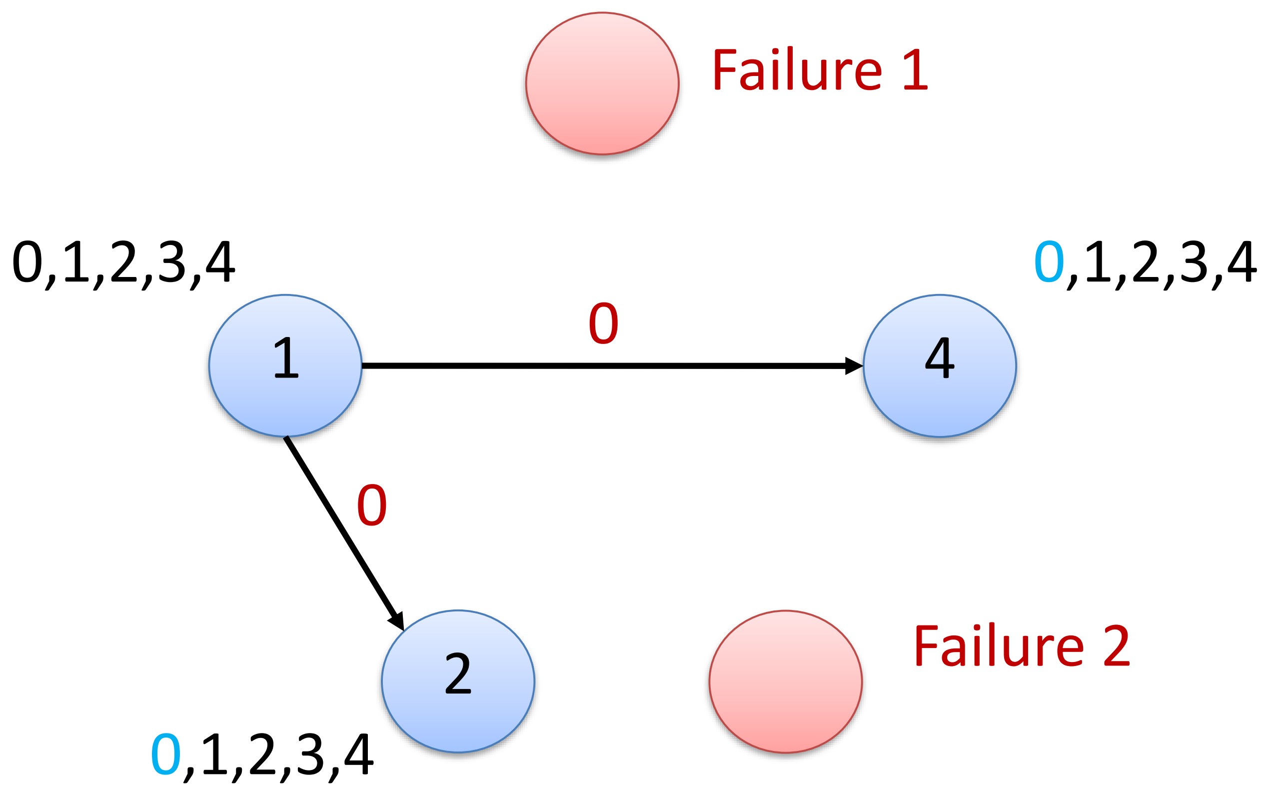
An f-Resilient Consensus Algorithm

- Round 2: Broadcast all new values to everybody



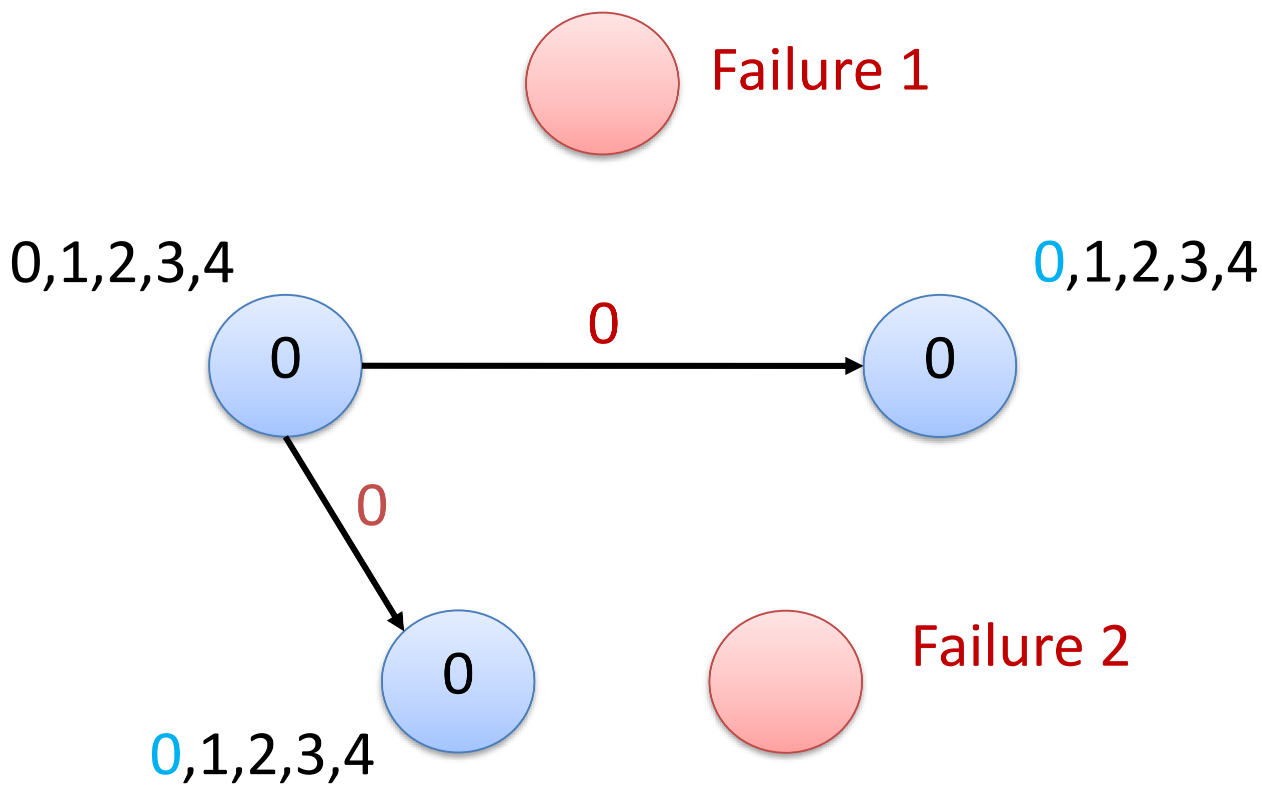
An f-Resilient Consensus Algorithm

- Round 3: Broadcast all new values to everybody



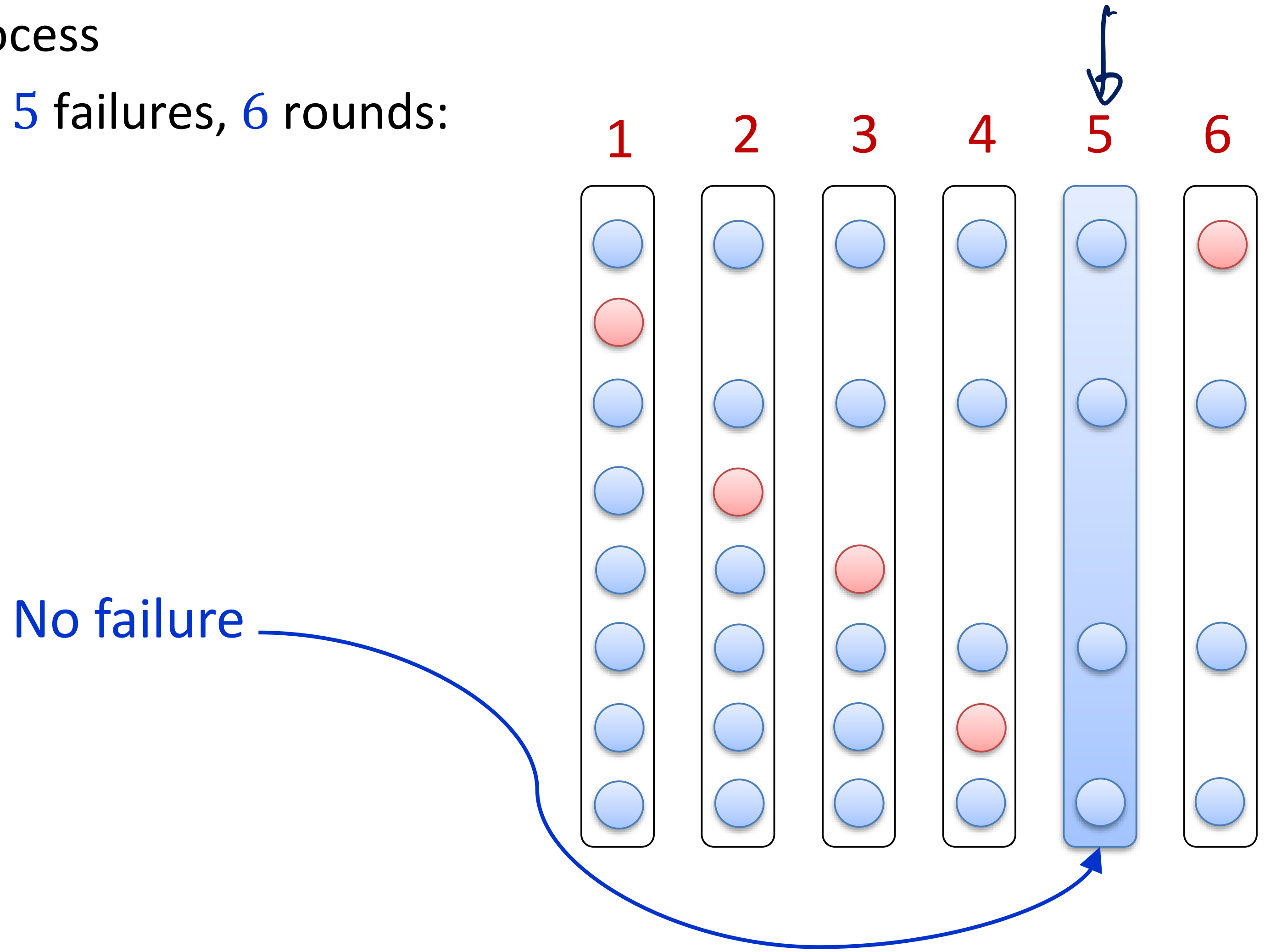
An f-Resilient Consensus Algorithm

- Decide on minimum → Consensus!



Analysis

