Chapter 1
Divide and Conquer

Part 2: Polynomial Multiplication

Algorithm Theory
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Polynomials

Real polynomial p in one variable x:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$$

Coefficients of p: $a_0, a_1, \ldots, a_n \in \mathbb{R}$

Degree of p: largest power of x in p (n in the above case)

Example:

$$p(x) = 3x^3 - 15x^2 + 18x$$

Set of all real-valued polynomials in x: $\mathbb{R}[x]$ (polynomial ring)

Operations: Addition

- Given: Polynomials $p, q \in \mathbb{R}[x]$ of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

- Compute sum $p(x) + q(x)$:

$$\begin{aligned} p(x) + q(x) &= (a_n x^n + \cdots + a_0) + (b_n x^n + \cdots + b_0) \\ &= (a_n + b_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0) \end{aligned}$$

Operations: Multiplication

- Given: Polynomials $p, q \in \mathbb{R}[x]$ of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

- Product $p(x) \cdot q(x)$:

$$p(x) \cdot q(x) = (a_n x^n + \cdots + a_0) \cdot (b_n x^n + \cdots + b_0)$$
$$= \underline{c_{2n}} x^{2n} + \underline{c_{2n-1}} x^{2n-1} + \cdots + \underline{c_1} x + \underline{c_0}$$

- Obtaining c_i: what products of monomials have degree i?

$$\text{For } 0 \leq i \leq 2n\text{: } c_i = \sum_{j=0}^{i} \underline{a_j b_{i-j}}$$

where $\underline{a_i = b_i = 0}$ for $i > n$.

Operations: Evaluation

- Given: Polynomial $p \in \mathbb{R}[x]$ of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- **Horner's method** for evaluation at specific value x_0:

$$p(x_0) = \Big(\ldots\big((a_n x_0 + a_{n-1})x_0 + a_{n-2}\big)x_0 + \cdots + a_1\Big)x_0 + a_0$$

- Pseudo-code:

$p := a_n;\ i := n;$
while $(i > 0)$ **do**
  $i := i - 1;$
  $p := p \cdot x_0 + a_i$
end

- Running time: $O(n)$

Representation of Polynomials

Coefficient representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree n is given by its $n + 1$ coefficients $\underline{a_0, \dots, a_n}$: $(a_0, a_1, \dots, a_n)$

$$p(x) = a_n x^n + \dots + a_1 x + a_0$$

- Example:

$$p(x) = 3x^3 - 15x^2 + 18x$$

$(0, 18, -15, 3)$

- The most typical (and probably most natural) representation of polynomials

Representation of Polynomials

Product of linear factors:

- Polynomial $p(x) \in \mathbb{C}[x]$of degree n is given by its n roots

$$p(x) = a_n \cdot (x - x_1) \cdot (x - x_2) \cdot \ldots \cdot (x - x_n)$$

- Example:

$$p(x) = 3x(x-2)(x-3)$$

- Every polynomial has exactly n roots $x_i \in \mathbb{C}$ for which $p(x_i) = 0$
  - Polynomial is uniquely defined by the n roots and a_n

- We will not use this representation…

Representation of Polynomials

Point-value representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree n is given by
  $n + 1$ point-value pairs:

$$p = \{(x_0, p(x_0)), (x_1, p(x_1)), \dots, (x_n, p(x_n))\}$$

  where $x_i \neq x_j$ for $i \neq j$.

- Example: The polynomial

$$p(x) = 3x(x-2)(x-3)$$

  is uniquely defined by the four point-value pairs
  $(0,0), (1,6), (2,0), (3,0)$.

Operations: Coefficient Representation

Deg.-n polynomials $p(x) = a_n x^n + \cdots + a_0$, $q(x) = b_n x^n + \cdots + b_0$

Addition:

$$p(x) + q(x) = (a_n + b_n)x^n + \cdots + (a_0 + b_0)$$

- Time: $O(n)$

Multiplication:

$$p(x) \cdot q(x) = c_{2n} x^{2n} + \cdots + c_0, \qquad \text{where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

- Naive solution: Need to compute product $a_i b_j$ for all $0 \le i, j \le n$
- Time: $O(n^2)$

Operations Point-Value Representation

Degree-n polynomials

$$p = \{(x_0, p(x_0)), \dots, (x_n, p(x_n))\}, q = \{(x_0, q(x_0)), \dots, (x_n, q(x_n))\}$$

