Chapter 1
Divide and Conquer

Algorithm Theory
WS 2012/13

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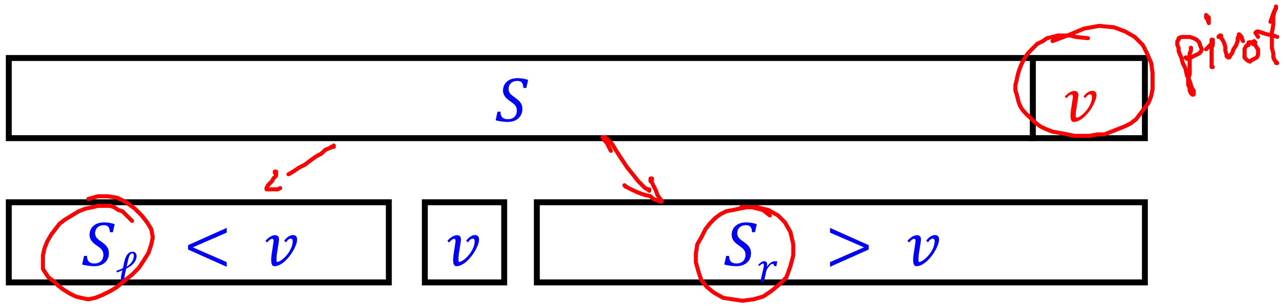
Divide-And-Conquer Principle

- Important algorithm design method

- Examples from Informatik 2:
  - Sorting: Mergesort, Quicksort
  - Binary search can be considered as a divide and conquer algorithm

- Further examples
  - Median
  - Comparison orders
  - Delaunay triangulation / Voronoi diagram
  - Closest pairs
  - Line intersections
  - Integer factorization / FFT
  - ...

Example 1: Quicksort



function Quick (S: sequence): sequence;

{returns the sorted sequence S}

begin

  if $\#S \leq 1$ then **return** S

  else { choose pivot element v in S;

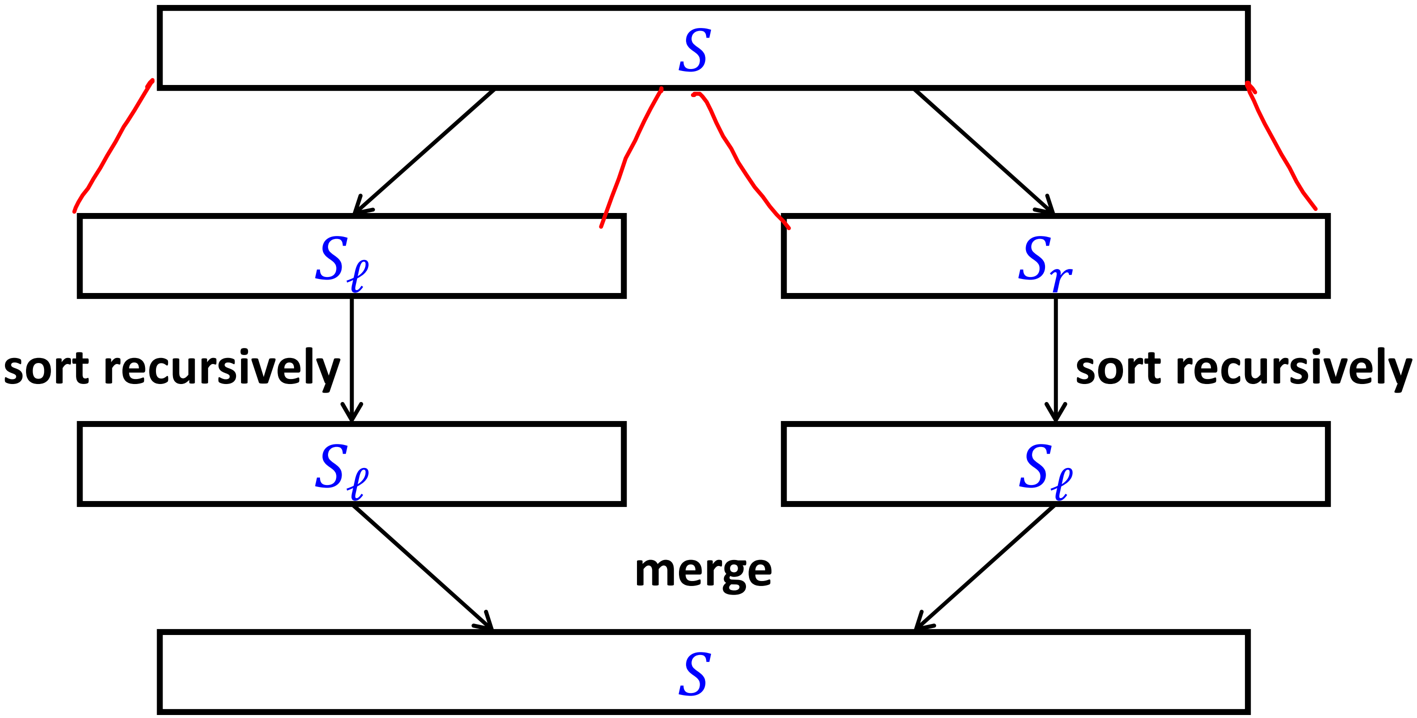
    partition S into S_ℓ with elements $< v$,

