University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
Z. Parsaeian

# Theoretical Computer Science - Bridging Course
# Sample Solution Exercise Sheet 1

**Due:** Tuesday, 23rd of April 2024, 12:00 pm

## Exercise 1: Validness of Mathematical Induction *(Bonus Points)*

To prove that a statement $P(n)$ is true for all $n \in \mathbb{N}$, mathematical induction can be stated as

$$(P(1) \land \forall k(P(k) \Rightarrow P(k+1))) \Rightarrow \forall n P(n)$$

$P(1)$ stands for the *Base Case*, and $P(k) \Rightarrow P(k+1)$ for *Induction Hypothesis*. The statement above is valid. *i.e) if Antecedent is true, then the Consequent can't be false.* ,which justifies the use of Mathematical Induction in this case. Using Contradiction, prove the validity of mathematical induction. In other words, *Using contradiction, prove that if* $P(1) \land \forall k(P(k) \Rightarrow P(k+1))$ *is true, then* $\forall n P(n)$ *necessarily follows.*
Use the *Well-Ordering Property* of natural numbers to help finding a contradiction.
(Hint : Well-Ordering Property of natural numbers states that every nonempty subset of natural numbers has a *least element.*)

## Sample Solution

By negating the consequent $\forall n P(n)$, assume there exists $n \in \mathbb{N}$ such that $P(n)$ is false.
Let $S$ be the set of natural numbers that make $P(n)$ false.
$S$ has a least element according to the well-ordering property of natural numbers. Let's call this least element $m$.
As $m \neq 1$, so $m > 1$, which makes $m - 1 \in \mathbb{N}$. $m - 1 \notin S$, as $m$ is the least element of $S$. According to the $\forall k(P(k) \Rightarrow P(k+1))$, $P(m)$ has to be true, and this contradicts the assumption we've made earlier.

## Exercise 2: Miscellaneous Mathematical Proofs *(2+3+3+1 Points)*

1. Let $S(n) = \sum_{i=1}^{n} i$ be the sum of the first $n$ natural numbers and $C(n) = \sum_{i=1}^{n} i^3$ be the sum of the first $n$ cubes. Use mathematical induction to prove the following interesting conclusion: $C(n) = S^2(n)$ for every $n$.

2. Let $A, B$, and $C$ be subsets of $U$. Which of the following statements is true? Justify.

   - If $A \cap B = A \cap C$, then $B = C$.
   - If $A \cup B = A \cup C$, then $B = C$.
   - $\overline{A \cup B} = \overline{A} \cap \overline{B}$, where $\overline{A}$ is the complement of $A$.

3. Let $A_1, A_2, ..., A_n$ be nonempty subsets of a Universal Set $U$, where $n$ is any positive integer, and $n \geq 2$. Using the result of above exercise, i.e. $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$ . Prove a generalized result

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$$

using induction.

4. Let $A_1, A_2, ..., A_k$ be nonempty subsets of $U$, where $k$ is any positive integer. Construct a nonempty subset $A \subseteq U$ such that $A \cap A_i \neq \phi$, for all $i \in \{1, 2, ..., k\}$.

## Sample Solution

1. Base case: for $n = 1$, $1^3 = (1)^2$ is true.

   Induction step: for each $k \geq 1$, we assume that the statement holds true for $k$ i.e. $C(k) = S^2(k)$ (induction hypothesis IH). Now, we need to prove that the statement holds true for $k+1$ i.e. we want to show that $C(k+1) = S^2(k+1)$.
   Indeed first, we recall that $S(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, hence $S^2(k+1) = (\frac{(k+1)(k+2)}{2})^2 = \frac{(k+1)^2(k+2)^2}{4}$.
   Next, we have that $C(k+1) = \sum_{i=1}^{k} i^3 + (k+1)^3 = C(k) + (k+1)^3 \overset{\text{IH}}{=} S^2(k) + (k+1)^3 = (\frac{k(k+1)}{2})^2 + (k+1)^3 = (\frac{k^2(k+1)^2}{4}) + (k+1)^3 = \frac{(k+1)^2}{4}(k^2 + 4k + 4) = \frac{(k+1)^2}{4}(k+2)^2 = S^2(k+1)$.
   Hence, the statement holds true for $k+1$, which ends our induction proof.