- Note: we use the same points $x_0, \dots, x_n$ for both polynomials

Addition:

$$p + q = \{(x_0, \underline{p(x_0) + q(x_0)}), \dots, (x_n, p(x_n) + q(x_n))\}$$

- Time: $O(n)$

Multiplication:

$$p \cdot q = \{(\underline{x_0}, \underline{p(x_0) \cdot q(x_0)}), \dots, (x_{2n}, p(x_{2n}) \cdot q(x_{2n}))\}$$

- Time: $O(n)$

Faster Multiplication?

- Multiplication is slow $\left(\Theta(n^2)\right)$ when using the standard coefficient representation

- Try divide-and-conquer to get a faster algorithm

- Assume: degree is $n-1$, n is even
- Divide polynomial $p(x) = a_{n-1}x^{n-1} + \cdots + a_0$ into 2 polynomials of degree $n/2 - 1$:

$$p_0(x) = a_{n/2-1}x^{n/2-1} + \cdots + a_0$$
$$p_1(x) = a_{n-1}x^{n/2-1} + \cdots + a_{n/2}$$
$$p(x) = p_1(x) \cdot x^{n/2} + p_0(x)$$

- Similarly: $q(x) = q_1(x) \cdot x^{n/2} + q_0(x)$

Use Divide-And-Conquer

- **Divide:**

  $p(x) = p_1(x) \cdot x^{n/2} + p_0(x), \qquad q(x) = q_1(x) \cdot x^{n/2} + q_0(x)$

- **Multiplication:**

  $$p(x)q(x) = p_1(x)q_1(x) \cdot x^n + \big(p_0(x)q_1(x) + p_1(x)q_0(x)\big) \cdot x^{n/2} + p_0(x)q_0(x)$$

- 4 multiplications of degree $n/2 - 1$ polynomials:

  $$T(n) = 4T(n/2) + O(n)$$

- Leads to $T(n) = \Theta(n^2)$ like the naive algorithm… (see exercises)

More Clever Recursive Solution

- **Recall that**

$$p(x)q(x) = \underbrace{p_1(x)q_1(x)}_{A} \cdot x^n + \underbrace{\left(p_0(x)q_1(x) + p_1(x)q_0(x)\right)}_{B} \cdot x^{n/2} + \underbrace{p_0(x)q_0(x)}_{C}$$

- Compute $r(x) = \underbrace{(p_0(x) + p_1(x))}_{\text{degree } n/2-1} \cdot (q_0(x) + q_1(x))$:

$$= \underbrace{p_1(x)q_1(x)}_{A} + \underbrace{p_0(x)q_1(x) + p_1(x)q_0(x)}_{B} + \underbrace{p_0(x)q_0(x)}_{C}$$

$$p(x)q(x) = Ax^n + (r(x) - A - C)x^{n/2} + C$$

$\longrightarrow$ 3 multiplications of poly. of degree $n/2 - 1$

Karatsuba Algorithm

- Recursive multiplication:

$$r(x) = (p_0(x) + p_1(x)) \cdot (q_0(x) + q_1(x))$$

$$\begin{aligned} p(x)q(x) = {} & p_1(x)q_1(x) \cdot x^n \\ & + (r(x) - p_0(x)q_0(x) + p_1(x)q_1(x)) \cdot x^{n/2} \\ & + p_0(x)q_0(x) \end{aligned}$$

- Recursively do 3 multiplications of degr. $(n/2 - 1)$-polynomials

$$T(n) = 3T(n/2) + O(n)$$

- Gives: $T(n) = O(n^{1.59})$ (see exercises)

Faster Polynomial Multiplication?

Multiplication is fast when using the point-value representation

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

Evaluation at points $x_0, x_1, \dots, x_{2n-1}$ Naively: $O(n^2)$

$2 \times 2n$ point-value pairs $(x_i, p(x_i))$ and $(x_i, q(x_i))$

Point-wise multiplication $O(n)$ time

$2n$ point-value pairs $(x_i, p(x_i)q(x_i))$

Interpolation ??