    and S_r with elements $> v$

    return | Quick(S_ℓ) | v | Quick(S_r) |

end;

Example 2: Mergesort



Formulation of the D&C principle

Divide-and-conquer method for solving a problem instance of size n:

1. Divide

$n \leq c$: Solve the problem directly.

$n > c$: Divide the problem into k subproblems of sizes $n_1, \ldots, n_k < n$ $(k \geq 2)$.

Quicksort (partition)

2. Conquer

Solve the k subproblems in the same way (recursively).

3. Combine

Combine the partial solutions to generate a solution for the original instance.

Mergesort (merge)

Analysis

Recurrence relation:

- $\boldsymbol{T(n)}$: max. number of steps necessary for solving an instance of size n
- $$\boldsymbol{T(n)} = \begin{cases} \boldsymbol{a} & \textbf{if } \boldsymbol{n \le c} \\ \boldsymbol{T(n_1) + \cdots + T(n_k)} & \textbf{if } \boldsymbol{n > c} \\ \textbf{+ cost for divide and combine} & \end{cases}$$
  (depends on n)

Special case: $\boldsymbol{k = 2, n_1 = n_2 = {}^{n}/_{2}}$

- cost for divide and combine: $\mathrm{DC}(n)$
- $T(1) = a$
- $T(n) = 2T(n/2) + \mathrm{DC}(n)$

Analysis, Example

Recurrence relation:

$$T(n) = 2 \cdot T(n/2) + cn^2, \qquad T(1) = a$$

Guess the solution by repeated substitution:

$$T(n) = 2T(n/2) + cn^2$$
$$= 2\left(2T(n/4) + c \cdot (n/2)^2\right) + cn^2$$
$$= 4T(n/4) + \left(c + \frac{c}{2}\right) \cdot n^2$$
$$= 4\left(2T(n/8) + c \cdot (n/4)^2\right) + \left(c + \frac{c}{2}\right)n^2$$
$$= 8 \cdot T(n/8) + \left(c + \frac{c}{2} + \frac{c}{4}\right)n^2$$
$$\vdots$$
$$= n \cdot T(1) + \underbrace{\left(c + \frac{c}{2} + \frac{c}{4} + \frac{c}{8} + \dots\right)}_{< 2c} n^2 \leq \underline{a \cdot n + 2cn^2}$$

Analysis, Example

Recurrence relation:

$$T(n) = 2 \cdot T(n/2) + cn^2, \qquad T(1) = a$$

Verify by induction:

Guess: $T(n) \leq an + 2cn^2$

Ind. Base: $n = 1$ ✓

Ind. Step: $T(n) \leq 2\left(a\left(\frac{n}{2}\right) + 2c\left(\frac{n}{2}\right)^2\right) + cn^2$

$= a \cdot n + 2cn^2$

□

Comparing Orders

- Many web systems maintain user preferences / rankings on things like books, movies, restaurants, …

- Collaborative filtering:
  - Predict user taste by comparing rankings of different users.
  - If the system finds users with similar tastes, it can make recommendations (e.g., Amazon)

- Core issue: Compare two rankings
  - Intuitively, two rankings (of movies) are more similar, the more pairs are ordered in the same way
  - Label the first user's movies from 1 to n according to ranking
  - Order labels according to second user's ranking
  - How far is this from the ascending order (of the first user)?

