Chapter 3
Dynamic Programming

Algorithm Theory
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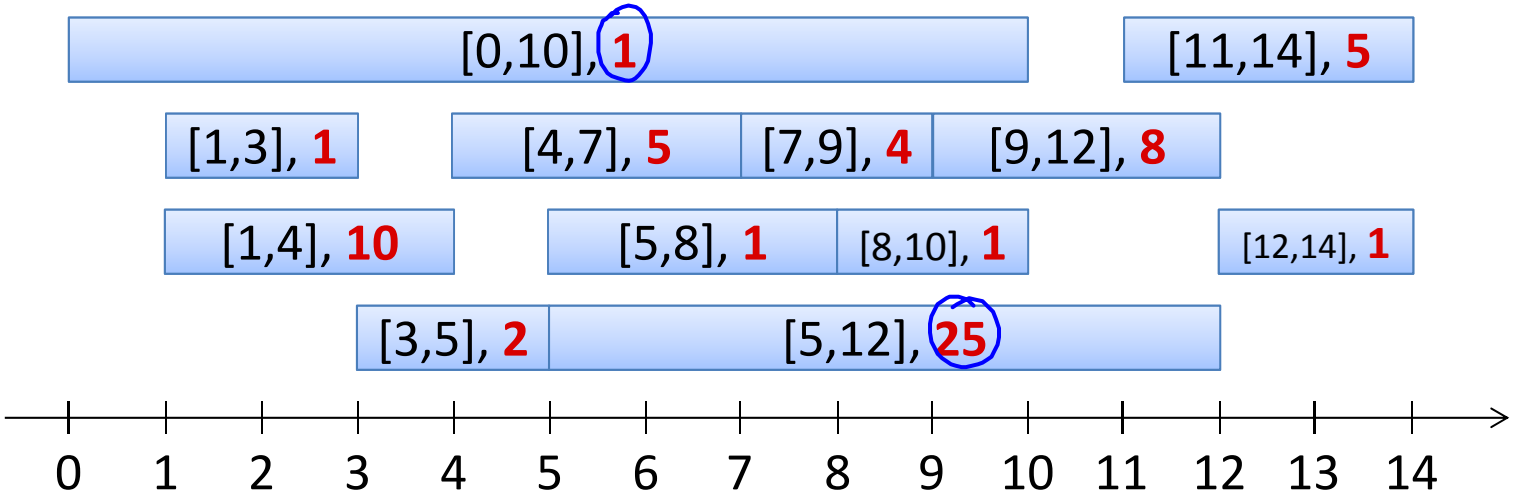
Weighted Interval Scheduling

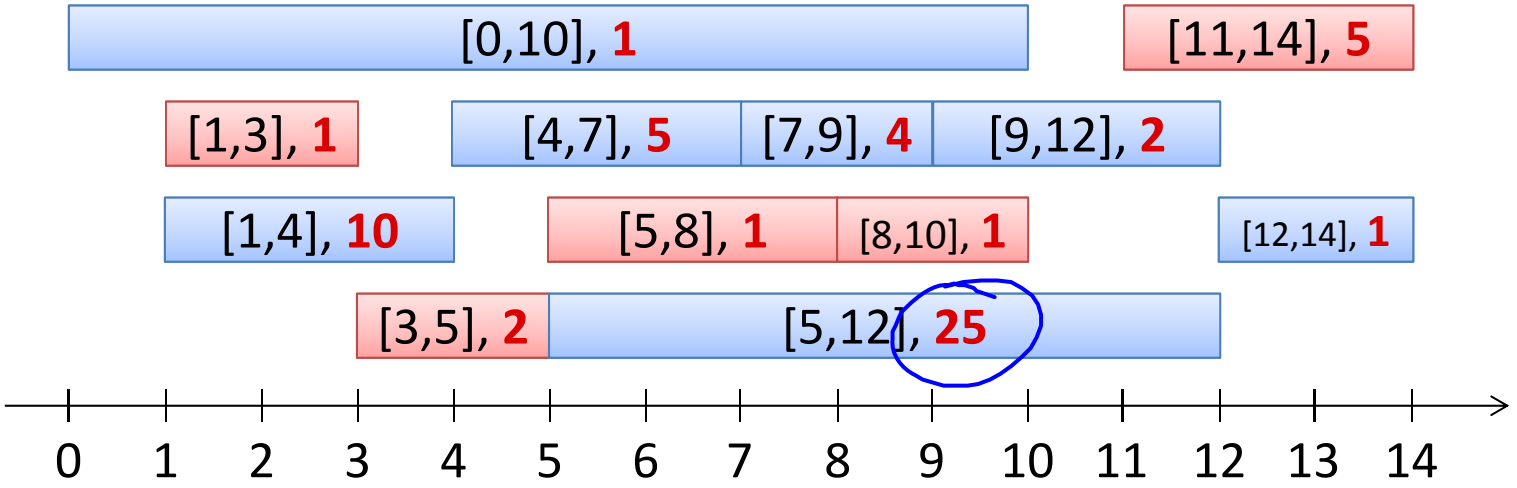
- **Given:** Set of intervals, e.g.
  [0,10],[1,3],[1,4],[3,5],[4,7],[5,8],[5,12],[7,9],[9,12],[8,10],[11,14],[12,14]
- Each interval has a **weight w**



- **Goal:** Non-overlapping set of intervals of largest possible weight
  - Overlap at boundary ok, i.e., [4,7] and [7,9] are non-overlapping
- **Example:** Intervals are room requests of different importance