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

Point-Value Representation of p, q

- Select points $x_0, x_1, \dots, x_{N-1}$ to evaluate p and q in a clever way

Consider the N powers of the principle Nth root of unity:

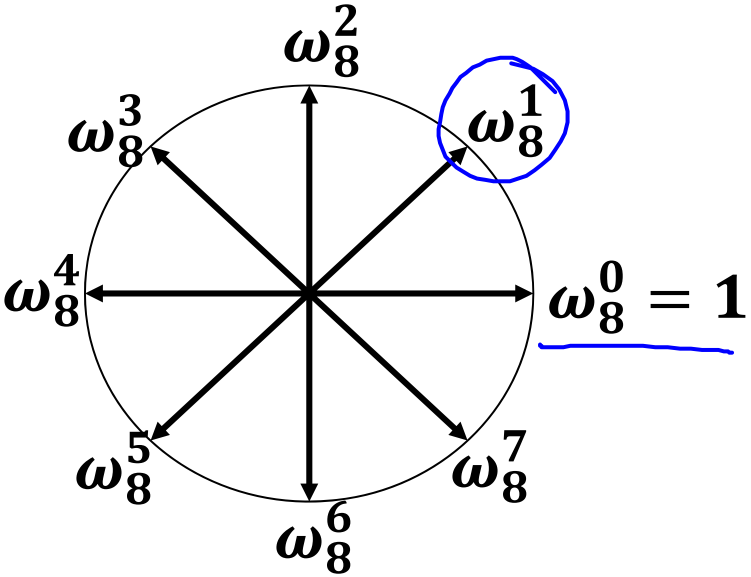
Principle root of unity: $\boldsymbol{\omega_N = e^{2\pi i/N}}$

$$\left(i = \sqrt{-1}, \qquad e^{2\pi i} = 1\right)$$

Powers of ω_n (roots of unity):

$$1 = \omega_N^0, \omega_N^1, \dots, \omega_N^{N-1}$$

$x_0, x_1 \cdot \dots, x_N$



Note: $\omega_N^k = e^{2\pi i k/N} = \cos\frac{2\pi k}{N} + i \cdot \sin\frac{2\pi k}{N}$

Discrete Fourier Transform

- The values $p(\omega_N^i)$ for $i = 0, \dots, N-1$ uniquely define a polynomial p of degree $< N$.

Discrete Fourier Transform (DFT):

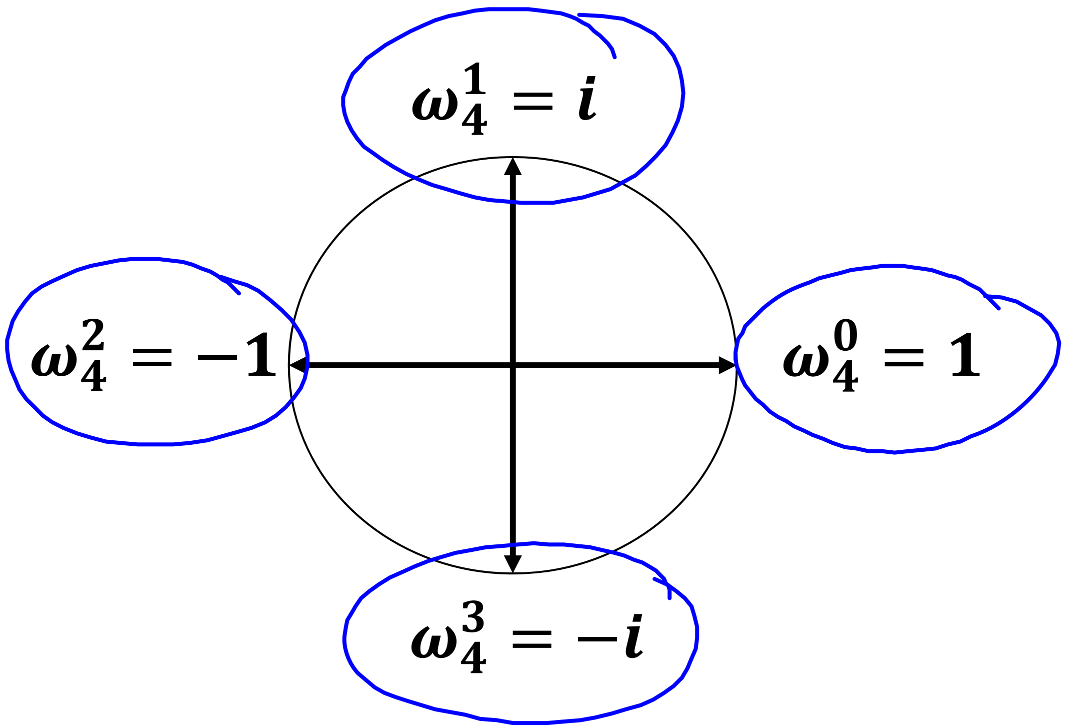
- Assume $a = (a_0, \dots, a_{N-1})$ is the coefficient vector of poly. p