- If there are f failures and $f+1$ rounds, then there is a round with no failed process
- Example: 5 failures, 6 rounds:



Analysis

- At the end of the round with no failure
  - Every (non faulty) process knows about all the values of all the other participating processes
  - This knowledge doesn't change until the end of the algorithm
- Therefore, everybody will decide on the same value
- However, as we don't know the exact position of this round, we have to let the algorithm execute for $f + 1$ rounds

- **Validity:** When all processes start with the same input value, then consensus is that value

Theorem

Theorem

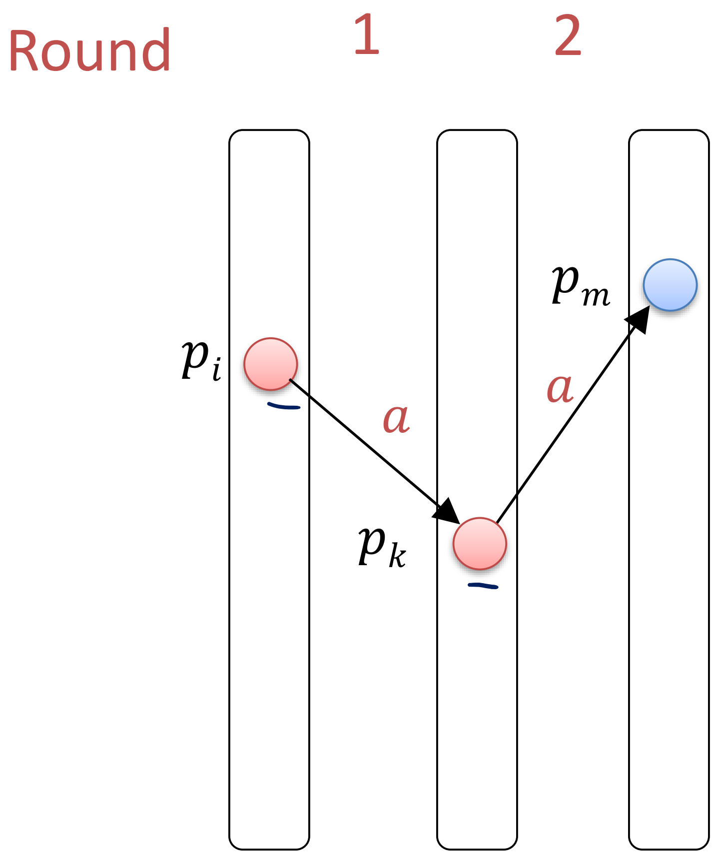
If at most $f \leq n - 2$ of n nodes of a synchronous message passing system can crash, at least $f + 1$ rounds are needed to solve consensus.