Number of Inversions

Formal problem:

- **Given**: array $A = [a_1, a_2, a_3, \dots, a_n]$ of distinct elements

  compare elements, global order $a_i < a_j$

- **Objective**: Compute number of inversions I

$$I := \left|\{0 \leq i < j \leq n \mid a_i > a_j)\}\right|$$

- **Example**: $A = [\ 4\ ,\ 1\ ,\ 5\ ,\ 2\ ,\ 7\ ,\ 10\ ,\ 6\ ]$

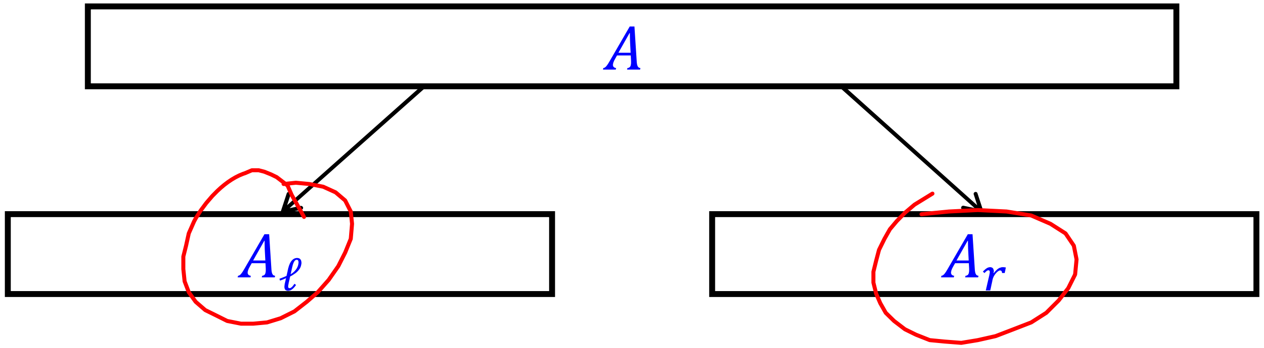
  5 inversions

- **Naive solution**:

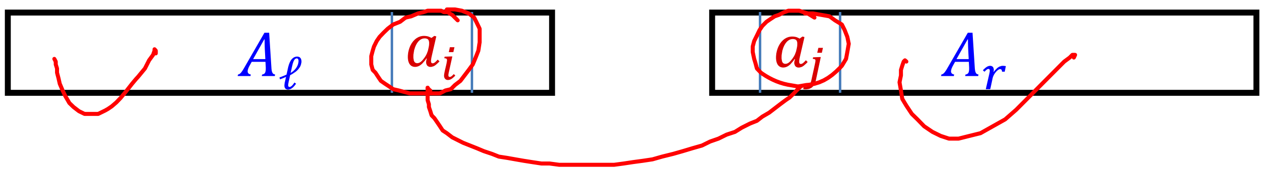
  compare all pairs

  time: $\Theta(n^2)$

Divide and conquer

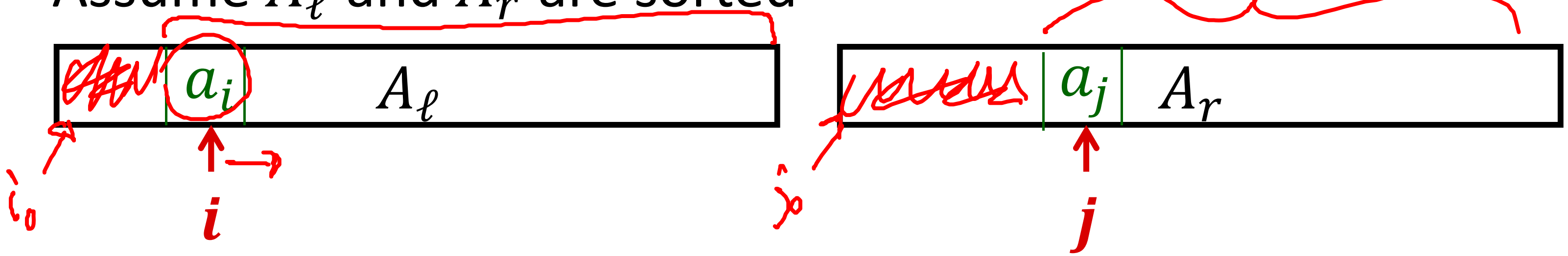


1. Divide array into 2 equal parts A_ℓ and A_r
2. Recursively compute #inversions in A_ℓ and A_r
3. Combine: add #pairs $a_i \in A_\ell, a_j \in A_r$ such that $a_i > a_j$