$$(p(x) = a_{N-1}x^{N-1} + \cdots + a_1 x + a_0)$$

$$\mathrm{DFT}_N(a) := \left(p\left(\omega_N^0\right), p(\omega_N^1), \dots, p(\omega_N^{N-1})\right)$$

Example

- Consider polynomial $p(x) = 3x^3 - 15x^2 + 18x$
- Choose $N = 4$
- Roots of unity:



Example

- Consider polynomial $p(x) = 3x^3 - 15x^2 + 18x$
- $N = 4$, roots of unity: $\omega_4^0 = 1, \omega_4^1 = i, \omega_4^2 = -1, \omega_4^3 = -i$
- Evaluate $p(x)$ at ω_4^k:

$$\left(\omega_4^0, p(\omega_4^0)\right) = \left(1, p(1)\right) = (1, 6)$$
$$\left(\omega_4^1, p(\omega_4^1)\right) = \left(i, p(i)\right) = (i, 15 + 15i)$$
$$\left(\omega_4^2, p(\omega_4^2)\right) = \left(-1, p(-1)\right) = (-1, -36)$$
$$\left(\omega_4^3, p(\omega_4^3)\right) = \left(-i, p(-i)\right) = (-i, 15 - 15i)$$

- For $a = (3, -15, 18, 0)$:

$$\mathbf{DFT_4}(\boldsymbol{a}) = (\mathbf{6}, \mathbf{15} + \mathbf{15}\boldsymbol{i}, -\mathbf{36}, \mathbf{15} - \mathbf{15}\boldsymbol{i})$$

Faster Polynomial Multiplication?

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

$\downarrow$ **Evaluation** at points $\boxed{\omega_{2n}^0, \omega_{2n}^1, \dots, \omega_{2n}^{2n-1}}$ DFT(a) DFT(b)

$2 \times 2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)\right)$ and $\left(\omega_{2n}^k, q\left(\omega_{2n}^k\right)\right)$

$\downarrow$ **Point-wise multiplication**

$2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)q\left(\omega_{2n}^k\right)\right)$

$\downarrow$ **Interpolation**

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

Properties of the Roots of Unity

- **Cancellation Lemma:**

  For all integers $n > 0$, $k \geq 0$, and $d > 0$, we have:

  $$\boldsymbol{\omega_{dn}^{dk} = \omega_n^k}, \qquad \boldsymbol{\omega_n^{k+n} = \omega_n^k}$$

- **Proof:**

$$\omega_{dn}^{dk} = e^{2\pi i \cdot dk/dn} = e^{\frac{2\pi i k}{n}} = \omega_n^k \quad \checkmark$$

$$\omega_n^{k+n} = e^{2\pi i (k+n)/n} = e^{\frac{2\pi i k}{n}} \cdot \underbrace{e^{\frac{2\pi i n}{n}}}_{1} = e^{\frac{2\pi i k}{n}} = \omega_n^k \quad \checkmark$$

Divide-and-Conquer Approach

- Divide $p(x)$ of degree $N-1$ (N is even) into 2 polynomials of degree $N/_2 - 1$ differently than in Karatsuba's algorithm

$P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{N-1} x^{N-1}$

- $p_0(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{N-2} x^{N/_2 - 1}$ (even coeff.)