2. 
   - False. We give a counterexample: take $A = \{1, 2, 3\}$, $B = \{1, 4\}$ and $C = \{1, 5\}$, hence $A \cap B = A \cap C$ and $B \neq C$.
   - False. We give a counterexample: take $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{2, 3\}$, hence $A \cup B = A \cup C$ and $B \neq C$.
   - (De Morgan's law). Indeed,

     $$x \in \overline{A \cup B} \iff x \notin A \cup B \iff x \notin A \text{ and } x \notin B \iff x \in \overline{A} \text{ and } x \in \overline{B} \iff x \in \overline{A} \cap \overline{B}$$

     hence, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

3. Base case: for n=2, $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$.
   Induction Hypothesis : for arbitrary $k \geq 2$, $\overline{\bigcup_{i=1}^{k} A_i} = \bigcap_{i=1}^{k} \overline{A_i}$, where $A_1, A_2 ... A_k$ are subsets of $U$. We assume this to be true for every possible collection of these k subsets of U. Using this, we want this to be true, also for every possible collection of k+1 subsets of U.
   Induction Step(one version) : Starting from induction hypothesis, pick arbitrary $A_{k+1}$ and add $\overline{A_{k+1}}$ for both sides.

   $$\overline{\bigcup_{i=1}^{k} A_i} \cap \overline{A_{k+1}} = \bigcap_{i=1}^{k} \overline{A_i} \cap \overline{A_{k+1}}$$

   In order to show $\overline{\bigcup_{i=1}^{k} A_i} \cap \overline{A_{k+1}} = \bigcap_{i=1}^{k+1} \overline{A_i}$,

   $$\overline{\bigcup_{i=1}^{k} A_i} \cap \overline{A_{k+1}} \overset{\text{IH}}{=} \bigcap_{i=1}^{k} \overline{A_i} \cap \overline{A_{k+1}} = \bigcap_{i=1}^{k+1} \overline{A_i}$$

   Note that for this problem, we have to show for all the possible statements $P(n)$. If you take a look at above way of proving it, it starts from induction hypothesis to build up $(k+1)$th object. If you do it this way, you need to make sure that you build all the possible objects to prove the statement. This problem was easy because the cardinality of $(k+1)$th object is just +1 from $(k)$th object. So we only need to pick one arbitrary subset to go to $(k+1)$th object.

But often, this method of building up from the induction hypothesis not always works well, simply because there could be many ways to build up $(k+1)$th objects, and you need to prove for all of them. This is cumbersome and this often leads you to an incorrect proof. So another version, which is stated below, would be a better and natural way to prove, as it starts from an arbitrary object $A_{k+1}$(So already covering all the $(k+1)$th objects) and try to decompose it so that we could utilize the induction hypothesis.

Induction Step(recommended version) : for some list of subsets $A_1, ...A_{k+1}$,
WTS: $\overline{\bigcup_{i=1}^{k+1} A_i} = \bigcap_{i=1}^{k+1} \overline{A_i}$

$$\overline{\bigcup_{i=1}^{k+1} A_i} = \overline{\bigcup_{i=1}^{k} A_i \cup A_{k+1}} = \overline{\bigcup_{i=1}^{k} A_i} \cap \overline{A_{k+1}} \overset{\text{IH}}{=} \bigcap_{i=1}^{k} \overline{A_i} \cap \overline{A_{k+1}} = \bigcap_{i=1}^{k+1} \overline{A_i}$$

4. We construct $A$ by choosing one element from each $A_i$, for all $i \in \{1, 2, ..., k\}$.

## Exercise 3: Graphs (Part 1) *(3+2 Points)*

A *simple graph* is a graph without self loops, i.e., every edge of the graph is an edge between two distinct nodes. The degree $d(v)$ of a node $v \in V$ in an undirected graph $G = (V, E)$ is the number of its neighbors, i.e, $d(v) = |\{u \in V \mid \{v, u\} \in E\}|$. Let $m \geq 0$ denote the number of edges in graph $G$.

1. Prove the handshaking lemma i.e. $\sum_{v \in V} d(v) = 2m$ via mathematical induction on $m$ *for any simple graph* $G = (V, E)$.
2. Show that every simple graph with an odd number of nodes contains a node with even degree.

## Sample Solution

1. We prove the handshaking lemma by mathematical induction on $m$.

   Base step: let $G = (V, E)$ be a graph where $|V| = n$ and $|E| = m = 0$. Notice that $G$ is the empty graph on $n$ nodes, hence $\sum_{v \in V} d(v) = 0$, thus the handshaking lemma is true on $G$.

   Induction step: for each $k$, we assume that the statement holds true for $k$ i.e. $\sum_{v \in V} d(v) = 2k$ for any graph $G = (V, E)$ where $|V| = n$ and $|E| = k$ (induction hypothesis IH).
   Now, we need to prove that the statement holds true for $k+1$ i.e. we want to show that $\sum_{v \in V} d(v) = 2(k+1)$ for any $G = (V, E)$ where $|V| = n$ and $|E| = k+1$.

   Indeed, first we consider a graph $G = (V, E)$ where $|V| = n$ and $|E| = k+1$. Let $\{u, v\}$ be an edge in $G$. Let $G' = (V, E')$ where $E' = E \backslash \{x, y\}$ i.e. $G'$ is the graph obtained after removing an edge $\{x, y\}$ from $G$. Note that we denote by $d_G(v), d_{G'}(v)$ the degree of node $v$ in $G$ and $G'$ respectively.