Proof idea:

- Show that f rounds are not enough if $n \geq f + 2$
- Before proving the theorem, we consider a

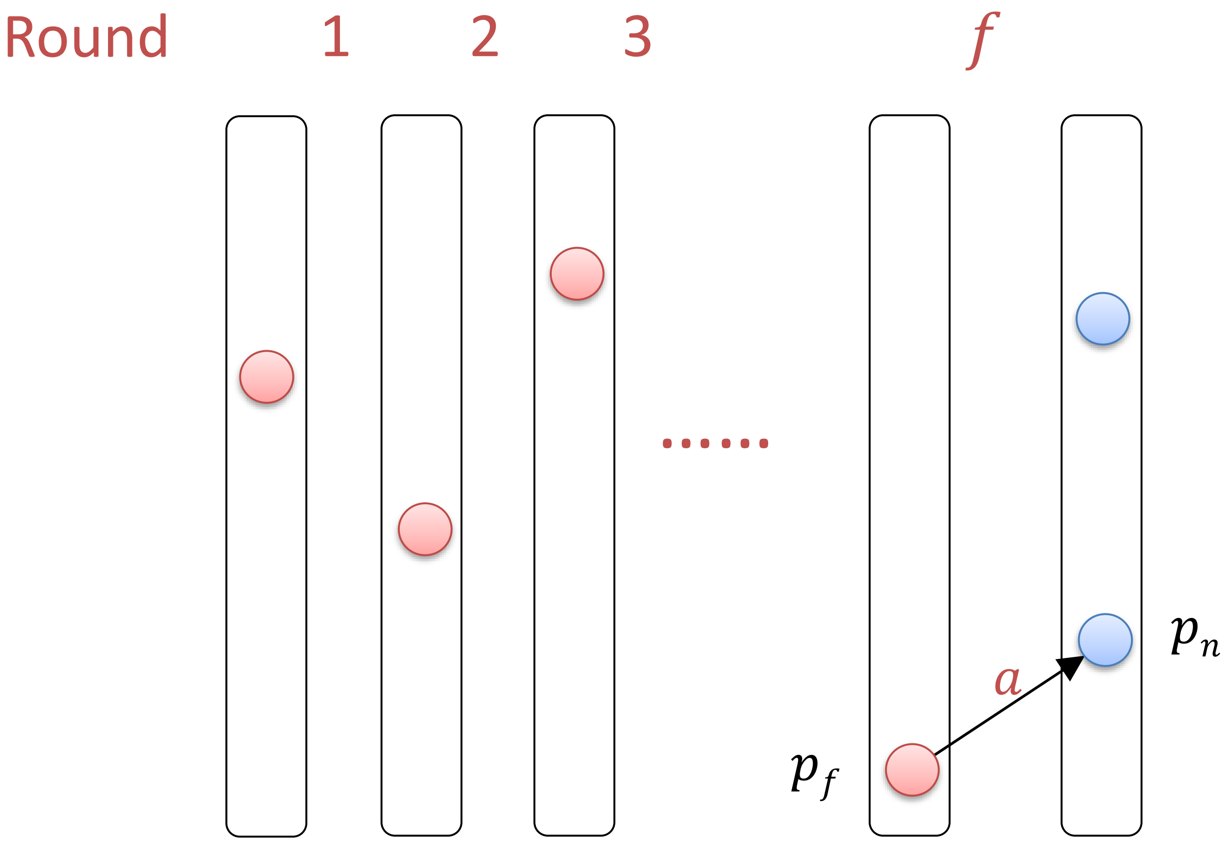
  "worst-case scenario": In each round one of the processes fails