  $p_1(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{N-1} x^{N/_2 - 1}$ (odd coeff.)

$$P(x) = a_0 + a_2 \cdot x^2 + a_4 x^4 + \dots + a_{N-2} x^{N-2} + a_1 x + a_3 x^3 + a_5 x^5 + \dots + a_{N-1} x^{N-1}$$

$$= P_0(x^2) + x \cdot P_1(x^2)$$

Discrete Fourier Transform

Evaluation for $k = 0, \dots, N-1$: $p(x) = p_0(x^2) + x\, p_1(x^2)$

$$\boxed{p(\omega_N^k)} = p_0\big((\omega_N^k)^2\big) + \omega_N^k \cdot p_1\big((\omega_N^k)^2\big) \qquad k = 0, \dots, N-1$$

$$= \begin{cases} p_0\big(\omega_{N/2}^k\big) + \omega_N^k \cdot p_1\big(\omega_{N/2}^k\big) & \text{if } k < N/2 \\ p_0\left(\omega_{N/2}^{k-N/2}\right) + \omega_N^k \cdot p_1\left(\omega_{N/2}^{k-N/2}\right) & \text{if } k \geq N/2 \end{cases}$$

$$\omega_{dN}^{dk} = \omega_N^k \qquad (\omega_N^k)^2 = \omega_N^{2k} = \omega_{N/2}^k = \omega_{N/2}^{k-N/2}$$

For the coefficient vector a of $p(x)$:

$$\mathrm{DFT}_N(a) = \left(p_0(\omega_{N/2}^0), \dots, p_0(\omega_{N/2}^0), p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right)\right)$$
$$+ \left(\omega_N^0 p_0(\omega_{N/2}^0), \dots, \omega_N^{N/2-1} p_0(\omega_{N/2}^0), \omega_N^{N/2} p_0(\omega_{N/2}^0), \dots, \omega_N^{N-1} p_0\left(\omega_{N/2}^{N/2-1}\right)\right)$$

Example

For the coefficient vector a of $p(x)$:

$$\mathrm{DFT}_N(a) = \left(p_0(\omega_{N/2}^0), \dots, p_0(\omega_{N/2}^0), p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right)\right)$$
$$+ \left(\omega_N^0 p_0(\omega_{N/2}^0), \dots, \omega_N^{N/2-1} p_0(\omega_{N/2}^0), \omega_N^{N/2} p_0(\omega_{N/2}^0), \dots, \omega_N^{N-1} p_0\left(\omega_{N/2}^{N/2-1}\right)\right)$$

$N = 4$:

$$\begin{aligned}
p(\omega_4^0) &= p_0(\omega_2^0) + \omega_4^0 p_1(\omega_2^0)\\
p(\omega_4^1) &= p_0(\omega_2^1) + \omega_4^1 p_1(\omega_2^1)\\
p(\omega_4^2) &= p_0(\omega_2^0) + \omega_4^2 p_1(\omega_2^0)\\
p(\omega_4^3) &= p_0(\omega_2^1) + \omega_4^3 p_1(\omega_2^1)
\end{aligned}$$

$p_0(\omega_2^2) = p_0(\omega_2^0)$

Need: $(p_0(\omega_2^0), p_0(\omega_2^1))$ and $(p_1(\omega_2^0), p_1(\omega_2^1))$

(DFTs of coefficient vectors of p_0 and p_1)

Recursive Structure

For simplicity, we abuse notation in the following:

- Poly. $p(x) = a_{N-1}x^{N-1} + \cdots + a_0$ with coefficient vector a

  Let $\mathrm{DFT}_N(p) \coloneqq \mathrm{DFT}_N(a)$

Recursive structure:

- For $N = 4$:

$$\big(\mathrm{DFT}_4(p)\big)_k = p\big(\omega_4^{\mathrm{k}}\big)$$

$k \in \{0, \ldots, 3\}$

$$= \big(\mathrm{DFT}_2(p_0)\big)_{k \bmod \cancel{4}\,2} + \omega_4^k \cdot \big(\mathrm{DFT}_2(p_1)\big)_{k \bmod \cancel{4}\,2}$$

- General N (assume N is even):

$$\big(\mathrm{DFT_N}(p)\big)_k = p\big(\omega_{\mathrm{N}}^{\mathrm{k}}\big)$$

$$= \big(\mathrm{DFT_{N/2}}(p_0)\big)_{k \bmod N/2} + \omega_N^k \cdot \big(\mathrm{DFT_{N/2}}(p_1)\big)_{k \bmod N/2}$$

Computation of DFT_N

- Divide-and-conquer algorithm for $\mathrm{DFT}_N(p)$: time $T(N)$

1. Divide

$N \leq 1$: $\mathrm{DFT}_1(p) = (a_0)$ $O(1)$

$N > 1$: Divide p into p_0 (even coeff.) and p_1 (odd coeff).