Combine Step

- Assume A_ℓ and A_r are sorted



- Pointers i and j, initially pointing to first elements of A_ℓ and A_r
- If $a_i < a_j$:
  - a_i is smallest among the remaining elements
  - No inversion of a_i and one of the remaining elements
  - Do not change count
- If $a_i > a_j$:
  - a_j is smallest among the remaining elements
  - a_j is smaller than all remaining elements in A_ℓ
  - Add number of remaining elements in A_ℓ to count
- Increment pointer pointing to smaller element

Combine Step

- Need sub-sequences in sorted order
- Then, combine step is like merging in merge sort

- **Idea**: Solve sorting and #inversions at the same time!
  1. Partition A into two equal parts A_ℓ and A_r
  2. Recursively compute #inversions and sort A_ℓ and A_r

     cost $2 \cdot T(n/2)$

  3. Merge A_ℓ and A_r to sorted sequence, at the same time, compute number of inversions between elements a_i in A_ℓ and a_j in A_r

     time: $O(n)$

Analysis, Example

Recurrence relation:

$$T(n) = 2 \cdot T(n/2) + c \cdot n, \qquad T(1) = c$$

Repeated substitution:

$$T(n) = 2T(n/2) + cn$$
$$= 2\left(2T(n/4) + c\cdot n/2\right) + cn = 4T(n/4) + 2cn$$
$$= 8T(n/8) + 3cn$$
$$\vdots$$
$$= c \cdot n \cdot \log_2(n)$$

Analysis, Example

Recurrence relation:

$$T(n) = 2 \cdot T(n/2) + c \cdot n, \qquad T(1) = c$$

Verify by induction:

Guess: $T(n) \leq c \cdot n(\log n + 1)$

Base: $n = 1$ ✓

Ind. Step: $T(n) \leq 2c \cdot \frac{n}{2}\left(\log\left(\frac{n}{2}\right) + 1\right) + cn$

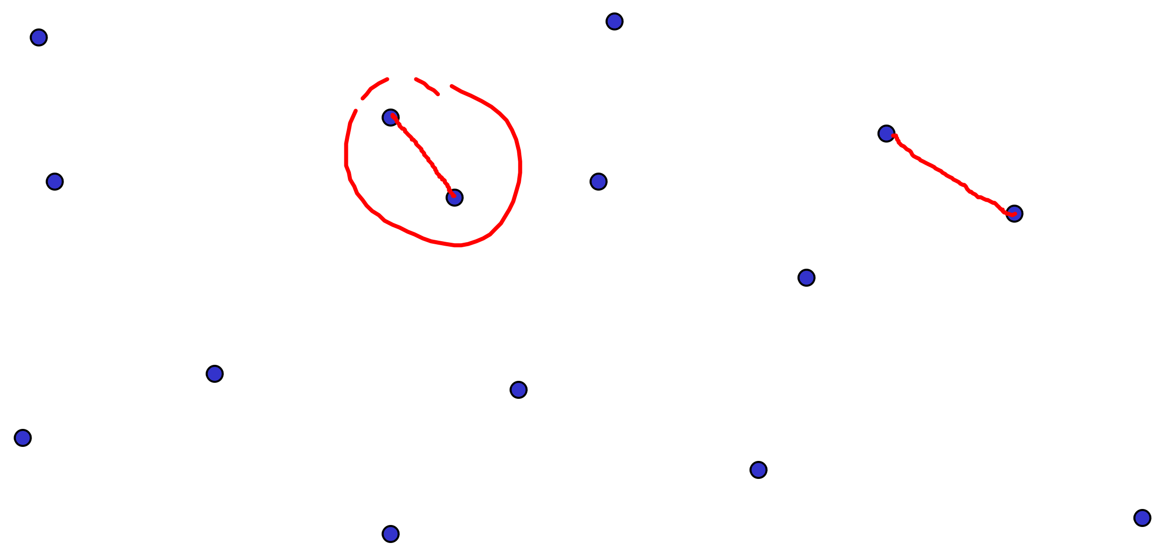
$= c \cdot n(\log n) + cn$

$= cn(\log n + 1)$

□

Geometric divide-and-conquer

Closest Pair Problem: Given a set S of n points, find a pair of points with the smallest distance.



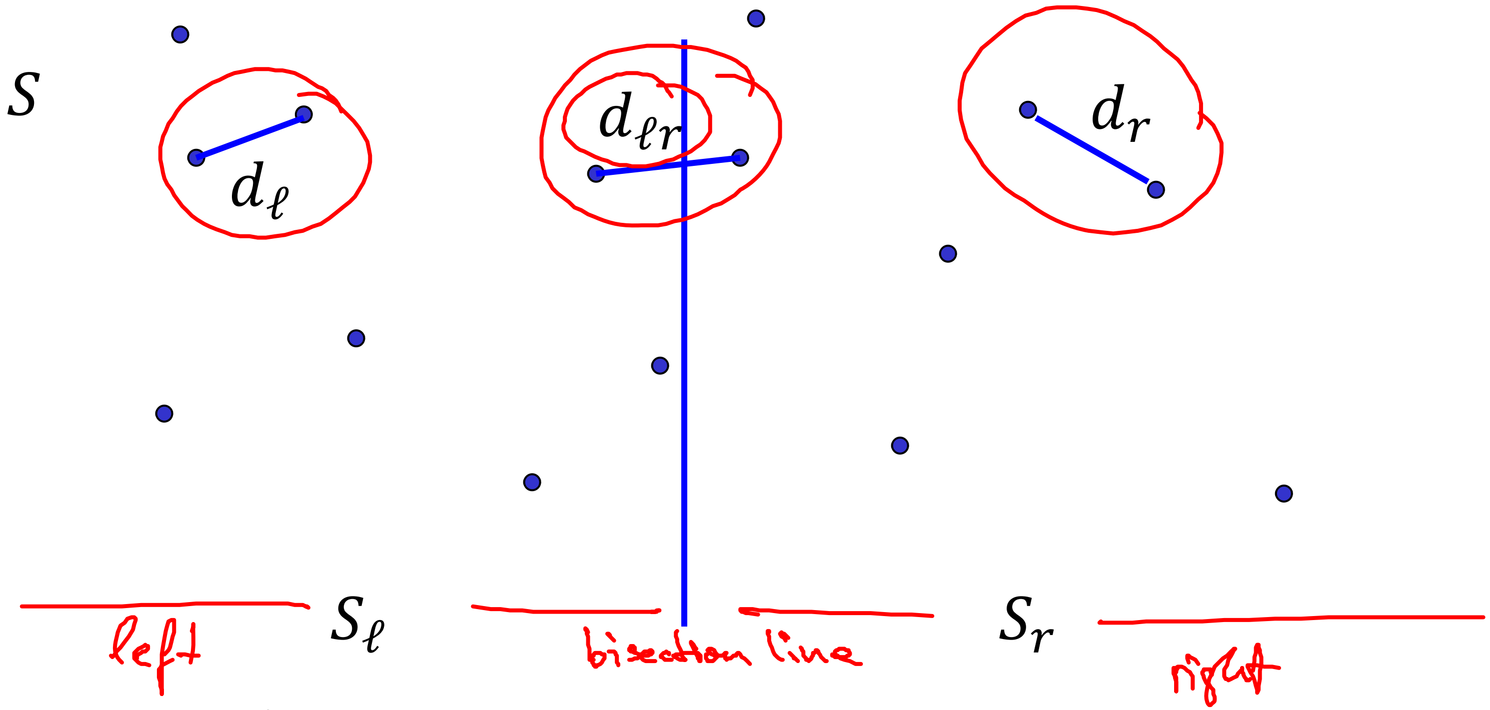
Naive solution:

compare all pairs

→ time $\Theta(n^2)$

Divide-and-conquer solution

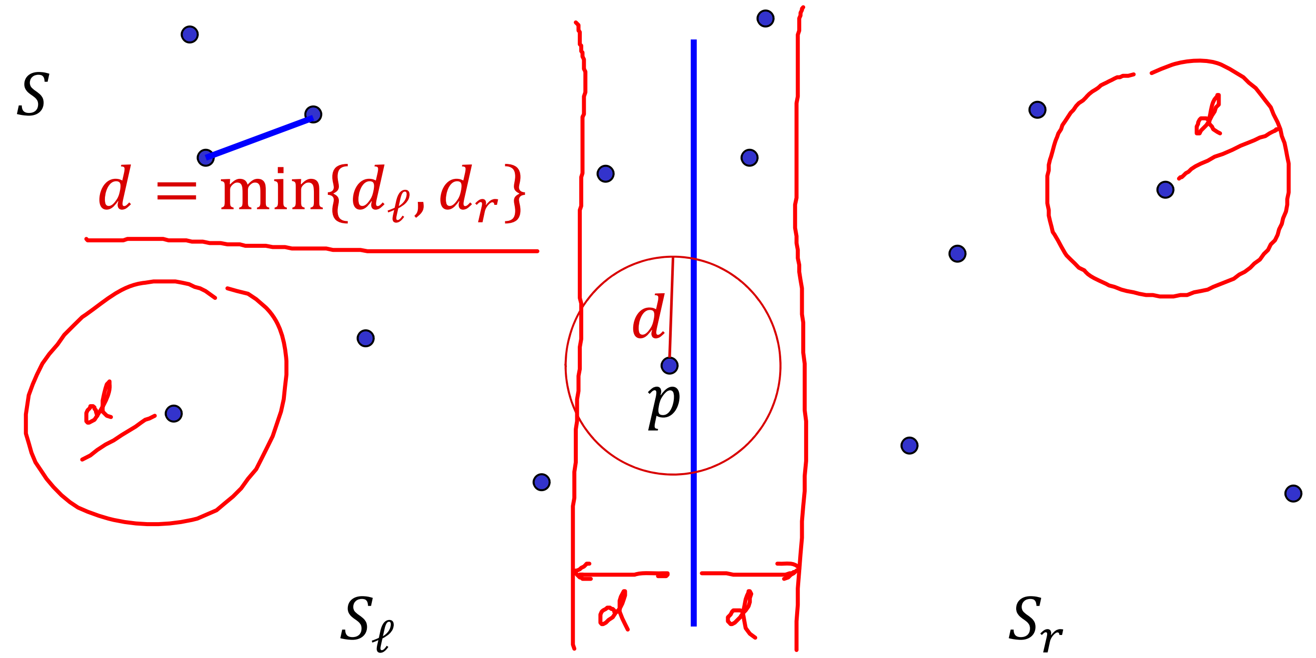
1. **Divide:** Divide S into two equal sized sets S_ℓ und S_r.
2. **Conquer:** $d_\ell = \text{mindist}(S_\ell)$ $d_r = \text{mindist}(S_r)$
3. **Combine:** $d_{\ell r} = \min\{d(p_\ell, p_r) \mid p_\ell \in S_\ell, p_r \in S_r\}$
   return $\min\{d_\ell, d_r, d_{\ell r}\}$



Divide-and-conquer solution

1. **Divide:** Divide S into two equal sized sets S_ℓ und S_r.
2. **Conquer:** $d_\ell = \text{mindist}(S_\ell)$ $d_r = \text{mindist}(Sr)$
3. **Combine:** $d_{\ell r} = \min\{d(p_\ell, p_r) \mid p_\ell \in S_\ell, p_r \in S_r\}$
   return $\min\{d_\ell, d_r, d_{\ell r}\}$

Computation of $d_{\ell r}$:

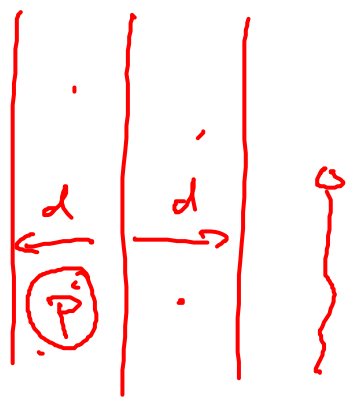


Merge step

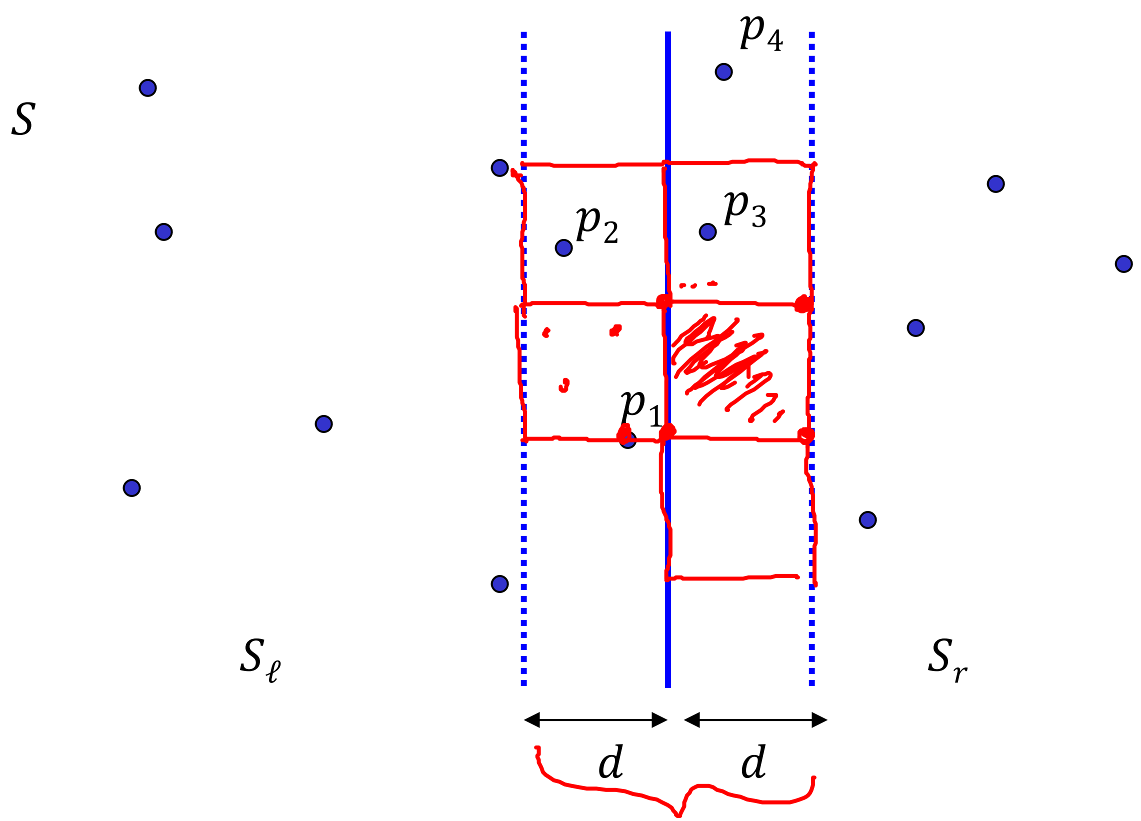
1. Consider only points within distance d of the bisection line, in the order of increasing y-coordinates.

2. For each point p consider all points q within y-distance at most d

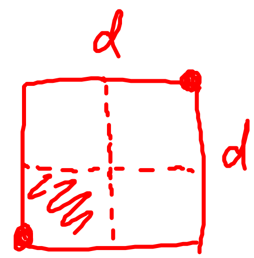
3. There are at most 7 such points.

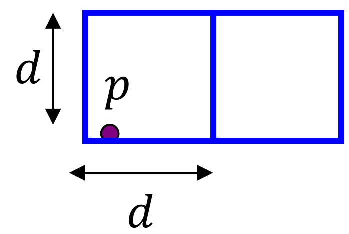


Combine step



$$d = \min\{d_\ell, dr\}$$





Implementation

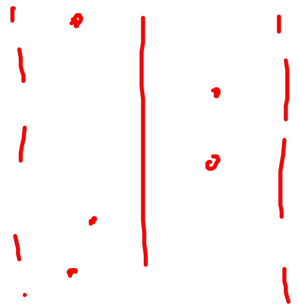
- Initially sort the points in S in order of increasing x-coordinates

  for partitioning

- While computing closest pair, also sort S according to y-coord.
  - Partition S into S_ℓ and S_r, solve and sort sub-problems recursively

    get d_ℓ, d_r, sorted S_ℓ, S_r (in y-coord.)
  - Merge to get sorted S according to y-coordinates

    cost $O(n)$
  - Center points: points within x-distance $d = \min\{d_\ell, d_r\}$ of center
  - Go through center points in S in order of incr. y-coordinates



overall time for combine: $O(n)$

Running Time

Recurrence relation:

$$T(n) = 2 \cdot T(n/2) + c \cdot n, \qquad T(1) = a$$

Solution:

- Same as for computing number of number of inversions, merge sort (and many others…)

$$T(n) = O(n \cdot \log n)$$