Lower Bound on Rounds: Intuition



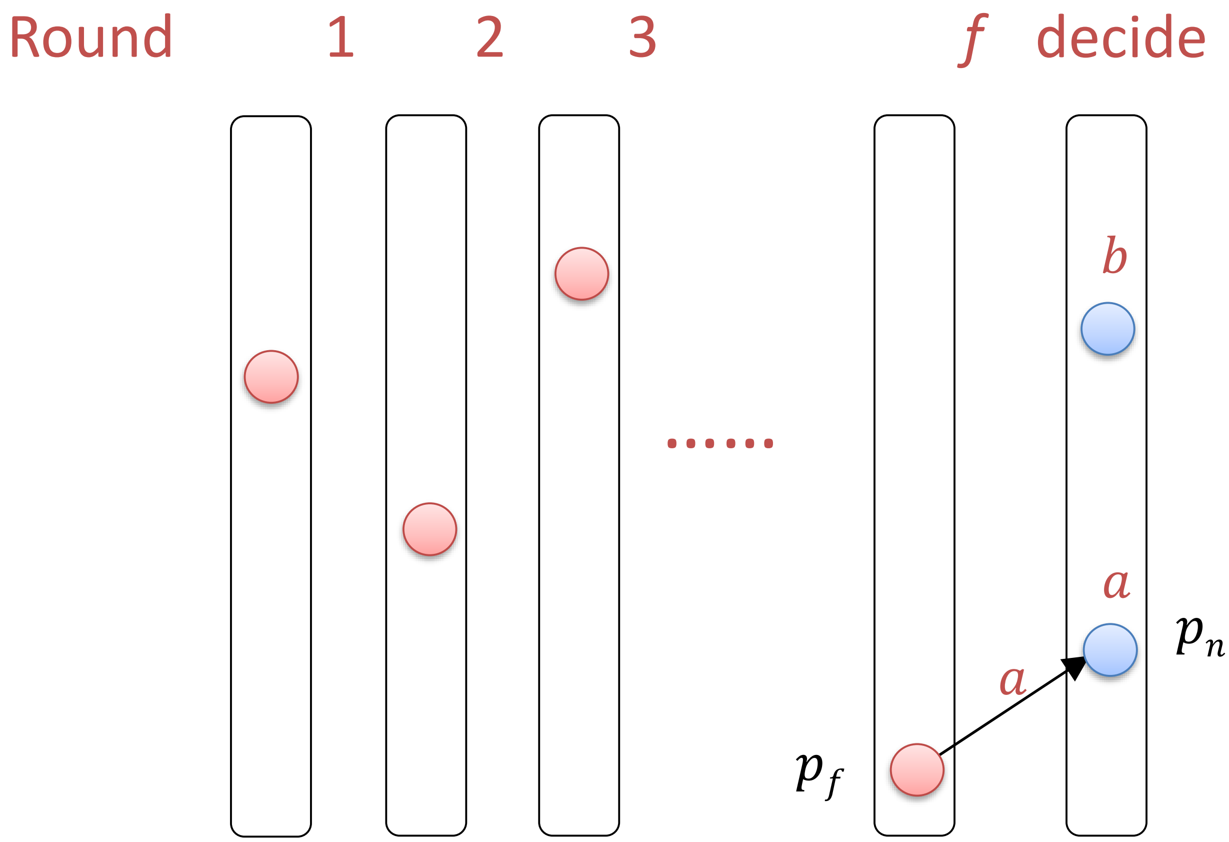
- Before process p_i fails, it sends its value a only to one process p_k
- Before process p_k fails, it sends its value a to only one process p_m

Lower Bound on Rounds: Intuition



- At the end of round f only one process p_n knows about value a

Lower Bound on Rounds: Intuition



- Process p_n may decide on a and all other processes may decide on another value b
- f rounds are not enough
  $\Longrightarrow$ at least $f+1$ rounds are needed