Chapter 3
Dynamic Programming

Algorithm Theory
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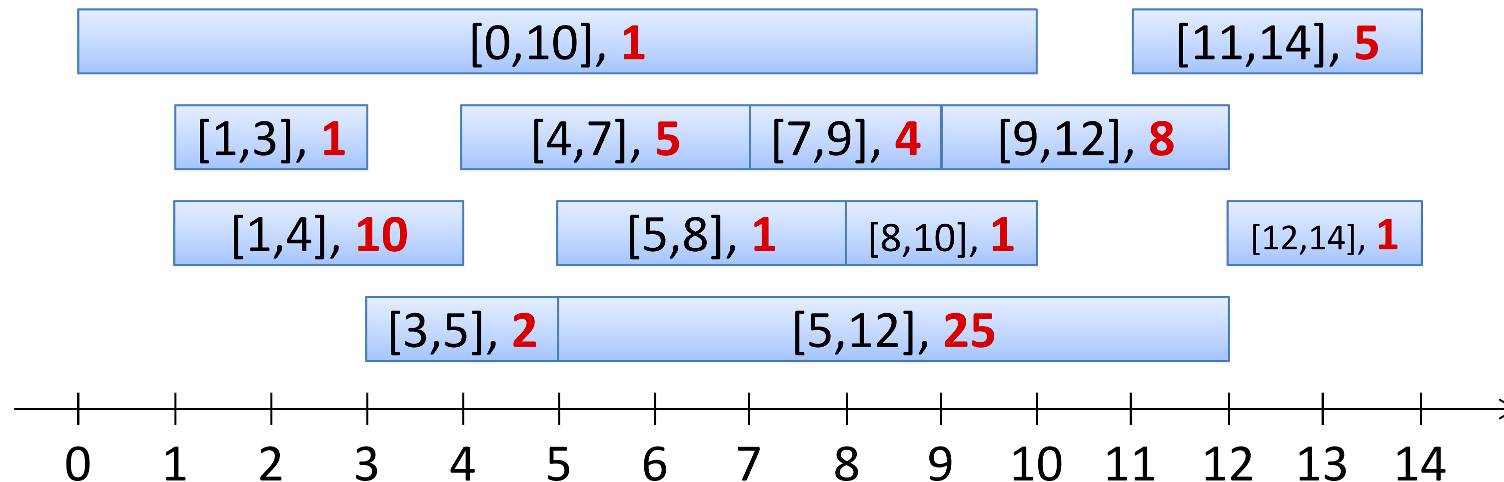
Weighted Interval Scheduling

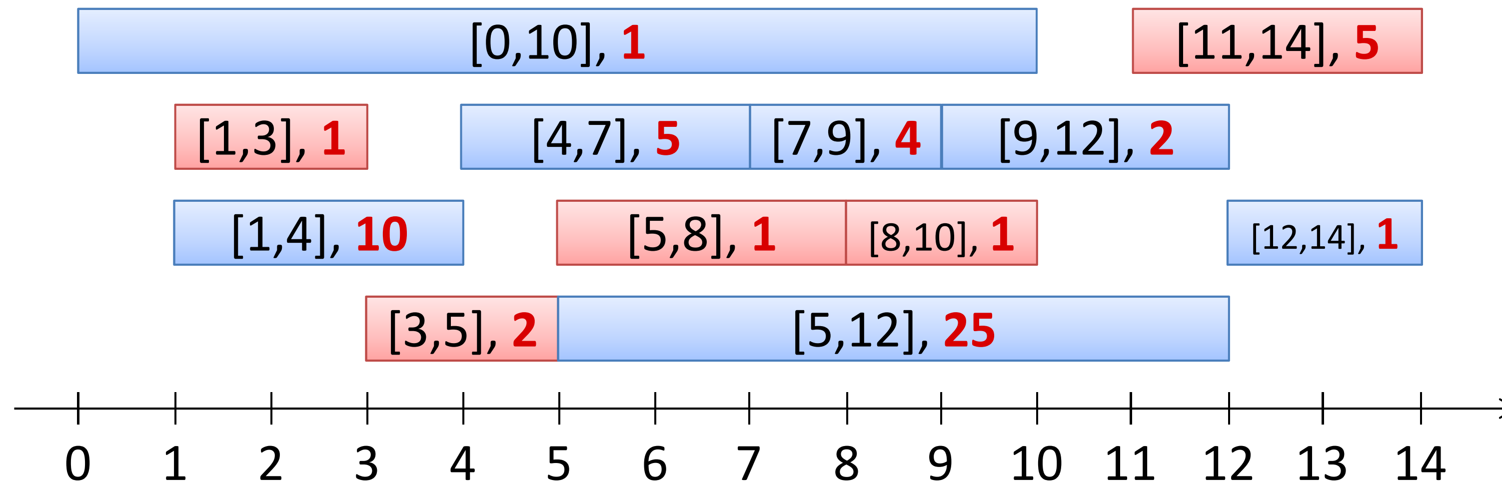
- **Given:** Set of intervals, e.g.
  [0,10],[1,3],[1,4],[3,5],[4,7],[5,8],[5,12],[7,9],[9,12],[8,10],[11,14],[12,14]
- Each interval has a **weight *w***



- **Goal:** Non-overlapping set of intervals of largest possible weight
  - Overlap at boundary ok, i.e., [4,7] and [7,9] are non-overlapping
- **Example:** Intervals are room requests of different importance

Greedy Algorithms

Choose available request with earliest finishing time:



- Algorithm is not optimal any more
  - It can even be arbitrarily bad…
- No greedy algorithm known that works

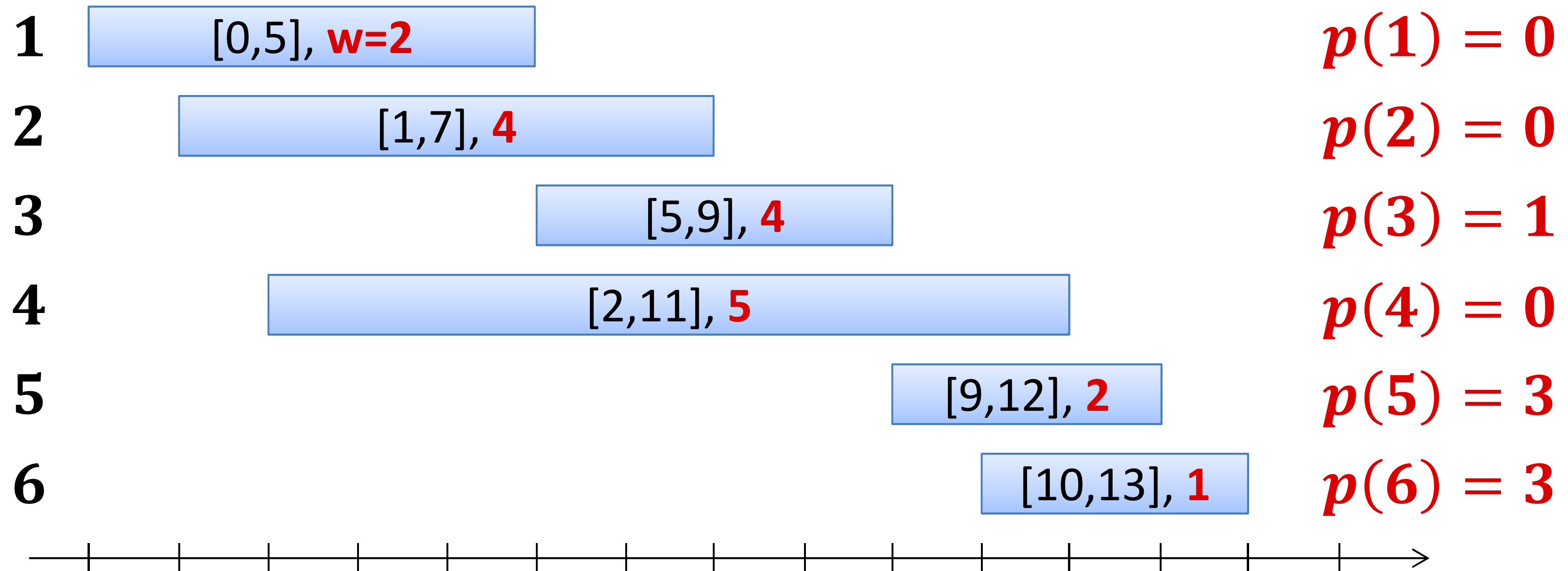
Solving Weighted Interval Scheduling

- Interval i: start time $s(i)$, finishing time: $t(i)$, weight: $w(i)$
- Assume intervals $1, \dots, n$ are sorted by increasing $t(i)$
  - $0 < f(1) \leq f(2) \leq \cdots \leq f(n)$, for convenience: $f(0) = 0$
- Simple observation:
  Opt. solution contains interval n or it doesn't contain interval i
- Weight of optimal solution for only intervals $1, \dots, k$: $W(k)$
  Define $p(k) \coloneqq \max\{i \in \{0, \dots, k-1\} : f(i) \leq s(k)\}$
- Opt. solution does not contain interval n: $\boldsymbol{W(n) = W(n-1)}$

  Opt. solution contains interval n: $\boldsymbol{W(n) = w(n) + W(p(n))}$

Example

Interval:



Recursive Definition of Optimal Solution

- Recall:
  - $W(k)$: weight of optimal solution with intervals $1, \ldots, k$
  - $p(k)$: last interval to finish before interval k starts

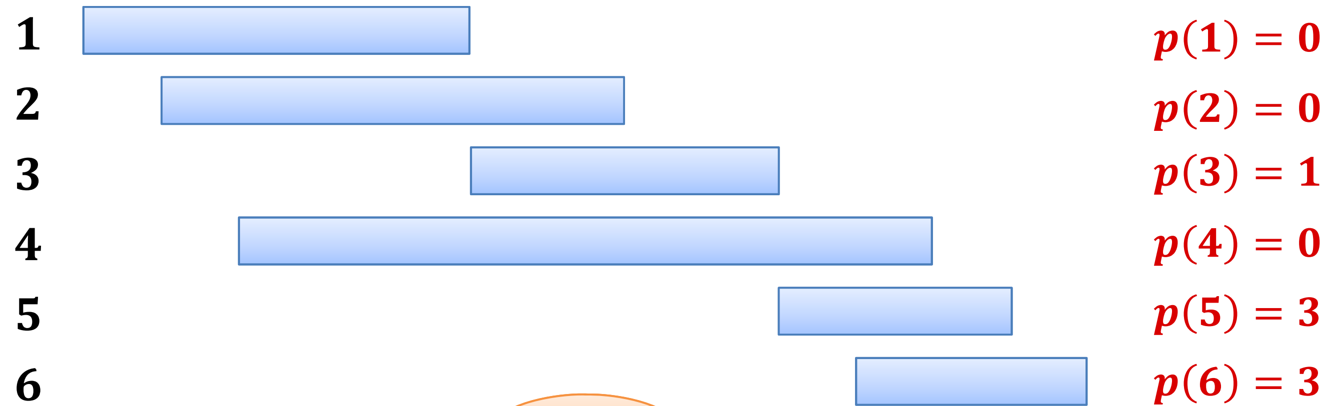
- Recursive definition of optimal weight:

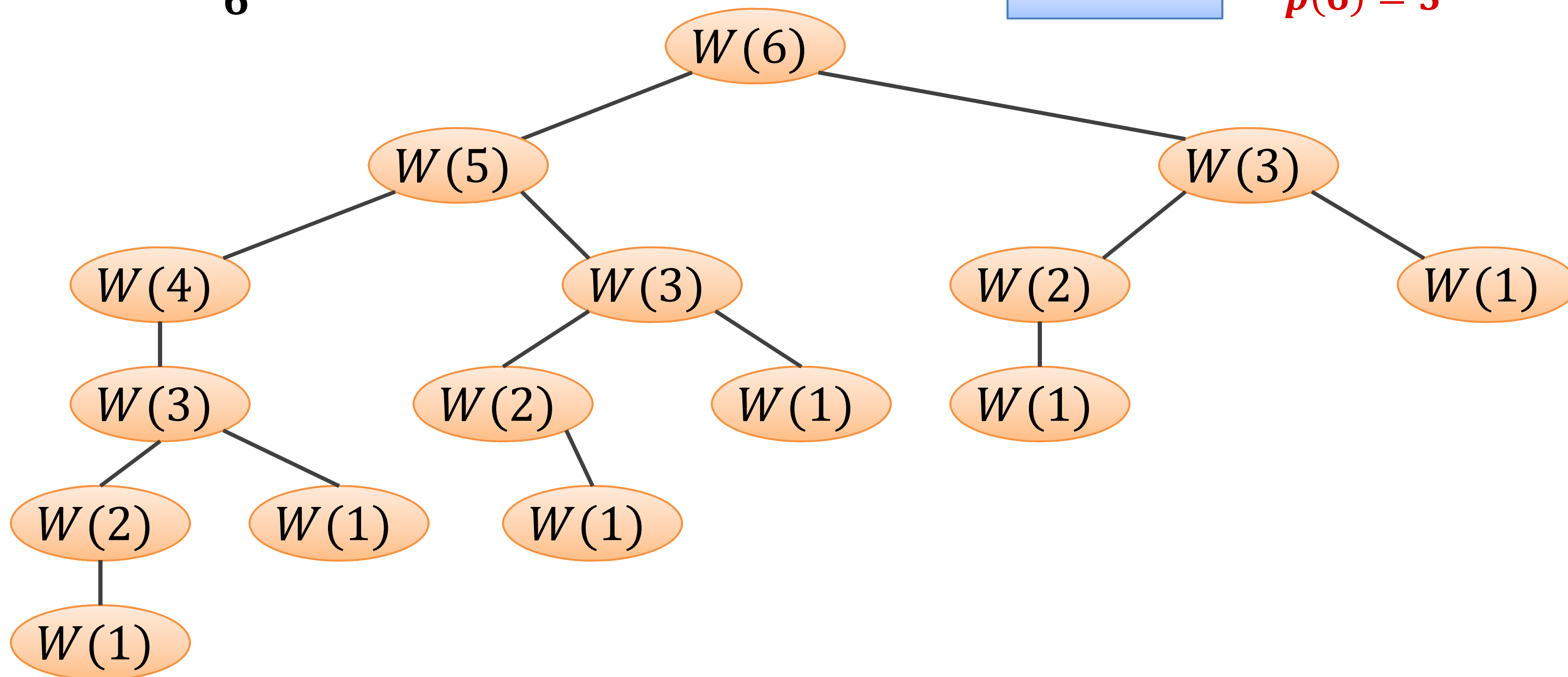
$$\forall k > 1: \; W(k) = \max\{W(k-1), w(k) + W(p(k))\}$$

$$W(1) = w(1)$$

- Immediately gives a simple, recursive algorithm

Running Time of Recursive Algorithm





Memoizing the Recursion

- Running time of recursive algorithm: exponential!
- But, alg. only solves n different sub=problems: $W(1), \dots, W(n)$
- There is no need to compute them multiple times

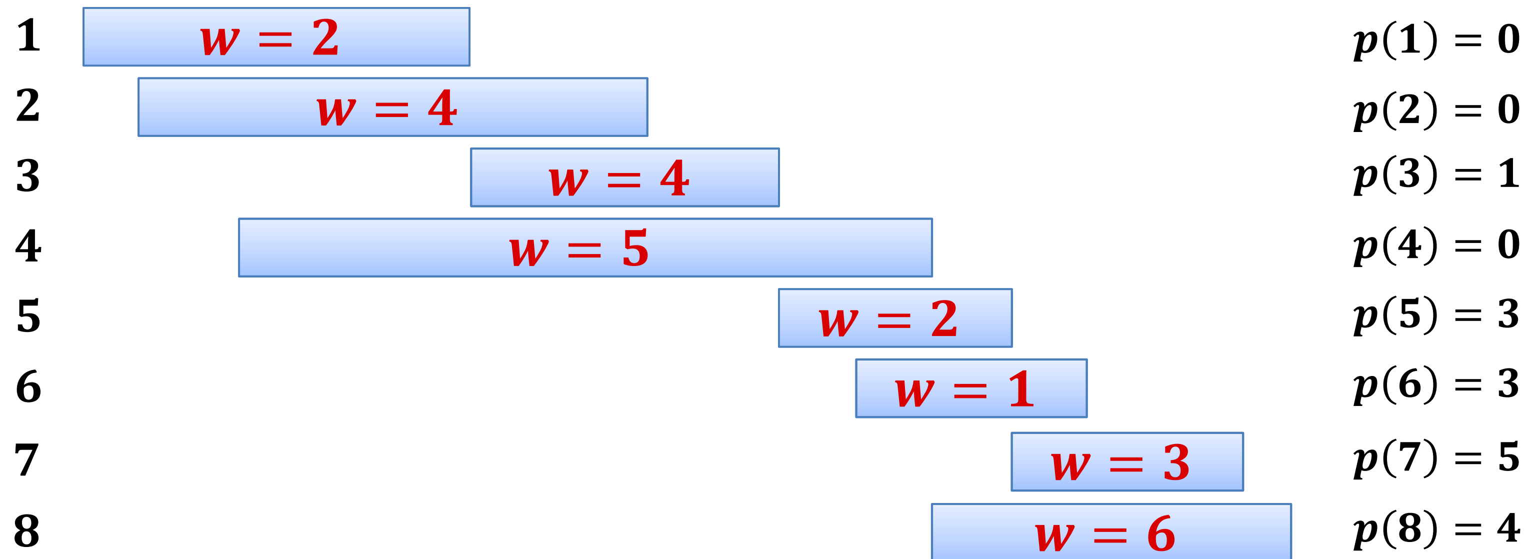
Memoization:

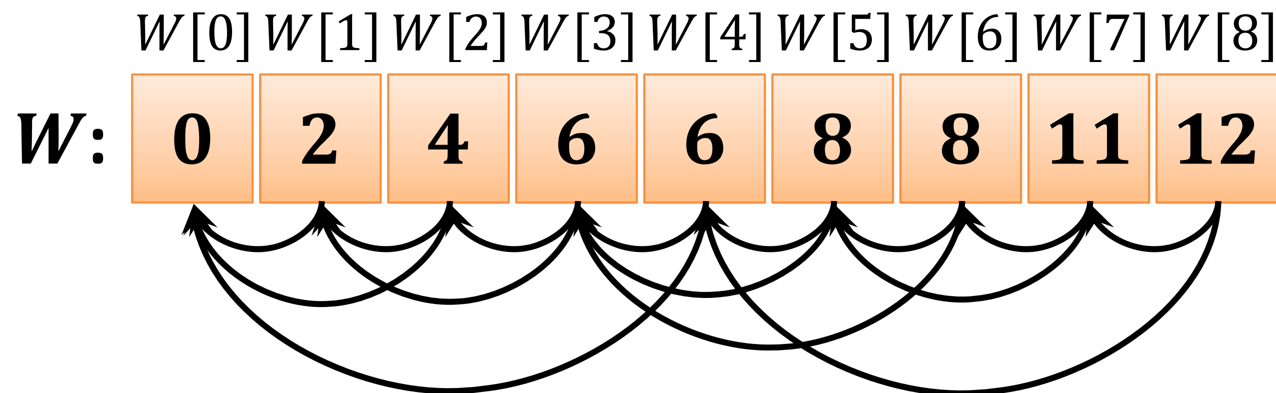
- **Store already computed values** for future use (recursive calls)

Efficient algorithm:

1. $W[0] \coloneqq 0$; compute values $p(i)$
2. **for** $i \coloneqq 1$ **to** n **do**
3. $\quad W[i] \coloneqq \max\{W[i-1], w(i) + W[p(i)]\}$
4. **end**

Example





Computing the schedule: store where you come from!

Matrix-chain multiplication

Given: sequence (chain) $\langle A_1, A_2, \ldots, An \rangle$ of matrices

Goal: compute the product $A_1 \cdot A_2 \cdot \ldots \cdot An$

Problem: Parenthesize the product in a way that minimizes the number of scalar multiplications.

Definition: A product of matrices is *fully parenthesized* if it is

- a single matrix
- or the product of two fully parenthesized matrix products, surrounded by parentheses.

Example

All possible fully parenthesized matrix products of the chain $\langle A_1, A_2, A_3, A_4 \rangle$:

$$(A_1(A_2(A_3A_4)))$$

$$(A_1((A_2A_3)A_4))$$

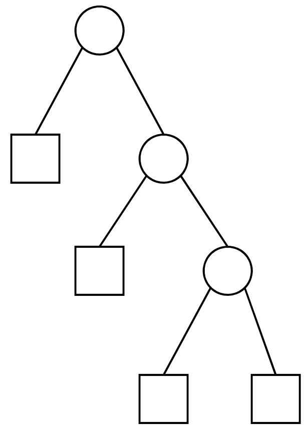
$$((A_1A_2)(A_3A_4))$$

$$((A_1(A_2A_3))A_4)$$

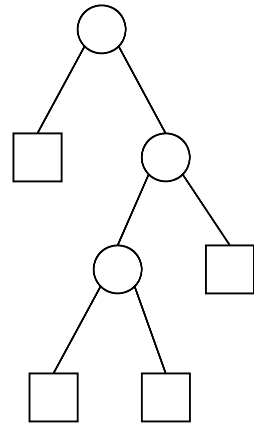
$$(((A_1A_2)A_3)A_4)$$

Different parenthesizations

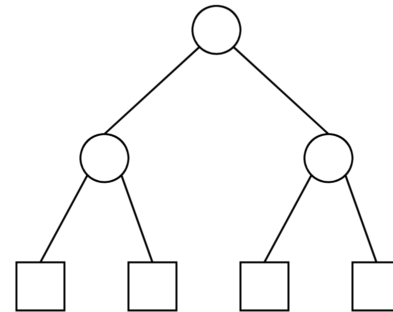
Different parenthesizations correspond to different trees:



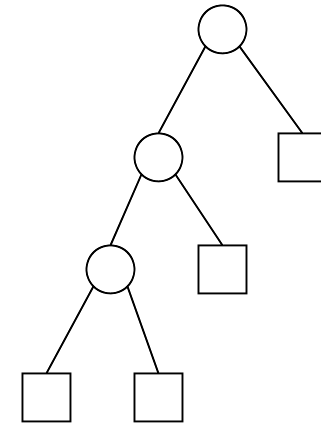
$(A_1(A_2(A_3A_4)))$



$(A_1((A_2A_3)A_4))$



$((A_1A_2)(A_3A_4))$



$(((A_1A_2)A_3)A_4)$

Number of different parenthesizations

- Let $P(n)$ be the number of alternative parenthesizations of the product $A_1 \cdot \ldots \cdot A_n$:
-

$$P(1) = 1$$

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k)\,, \qquad \text{for } n \geq 2$$

$$P(n+1) = \frac{1}{n+1}\binom{2n}{n} \approx \frac{4^n}{n\sqrt{\pi n}} + O\left(\frac{4^n}{\sqrt{n^5}}\right)$$

$$P(n+1) = C_n \qquad (n^{th} \text{ Catalan number})$$

- Thus: Exhaustive search needs exponential time!

Multiplying Two Matrices

$$A = \left(a_{ij}\right)_{p\times q}, \qquad B = \left(b_{ij}\right)_{q\times r}, \qquad A \cdot B = C = \left(c_{ij}\right)_{p\times r}$$

$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}$$

Algorithm *Matrix-Mult*
Input: $(p \quad q)$ matrix A, $(q \times r)$ matrix B
Output: $(p \times r)$ matrix $C = A \cdot B$
1 **for** $i := 1$ **to** p **do**
2   **for** $j := 1$ **to** r **do**
3     $C[i,j] := 0;$
4   **for** $k := 1$ **to** q **do**
5       $C[i,j] := C[i,j] + A[i,k] \cdot B[k,j]$

Remark:

Using this algorithm, multiplying two $(n \times n)$ matrices requires n^3 multiplications. This can also be done using $O(n^{2.376})$ multiplications.

Number of multiplications and additions: $\boldsymbol{p \cdot q \cdot r}$

Matrix-chain multiplication: Example

Computation of the product $A_1 A_2 A_3$, where

A_1 : (50×5) matrix
A_2 : (5×100) matrix
A_3 : (100×10) matrix

a) Parenthesization $((A_1 A_2)A_3)$ and $(A_1(A_2 A_3))$ require:

$A' = (A_1 A_2)$:

$A'' = (A_2 A_3)$:

$A' A_3$:

$A_1 A''$:

Sum:

Structure of an Optimal Parenthesization

- $(A_{\ell\ldots r})$: optimal parenthesization of $A_\ell \cdot \ldots \cdot A_r$

  For some $1 \leq k < n$: $\boldsymbol{(A_{1\ldots n}) = \big((A_{1\ldots k}) \cdot (A_{k+1\ldots n})\big)}$

- Any optimal solution contains optimal solutions for sub-problems

- Assume matrix A_i is a $(d_{i-1} \times d_i)$-matrix

- Cost to solve sub-problem $A_\ell \cdot \ldots \cdot A_r, \ell \leq r$ optimally: $C(\ell, r)$

- Then:

$$\boldsymbol{C(a,b) = \min_{a \leq k < b} C(a,k) + C(k+1,b) + d_{a-1} d_k d_b}$$

$$\boldsymbol{C(a,a) = 0}$$

Recursive Computation of Opt. Solution

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:

$C(1,5)$

$C(1,2)$ $C(1,3)$ $C(1,4)$ $C(2,5)$ $C(3,5)$ $C(4,5)$

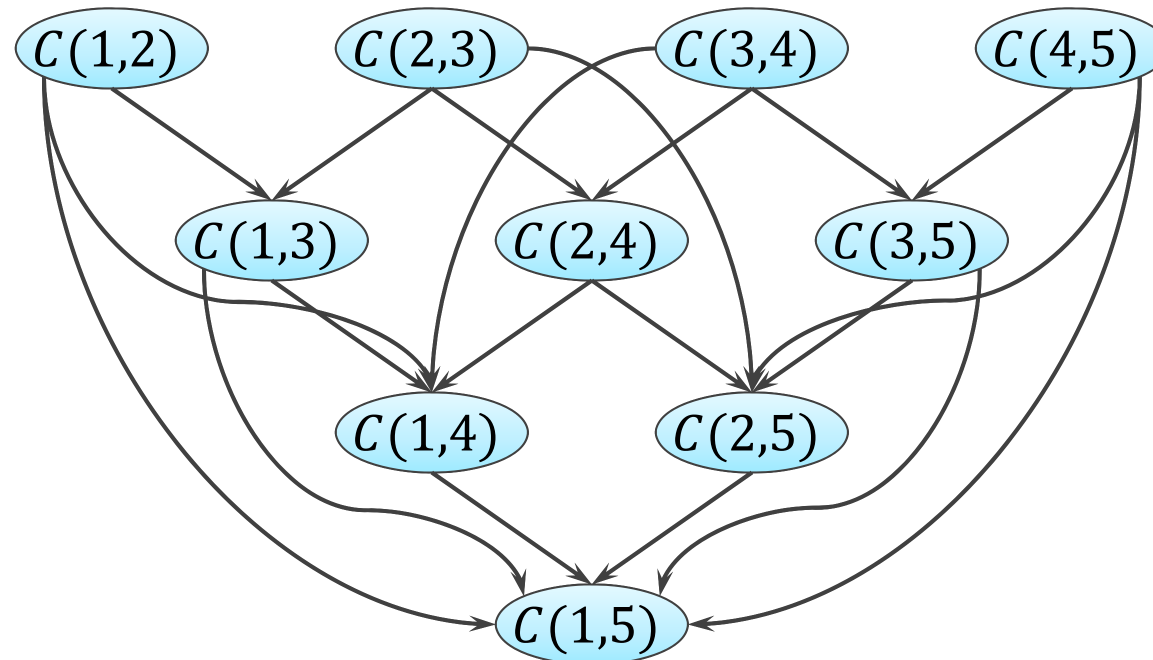
$C(1,2)$ $C(2,3)$ $C(3,4)$ $C(4,5)$

$C(1,2)$ $C(1,3)$ $C(2,3)$ $C(2,4)$ $C(2,4)$ $C(2,3)$ $C(3,5)$ $C(4,5)$

$C(1,2)$ $C(2,3)$ $C(2,3)$ $C(3,4)$ $C(2,3)$ $C(3,4)$ $C(3,4)$ $C(4,5)$

Using Meomization

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:



Compute $A_1 \cdot \ldots \cdot A_n$:

- Each $C(i,j), i < j$ is computed exactly once → $O(n^2)$ values
- Each $C(i,j)$ dir. depends on $C(i,k), C(k,j)$ for $i < k < j$

Cost for each $C(i,j)$: $O(n)$ → overall time: $\boldsymbol{O(n^3)}$

Dynamic Programming

„Memoization“ for increasing the efficiency of a recursive solution:

- Only the *first time* a sub-problem is encountered, its solution is computed and then stored in a table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned
  (without repeated computation!).

- *Computing the solution*: For each sub-problem, store how the value is obtained (according to which recursive rule).

Dynamic Programming

Dynamic programming / memoization can be applied if

- Optimal solution contains optimal solutions to sub-problems (recursive structure)

- Number of sub-problems that need to be considered is small

Remarks about matrix-chain multiplication

1. There is an algorithm that determines an optimal parenthesization in time

$$O(n \cdot \log n).$$

2. There is a linear time algorithm that determines a parenthesization using at most

$$1.155 \cdot C(1, n)$$

multiplications.

Knapsack

- n items $1, \dots, n$, each item has weight w_i and value v_i
- Knapsack (bag) of capacity W

- Goal: pack items into knapsack such that total weight is at most W and total value is maximized:

$$\max \sum_{i \in S} v_i$$

$$\text{s.t. } S \subseteq \{1, \dots, n\} \text{ and } \sum_{i \in S} w_i \leq W$$

- E.g.: jobs of length w_i and value v_i, server available for W time units, try to execute a set of jobs that maximizes the total value

Recursive Structure?

- Optimal solution: $\mathcal{O}$
- If $n \notin \mathcal{O}$: $\mathrm{OPT}(n) = \mathrm{OPT}(n-1)$
- What if $n \in \mathcal{O}$?
  - Taking n gives value v_n
  - But, n also occupies space w_n in the bag (knapsack)
  - There is space for $W - w_n$ total weight left!

$$\mathrm{OPT}(n) = w_n + \text{optimal solution with first } n-1 \text{ items and knapsack of capacity } W - w_n$$

A More Complicated Recursion

$\mathbf{OPT}(\boldsymbol{k}, \boldsymbol{x})$: value of optimal solution with items $1, \dots, k$
and knapsack of capacity x

Recursion:

Dynamic Programming Algorithm

Set up table for all possible $\mathrm{OPT}(k, x)$-values

- Assume that all weights w_i are integers!

	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$			$\cdots$						$\boldsymbol{W}$
$\mathbf{1}$												
$\mathbf{2}$												
$\mathbf{3}$												
$\vdots$												
$\boldsymbol{n}$												

Row $\boldsymbol{i}$, column $\boldsymbol{j}$:

$\boldsymbol{OPT(i, j)}$

Example

- 8 items: (3,2), (2,4), (4,1), (5,6), (3,3), (4,3), (5, 4), (6,6)
  Knapsack capacity: 12
  weight **value**
- $OPT(k, x) = \max\{OPT(k-1, x), OPT(k-1, x-w_k) + v_k\}$

	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												
4												
5												
6												
7												
8												

Running Time of Knapsack Algorithm

- **Size of table:** $O(n \cdot W)$
- Time per table entry: $O(1)$ → **overall time:** $\boldsymbol{O(nW)}$
- Computing solution (set of items to pick):
  Follow $\leq n$ arrows → $O(n)$ time (after filling table)
- Note: Time depends on W → can be exponential in n…
- And it is problematic if weights are not integers.