Chapter 7
Approximation Algorithms
Knapsack Approximation

Algorithm Theory
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Knapsack

- n items $1, \dots, n$, each item has weight $w_i > 0$ and value $v_i > 0$
- Knapsack (bag) of capacity W

- Goal: pack items into knapsack such that total weight is at most W and total value is maximized:

$$\max \sum_{i \in S} v_i$$

$$\text{s.t. } S \subseteq \{1, \dots, n\} \text{ and } \sum_{i \in S} w_i \leq W$$

- E.g.: jobs of length w_i and value v_i, server available for W time units, try to execute a set of jobs that maximizes the total value

Knapsack: Dynamic Programming Alg.

We have shown:

- If all item weights w_i are integers, using dynamic programming, the knapsack problem can be solved in time $O(nW)$
- If all values v_i are integers, there is another dynamic progr. algorithm that runs in time $O(n^2V)$, where V is the max. value.

Problems:

- If W and V are large, the algorithms are not polynomial in n
- If the values or weights are not integers, things are even worse (and in general, the algorithms cannot even be applied at all)

Idea:

- Can we adapt one the algorithms to at least compute an approximate solution?

Approximation Algorithm

- The algorithm has a parameter $\varepsilon > 0$
- We assume that each item alone fits into the knapsack
- We define:

$$V := \max_{1 \leq i \leq n} v_i, \qquad \forall i: \widehat{v}_i := \left\lceil \frac{v_i n}{\varepsilon V} \right\rceil, \qquad \widehat{V} := \max_{1 \leq i \leq n} \widehat{v}_i$$

- We solve the problem with values $\widehat{v}_i$ and weights w_i using dynamic programming in time $O(n^2 \cdot \widehat{V})$

Theorem: The described algorithm runs in time $O(n^3/\varepsilon)$.

Proof:

$$\widehat{V} = \max_{1 \leq i \leq n} \widehat{v}_i = \max_{1 \leq i \leq n} \left\lceil \frac{v_i n}{\varepsilon V} \right\rceil = \left\lceil \frac{V n}{\varepsilon V} \right\rceil = \left\lceil \frac{n}{\varepsilon} \right\rceil$$

Approximation Algorithm

Theorem: The approximation algorithm computes a feasible solution with approximation ratio at most $1 + \varepsilon$.

Proof:

- Define the set of all feasible solutions

$$\mathcal{S} := \left\{ S \subseteq \{1, \dots, n\} : \sum_{i \in S} w_i \leq W \right\}$$

- Let S^* be an optimal solution and $\hat{S}$ be the solution computed by the approximation algorithm.
- We have

$$S^* = \max_{S \in \mathcal{S}} \sum_{i \in S} v_i\,, \qquad \hat{S} = \max_{S \in \mathcal{S}} \sum_{S \in \mathcal{S}} \widehat{v_i}$$

- Hence, $\hat{S}$ is a feasible solution

Approximation Algorithm

Theorem: The approximation algorithm computes a feasible solution with approximation ratio at most $1 + \varepsilon$.

Proof:

- Because every item fits into the knapsack, we have

$$\forall i \in \{1, \dots, n\}: v_i \leq \sum_{j \in S^*} v_j$$

- For the solution of the algorithm, we get

$$\hat{v}_i = \left\lfloor \frac{v_i n}{\varepsilon V} \right\rfloor \implies v_i \leq \frac{\varepsilon V}{n} \cdot \hat{v}_i$$

- Therefore

$$\sum_{i \in S^*} v_i \leq \frac{\varepsilon V}{n} \cdot \sum_{i \in S^*} \hat{v}_i \leq \frac{\varepsilon V}{n} \cdot \sum_{i \in \hat{S}} \hat{v}_i \leq \frac{\varepsilon V}{n} \cdot \sum_{i \in \hat{S}} \left(\frac{v_i n}{\varepsilon V} + 1 \right)$$

value of opt. solution (orig. problem)

value of the opt. solution with $\hat{v}_i$

Approximation Algorithm

Theorem: The approximation algorithm computes a feasible solution with approximation ratio at most $1 + \varepsilon$.

Proof:

- We have

$$\sum_{i\in S^*} v_i \leq \frac{\varepsilon V}{n} \cdot \sum_{i\in S^*} \hat{v}_i \leq \frac{\varepsilon V}{n} \cdot \sum_{i\in \hat{S}} \hat{v}_i \leq \frac{\varepsilon V}{n} \cdot \sum_{i\in \hat{S}} \left(\frac{v_i n}{\varepsilon V} + 1\right)$$

- Therefore

$$\sum_{i\in S^*} v_i \leq \sum_{i\in \hat{S}} v_i + \frac{\varepsilon V}{n} \cdot \left|\hat{S}\right| \leq \varepsilon V + \sum_{i\in \hat{S}} v_i$$

- Because V is a lower bound on the optimal solution:

$$\sum_{i\in S^*} v_i \leq (1+\varepsilon) \cdot \sum_{i\in \hat{S}} v_i$$

$V \leq \sum_{i\in\hat{S}} v_i$ OPT ALG $\frac{OPT}{ALG} \leq 1+\varepsilon$

Approximation Schemes

$O(\text{poly}(n) \cdot 2^{1/\varepsilon})$

- For every parameter $\varepsilon > 0$, the knapsack algorithm computes a $(1+\varepsilon)$-approximation in time $O(n^3/\varepsilon)$.
- For every fixed ε, we therefore get a polynomial time approximation algorithm
- An algorithm that computes an $(1+\varepsilon)$-approximation for every $\varepsilon > 0$ is called an approximation scheme.
- If the running time is polynomial for every fixed ε, we say that the algorithm is a polynomial time approximation scheme (PTAS)
- If the running time is also polynomial in $1/\varepsilon$, the algorithm is a fully polynomial time approximation scheme (FPTAS)
- Thus, the described alg. is an FPTAS for the knapsack problem