   First we notice that $G'$ has $k$ edges, hence by IH $\sum_{v \in V} d_{G'}(v) = 2k$.

   Moreover, $\sum_{v \in V} d_{G'}(v) = \sum_{v \in V \backslash \{x,y\}} d_{G'}(v) + d_{G'}(x) + d_{G'}(y) = \sum_{v \in V \backslash \{x,y\}} d_G(v) + (d_G(x) - 1) + (d_G(y) - 1) = \sum_{v \in V \backslash \{x,y\}} d_G(v) + d_G(x) + d_G(y) - 2 = \sum_{v \in V} d_G(v) - 2$.

   Thus $\sum_{v \in V} d_G(v) = \sum_{v \in V} d_{G'}(v) + 2 \overset{\text{IH}}{=} 2k + 2 = 2(k+1)$

   Hence, the statement holds true for $k+1$, which ends our induction proof.

   (Note that how many cases we should divide into, if we have started off from induction hypothesis to build up (k+1)th statement.)

2. Let $G = (V, E)$ be a graph. We argue by contradiction. Assume that $\forall v \in V$, $d(v)$ is odd. Now since $G$ has odd number of nodes, we notice that $\sum_{v \in V} d(v)$ is the sum of an odd number of odd numbers, which is odd. But by the handshaking lemma $\sum_{v \in V} d(v)$ must be even. This is a contradiction. Thus our assumption must have been false and hence there must exist a node in $G$ with even degree.

## Exercise 4: Graphs (Part 2) *(2+4 Points)*

A graph $G = (V, E)$ is said to be *connected* if for every pair of vertices $u, v \in V$ such that $u \neq v$ there exists a path in $G$ connecting $u$ to $v$.

1. Prove that if $G$ is connected, then for any two nonempty subsets $V_1$ and $V_2$ of $V$ such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$, there exists an edge joining a vertex in $V_1$ to a vertex in $V_2$.

2. Let $G$ be a simple, connected graph and $P$ be a path of the longest length $\ell$ in $G$. Show that if the two ends of $P$ are adjacent, then $V = V(P)$, where $V(P)$ is the set of vertices of $P$.
   *Hint: Try to argue by contradiction.*

## Sample Solution

*Definition*: a family of sets $V_1, V_2, ..., V_k$, where $k$ is some positive integer is called a *partition* of $V$ if and only if all of the following conditions hold:

- For all $i \in \{1, 2..., k\}$, $V_i$ is a nonempty subset of $V$
- $\bigcup_{i=1}^{k} V_i = V$
- $V_i \cap V_j = \phi$ for all $i, j \in \{1, 2..., k\}$ such that $i \neq j$

*Intuitively* you can think of a partition of a set $V$ as a set of non-empty subsets of $V$ such that every element $x \in V$ is in exactly one of these subsets.

1. Let $V_1$ and $V_2$ be any two non empty subsets of $V$ such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$ (i.e. $V_1$ and $V_2$ is a partition of the vertex set $V$). Let $u \in V_1$ and $v \in V_2$. Since $G$ is connected, there exists a path in $G$ joining $u$ to $v$. For this to happen, there must then exist an edge joining some vertex in $V_1$ to some other vertex in $V_2$, which ends our proof.

2. *Notations and definitions:* A path $P$ on $n$ vertices say $\{v_1, v_2, ..., v_n\}$ is a graph whose set of edges is $\{\{v_i, v_{i+1}\}; 1 \leq i \leq n-1\}$ and to describe it we write $P = v_1 v_2 ... v_n$.
   Let $v_i$ and $v_j$ be any two vertices of $P$, where $1 \leq i \leq j \leq n$, then we denote by $P_{[v_i, v_j]} = v_i v_{i+1} ... v_j$ the subpath of $P$ with ends $v_i$ and $v_j$.

   *Solution:* We argue by contradiction. Suppose $V \neq V(P)$, where we define $V(P) := \{v_1, v_2, ..., v_{\ell+1}\}$, then there exists at least one vertex in $V$ that is not in $V(P)$. Hence, we can define $V_1 := V \backslash V(P) \neq \phi$ and $V_2 := V(P) \neq \phi$. Notice that $V_1$ and $V_2$ from a partition of $V$. Moreover since $G$ is connected, by the previous part we deduce that there exists an edge joining a vertex in $V_1$ ( call it $x$) to a vertex $v_k$ in $V_2 = V(P)$, where $1 \leq k \leq \ell + 1$. Let $P = v_1 v_2 ... v_{\ell+1}$ and $e = \{x, v_k\}$. Since the two ends of $P$ are adjacent i.e. $\{v_1, v_{\ell+1}\} \in E$, we can define another path $P' = x v_k P_{[v_{k+1}, v_\ell]} v_{\ell+1} v_1 P_{[v_2, v_{k-1}]}$. Notice that $P'$ is a path in $G$ of length $\ell + 1$, which is a contradiction. Hence, our supposition is incorrect. Thus, $V = V(P)$.