time: $O(N)$

2. Conquer

Solve $\mathrm{DFT}_{N/2}(p_0)$ and $\mathrm{DFT}_{N/2}(p_1)$ recursively time: $2T(N/2)$

3. Combine

Compute $\mathrm{DFT}_N(p)$ based on $\mathrm{DFT}_{N/2}(p_0)$ and $\mathrm{DFT}_{N/2}(p_1)$

for every comp. of $\mathrm{DFT}_N(p)$, $O(1)$ time

time: $O(N)$

Analysis

- $T(N)$: max. time to compute $\mathrm{DFT}_N(p)$:

$$T(N) = \underline{2T\left(\frac{N}{2}\right)} + O(N), \qquad T(1) = O(1)$$

- As for mergesort, comparing orders, closest pair of points:

$$\boxed{T(N) = O(N \cdot \log N)}$$

Small Improvement

Polynomial p of degree $N-1$:

$$p(\omega_N^k) = \begin{cases} p_0(\omega_{N/2}^k) + \omega_N^k \cdot p_1(\omega_{N/2}^k) & \text{if } k < {}^N/_2 \\ p_0\left(\omega_{N/2}^{k-N/2}\right) + \omega_N^{k} \cdot p_1\left(\omega_{N/2}^{k-N/2}\right) & \text{if } k \geq {}^N/_2 \end{cases}$$

$$= \begin{cases} p_0(\omega_{N/2}^k) + \omega_N^k \cdot p_1(\omega_{N/2}^k) & \text{if } k < {}^N/_2 \\ p_0\left(\omega_{N/2}^{k-N/2}\right) - \omega_N^{k-N/2} \cdot p_1\left(\omega_{N/2}^{k-N/2}\right) & \text{if } k \geq {}^N/_2 \end{cases}$$

$\omega_N^k = -\omega_N^{k-N/2}$ $\qquad$ $\omega_N^{N/2} = \omega_2^1 = -1$

Need to compute $p_0(\omega_{N/2}^k)$ and $\omega_N^k \cdot p_1(\omega_{N/2}^k)$ for $0 \leq k < {}^N/_2$.

Example

$$p(\omega_4^0) = p_0(\omega_2^0) + \omega_4^0 \cdot p_1(\omega_2^0)$$

$$p(\omega_4^1) = p_0(\omega_2^1) + \omega_4^1 \cdot p_1(\omega_2^0)$$

$$p(\omega_4^2) = p_0(\omega_2^0) - \omega_4^0 \cdot p_1(\omega_2^0)$$

$$p(\omega_4^3) = p_0(\omega_2^1) - \omega_4^1 \cdot p_1(\omega_2^1)$$

Fast Fourier Transform (FFT) Algorithm

Algorithm FFT(a)

- Input: Array a of length N, where N is a power of 2
- Output: $\mathrm{DFT}_N(a)$

if $n = 1$ **then return** a_0; // $a = [a_0]$

$d^{[0]} := \mathrm{FFT}([a_0, a_2, \dots, a_{N-2}])$;

$d^{[1]} := \mathrm{FFT}([a_1, a_3, \dots, a_{N-1}])$;

$\omega_N := e^{2\pi i/N}$; $\omega := 1$;

for $k = 0$ **to** $N/2 - 1$ **do** // $\omega = \omega_N^k$

$\quad x := \omega \cdot d_k^{[1]}$;

$\quad d_k := d_k^{[0]} + x; d_{k+N/2} := d_k^{[0]} - x$;

$\quad \omega := \omega \cdot \omega_N$

end;

return $d = [d_0, d_1, \dots, d_{N-1}]$;

Example

- $p(x) = 3x^3 - 15x^2 + 18x + 0$, $a = [0,18,-15,3]$

Faster Polynomial Multiplication?

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

Evaluation at $\omega_{2n}^0, \omega_{2n}^1, \dots, \omega_{2n}^{2n-1}$ using **FFT**

$2 \times 2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)\right)$ and $\left(\omega_{2n}^k, q\left(\omega_{2n}^k\right)\right)$

Point-wise multiplication

$2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)q\left(\omega_{2n}^k\right)\right)$

Interpolation

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients