Chapter 8
Online Algorithms

Algorithm Theory
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Online Computations

- Sometimes, an algorithm has to start processing the input before the complete input is known

- For example, when storing data in a data structure, the sequence of operations on the data structure is not known

Online Algorithm: An algorithm that has to produce the output step-by-step when new parts of the input become available.

Offline Algorithm: An algorithm that has access to the whole input before computing the output.

- Some problems are inherently online
  - Especially when real-time requests have to be processed over a significant period of time

Competitive Ratio

- Let's again consider optimization problems
  - For simplicity, assume, we have a minimization problem

Optimal offline solution $\mathbf{OPT}(\boldsymbol{I})$:

- Best objective value that an offline algorithm can achieve for a given input sequence I

Online solution $\mathbf{ALG}(\boldsymbol{I})$:

- Objective value achieved by an online algorithm ALG on I

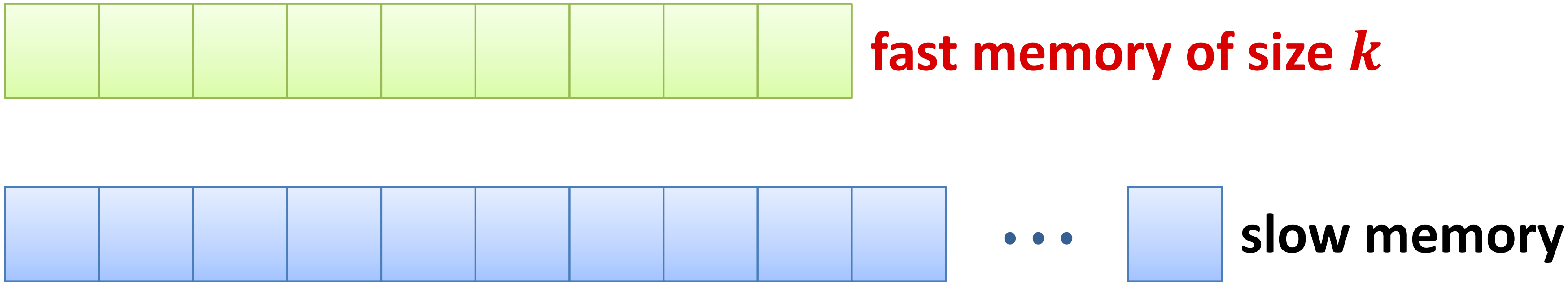
Competitive Ratio: An algorithm has competitive ratio $c \geq 1$ if

$$\mathbf{ALG}(\boldsymbol{I}) \leq \boldsymbol{c} \cdot \mathbf{OPT}(\boldsymbol{I}) + \boldsymbol{\alpha}.$$

- If $\alpha \leq 0$, we say that ALG is strictly c-competitive.

Paging Algorithm

Assume a simple memory hierarchy:



If a memory page has to be accessed:

- Page in fast memory (hit): take page from there
- Page not fast memory (miss): leads to a page fault
- Page fualt: the page is loaded into the fast memory and some page has to be evicted from the fast memory
- Paging algorithm: decides which page to evict
- Classical online problem: we don't know the future accesses

Paging Strategies

Least Recently Used (LRU):

- Replace the page that hasn't been used for the longest time

First In First Out (FIFO):

- Replace the page that has been in the fast memory longest

Last In First Out (LIFO):

- Replace the page most recently moved to fast memory

Least Frequently Used (LFU):

- Replace the page that has been used the least

Longest Forward Distance (LFD):

- Replace the page whose next request is latest (in the future)
- LFD is **not** an online strategy!

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

- For contradiction, assume that LFD is not optimal
- Then there exists a finite input sequence σ on which LFD is not optimal (assume that the length of σ is $|\sigma| = n$)
- Let OPT be an optimal solution for σ such that
  - OPT processes requests $1, \dots, i$ in exactly the same way as LFD
  - OPT processes request $i + 1$ differently than LFD
  - Any other optimal strategy processes one of the first $i + 1$ requests differently than LDF
- Hence, OPT is the optimal solution that behaves in the same way as LFD for as long as possible → we have $i < n$
- Goal: Construct OPT' that is identical with LFD for req. $1, \dots, i + 1$

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

Case 1: Request $i + 1$ does **not** lead to a page fault

- LFD does not change the content of the fast memory
- OPT behaves differently than LFD
  → OPT replaces some page in the fast memory
  - As up to request $i + 1$, both algorithms behave in the same way, they also have the same fast memory content
  - OPT therefore does not require the new page for request $i + 1$
  - Hence, OPT can also load that page later (without extra cost) → OPT′

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

Case 2: Request $i + 1$ does lead to a **page fault**

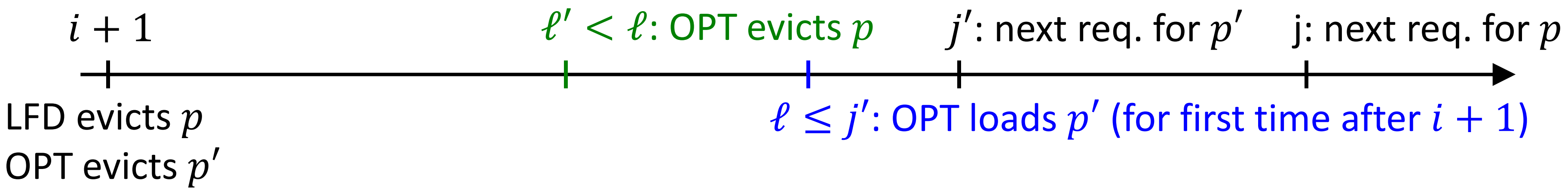
- LFD and OPT move the same page into the fast memory, but they evict different pages
  - If OPT loads more than one page, all pages that are not required for request $i + 1$ can also be loaded later
- Say, LFD evicts page p and OPT evicts page p'
- By the definition of LFD, p' is required again before page p

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

Case 2: Request $i + 1$ does lead to a **page fault**



$i + 1$

$\ell' < \ell$: OPT evicts p

j': next req. for p'

j: next req. for p

LFD evicts p
OPT evicts p'

$\ell \leq j'$: OPT loads p' (for first time after $i + 1$)

a) OPT keeps p in fast memory until request ℓ
   - Evict p at request $i + 1$, keep p' instead and load p (instead of p') back into the fast memory at request ℓ

b) OPT evicts p at request $\ell' < \ell$
   - Evict p at request $i + 1$ and p' at request ℓ' (switch evictions of p and p')

Phase Partition

We partition a given request sequence σ into phases as follows:

- **Phase $\mathbf{0}$**: empty sequence
- **Phase $\boldsymbol{i}$** : maximal sequence that immediately follows phase $i-1$ and contains at most k distinct page requests

Example sequence ($\boldsymbol{k=4}$):

$$2, 5, 12, 5, 4, 2, 10, 8, 3, 6, 2, 2, 6, 6, 8, 3, 2, 6, 9, 10, 6, 3, 10, 2, 1, 3, 5$$

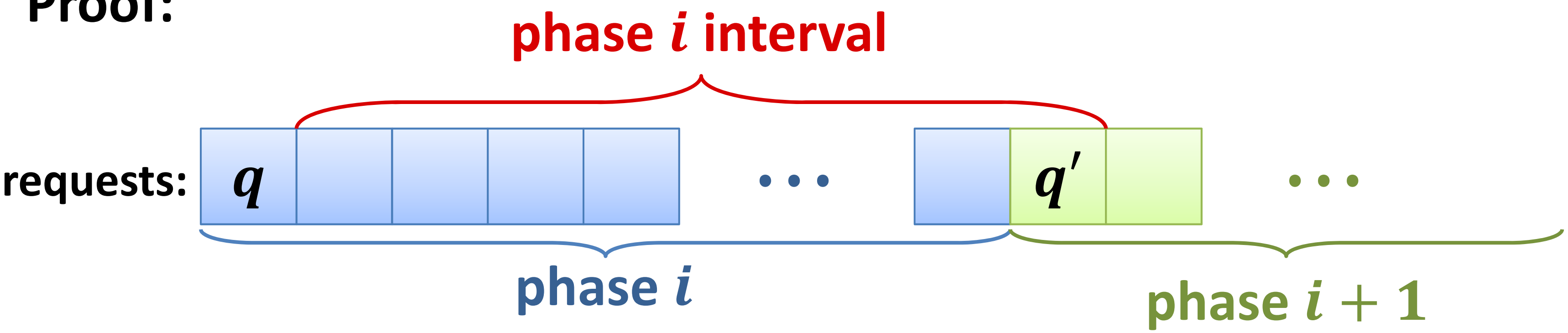
Phase $\boldsymbol{i}$ Interval: interval starting with the second request of phase i and ending with the first request of phase $i+1$

- If the last phase is phase p, phase-interval i is defined for $i = 1, \dots, p-1$

Optimal Algorithm

Lemma: Algorithm LFD has at least one page fault in each phase i interval (for $i = 1, \dots, p - 1$, where p is the number of phases).

Proof:



- q is in fast memory after first request of phase i
- Number of distinct requests in phase i: k
- By maximality of phase i: q' does not occur in phase i
- Number of distinct requests $\neq q$ in phase interval i: k

→ at least one page fault

LRU and FIFO Algorithms

Lemma: Algorithm LFD has at least one page fault in each phase interval i (for $i = 1, \ldots, p-1$, where p is the number of phases).

Corollary: The number of page faults of an optimal offline algorithm is at least $p - 1$, where p is the number of phases

Theorem: The LRU and the FIFO algorithms both have a competitive ratio of at most k.

Proof:

- In phase i only pages from phases before phase i are evicted from the fast memory → $\leq k$ page faults per phase
  - As long as not all k pages from phase i have been requested, the least recently used and the first inserted are from phases before i
  - When all k pages have been requested, the k pages of phase i are in fast memory and there are no more page faults in phase i

Lower Bound

Theorem: Even if the slow memory contains only $k + 1$ pages, any deterministic algorithm has competitive ratio at least k.

Proof:

- Consider some given deterministic algorithm ALG
- Because ALG is deterministic, the content of the fast memory after the first i requests is determined by the first i requests.
- Construct a request sequence inductively as follows:
  - Assume some initial slow memory content
  - The $(i + 1)^{\text{st}}$ request is for the page which is not in fast memory after the first i requests (throughout we only use $k + 1$ different pages)
- There is a page fault for every request
- OPT has a page fault at most every k requests
  - There is always a page that is not required for the next $k - 1$ requests

Randomized Algorithms

- We have seen that deterministic paging algorithms cannot be better than k-competitive
- Does it help to use randomization?

Competitive Ratio: A randomized online algorithm has competitive ratio $c \geq 1$ if for all inputs I,

$$\mathbb{E}[\mathbf{ALG}(\boldsymbol{I})] \leq \boldsymbol{c} \cdot \mathbf{OPT}(\boldsymbol{I}) + \boldsymbol{\alpha}.$$

- If $\alpha \leq 0$, we say that ALG is strictly c-competitive.

Adversaries

- For randomized algorithm, we need to distinguish between different kinds of adversaries (providing the input)

Oblivious Adversary:

- Has to determine the complete input sequence before the algorithm starts
  - The adversary cannot adapt to random decisions of the algorithm

Adaptive Adversary:

- The adversary knows how the algorithm reacted to earlier inputs
- online adaptive: adversary has no access to the randomness used to react to the current input
- offline adaptive: adversary knows the random bits used by the algorithm to serve the current input

Lower Bound

The adversaries can be ordered according to their strength

$$\text{oblivious} < \text{online adaptive} < \text{offline adaptive}$$

- An algorithm that works with an adaptive adversary also works with an oblivious one
- A lower bound that holds against an oblivious adversary also holds for the other 2
- …

Theorem: No randomized paging algorithm can be better than k-competitive against an online (or offline) adaptive adversary.

Proof: The same proof as for deterministic algorithms works.

- Are there better algorithms with an oblivious adversary?

The Randomized Marking Algorithm

- Every entry in fast memory has a marked flag

- Initially, all entries are unmarked.
- If a page in fast memory is accessed, it gets marked
- When a page fault occurs:
  - If all k pages in fast memory are marked, all marked bits are set to 0
  - The page to be evicted is chosen uniformly at random among the unmarked pages
  - The marked bit of the new page in fast memory is set to 1

Example

Input Sequence (k=6):

$$\underbrace{2, 5, 3, 3, 6, 8, 2, 9, 5}_{\text{phase 1}}, \underbrace{7, 1, 2, 5, 2, 3, 7, 4}_{\text{phase 2}}, \underbrace{8, 1, 2, 7, 5,3}_{\text{phase 3}},\underbrace{6,9,6,10,4,1,2}_{\text{phase 4}} \ldots$$

Fast Memory:

10	6	1	9	4	2

Observations:

- At the end of a phase, the fast memory entries are exactly the k pages of that phase
- At the beginning of a phase, all entries get unmarked
- #page faults depends on #new pages in a phase

Page Faults per Phase

Consider a fixed phase $\boldsymbol{i}$:

- Assume that of the k pages of phase i, m_i are new and $k - m_i$ are old (i.e., they already appear in phase $i - 1$)
- All m_i new pages lead to page faults (when they are requested for the first time)
- When requested for the first time, an old page leads to a page fault, if the page was evicted in one of the previous page faults
- We need to count the number of page faults for old pages

Page Faults per Phase

Phase $\boldsymbol{i}$, $\boldsymbol{j}^{\text{th}}$ old page that is requested (for the first time):

- There is a page fault if the page has been evicted
- There have been at most $m_i + j - 1$ distinct requests before
- The old places of the $j - 1$ first old pages are occupied
- The other $\leq m_i$ pages are at uniformly random places among the remaining $k - (j - 1)$ places (oblivious adv.)
- Probability that the old place of the j^{th} old page is taken:

$$\leq \frac{m_i}{k - (j - 1)}$$

Page Faults per Phase

Phase $i > 1$, j^{th} old page that is requested (for the first time):

- Probability that there is a page fault:

$$\leq \frac{m_i}{k-(j-1)}$$

Number of page faults for old pages in phase i: F_i

$$\mathbb{E}[F_i] = \sum_{j=1}^{k-m_i} \mathbb{P}\big(j^{\text{th}} \text{ old page incurs page fault}\big)$$

$$\leq \sum_{j=1}^{k-m_i} \frac{m_i}{k-(j-1)} = m_i \cdot \sum_{\ell=m_i+1}^{k} \frac{1}{\ell}$$

$$= m_i \cdot \big(H(k) - H(m_i)\big) \leq m_i \cdot (H(k) - 1)$$

Competitive Ratio

Theorem: Against an oblivious adversary, the randomized marking algorithm has a competitive ratio of at most $2H(k) \leq 2\ln(k) + 2$.

Proof:

- Assume that there are p phases
- #page faults of rand. marking algorithm in phase i: $F_i + m_i$
- We have seen that

$$\mathbb{E}[F_i] \leq m_i \cdot (H(k) - 1) \leq m_i \cdot \ln(k)$$

- Let F be the total number of page faults of the algorithm:

$$\mathbb{E}[F] \leq \sum_{i=1}^{p} (\mathbb{E}[F_i] + m_i) \leq H(k) \cdot \sum_{i=1}^{p} m_i$$

Competitive Ratio

Theorem: Against an oblivious adversary, the randomized marking algorithm has a competitive ratio of at most $2H(k) \leq 2\ln(k) + 2$.

Proof:

- Let F_i^* be the number of page faults in phase i in an opt. exec.
- Phase 1: m_1 pages have to be replaces → $F_1^* \geq m_1$
- Phase $i > 1$:
  - Number of distinct page requests in phases $i - 1$ and i: $\boldsymbol{k + m_i}$
  - Therefore, $\boldsymbol{F_{i-1}^* + F_i^* \geq m_i}$
- Total number of page requests F^*:

$$F^* = \sum_{i=1}^{p} F_i^* \geq \frac{1}{2} \cdot \left(F_1^* + \sum_{i=2}^{p} (F_{i-1}^* + F_i^*) \right) \geq \frac{1}{2} \cdot \sum_{i=1}^{p} m_i$$

Competitive Ratio

Theorem: Against an oblivious adversary, the randomized marking algorithm has a competitive ratio of at most $2H(k) \leq 2\ln(k) + 2$.

Proof:

- Randomized marking algorithm:

$$\mathbb{E}[F] \leq H(k) \cdot \sum_{i=1}^{p} m_i$$

- Optimal algorithm:

$$\mathrm{F}^* \geq \frac{1}{2} \cdot \sum_{i=1}^{p} m_i$$

Remark: It can be shown that no randomized algorithm has a competitive ratio better than $H(k)$ (against an obl. adversary)

Self-Adjusting Lists

- Linked lists are often inefficient
  - Cost of accessing an item at position i is linear in i
- But, linked lists are extremely simple
  - And therefore nevertheless interesting

- Can we at least improve the behavior of linked lists?

- In practical applications, not all items are accessed equally often and not equally distributed over time
  - The same items might be used several times over a short period of time

- **Idea:** rearrange list after accesses to optimize the structure for future accesses
- **Problem:** We don't know the future accesses
  - The list rearrangement problems is an online problem!

Model

- Only find operations (i.e., access some item)
  - Let's ignore insert and delete operations
  - Results can be generalized to cover insertions and deletions

Cost Model:

- Accessing item at position i costs i
- The only operation allowed for rearranging the list is swapping two adjacent list items
- Swapping any two adjacent items costs 1

Rearranging The List

Frequency Count (FC):

- For each item keep a count of how many times it was accessed
- Keep items in non-increasing order of these counts
- After accessing an item, increase its count and move it forward past items with smaller count

Move-To-Front (MTF):

- Whenever an item is accessed, move it all the way to the front

Transpose (TR):

- After accessing an item, swap it with its predecessor

Cost

Cost when accessing item at position $\boldsymbol{i}$:

- Frequency Count (FC): between i and $2i - 1$
- Move-To-Front (MTF): $2i - 1$
- Transpose (TR): $i + 1$

Random Accesses:

- If each item x has an access probability p_x and the items are accessed independently at random using these probabilities, FC and TR are asymptotically optimal

Real access patterns are not random, TR usually behaves badly and the much simpler MTF often beats FC

Move-To-Front

- We will see that MTF is competitive
- To analyze MTF we need competitive analysis and amortized analysis

Operation k:

- Assume, the operation accesses item x at position i
- c_k: actual cost of the MTF algorithm

$$c_k = 2i - 1$$

- a_k: amortized cost of the MTF algorithm
- c_k^*: actual cost of an optimal offline strategy
  - Let's call the optimal offline strategy OPT

Potential Function

Potential Function $\boldsymbol{\Phi_k}$:

- **Twice** the number of ***inversions*** between the lists of MTF and OPT after the first k operations
- Measure for the difference between the lists after k operations
- **Inversion:** pair of items x and y such that x precedes y in one list and y precedes x in the other list

Initially, the two lists are identical: $\boldsymbol{\Phi_0 = 0}$

For all k, it holds that $\boldsymbol{0 \leq \Phi_k \leq 2 \cdot \binom{n}{2} = n(n-1)}$

To show that MTF is α-competitive, we need to show that

$$\forall \boldsymbol{k}:\ \ \boldsymbol{a_k = c_k + \Phi_k - \Phi_{k-1} \leq \alpha \cdot c_k^*}$$

Competitive Analysis

Theorem: MTF is **4-competitive**.

Proof:

- Need that $a_k = c_k + \Phi_k - \Phi_{k-1} \leq 4c_k^*$

- Position of x in list of OPT: i^*
- Number of swaps of OPT: s^*

- In MTF list, position of x is changed w.r.t. to the $i - 1$ preceding items (nothing else is changed)
- For each of these items, either an inversion is created or one is destroyed (before the s^* swaps of OPT)
- Number of new inversions (before OPT's swaps) $\leq i^* - 1$:
  - Before op. k, only $i^* - 1$ items are before x in OPT's list
  - With all other items, x is ordered the same as in OPT's list after moving it to the front

Competitive Analysis

Theorem: MTF is **4-competitive**.

Proof:

- Need that $a_k = c_k + \Phi_k - \Phi_{k-1} \leq 4c_k^*$
- $c_k = 2i - 1, \quad c_k^* = i^* + s^*$
- Number of inversions created: $\leq i^* - 1 + s^*$
- Number of inversions destroyed: $\geq i - i^*$

Competitive Analysis

Theorem: MTF is **4-competitive**.

Proof:

- Need that $a_k = c_k + \Phi_k - \Phi_{k-1} \le 4c_k^*$
- $c_k = 2i - 1, \quad c_k^* = i^* + s^*$
- Number of inversions created: $\le i^* - 1 + s^*$
- Number of inversions destroyed: $\ge i - i^*$