Greedy Algorithms

Choose available request with earliest finishing time:



- Algorithm is not optimal any more
  - It can even be arbitrarily bad…
- No greedy algorithm known that works

Solving Weighted Interval Scheduling

- Interval i: start time $s(i)$, finishing time: $f(i)$, weight: $w(i)$
- Assume intervals $1, \dots, n$ are sorted by increasing $f(i)$
  - $0 < f(1) \leq f(2) \leq \cdots \leq f(n)$, for convenience: $f(0) = 0$
- Simple observation:

  Opt. solution contains interval n or it doesn't contain interval n

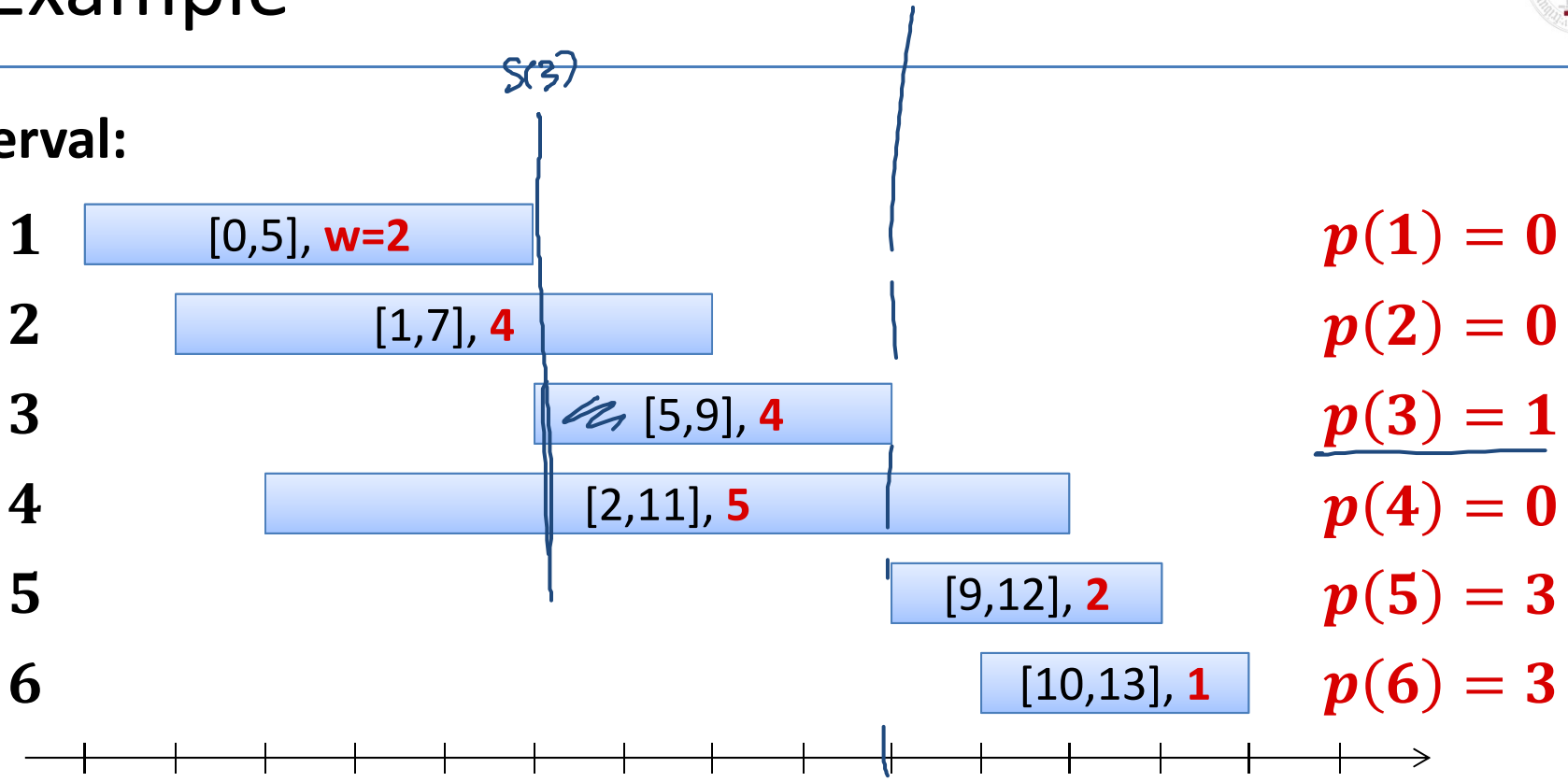
  $W(n)$: weight of an opt. sol.
- Weight of optimal solution for only intervals $1, \dots, k$: $W(k)$

  Define $p(k) := \max\{i \in \{0, \dots, k-1\} : f(i) \leq s(k)\}$

  $p(k) = k-2$
- Opt. solution does not contain interval n: $\boldsymbol{W(n) = W(n-1)}$

  Opt. solution contains interval n: $\boldsymbol{W(n) = w(n) + W(p(n))}$

Example


s(3)
Interval:
1
[0,5], w=2
2
[1,7], 4
3
[5,9], 4
4
[2,11], 5
5
[9,12], 2
6
[10,13], 1
$p(1) = 0$
$p(2) = 0$
$p(3) = 1$
$p(4) = 0$
$p(5) = 3$
$p(6) = 3$

Recursive Definition of Optimal Solution

- Recall:
  - $W(k)$: weight of optimal solution with intervals $1, \dots, k$
  - $p(k)$: last interval to finish before interval k starts

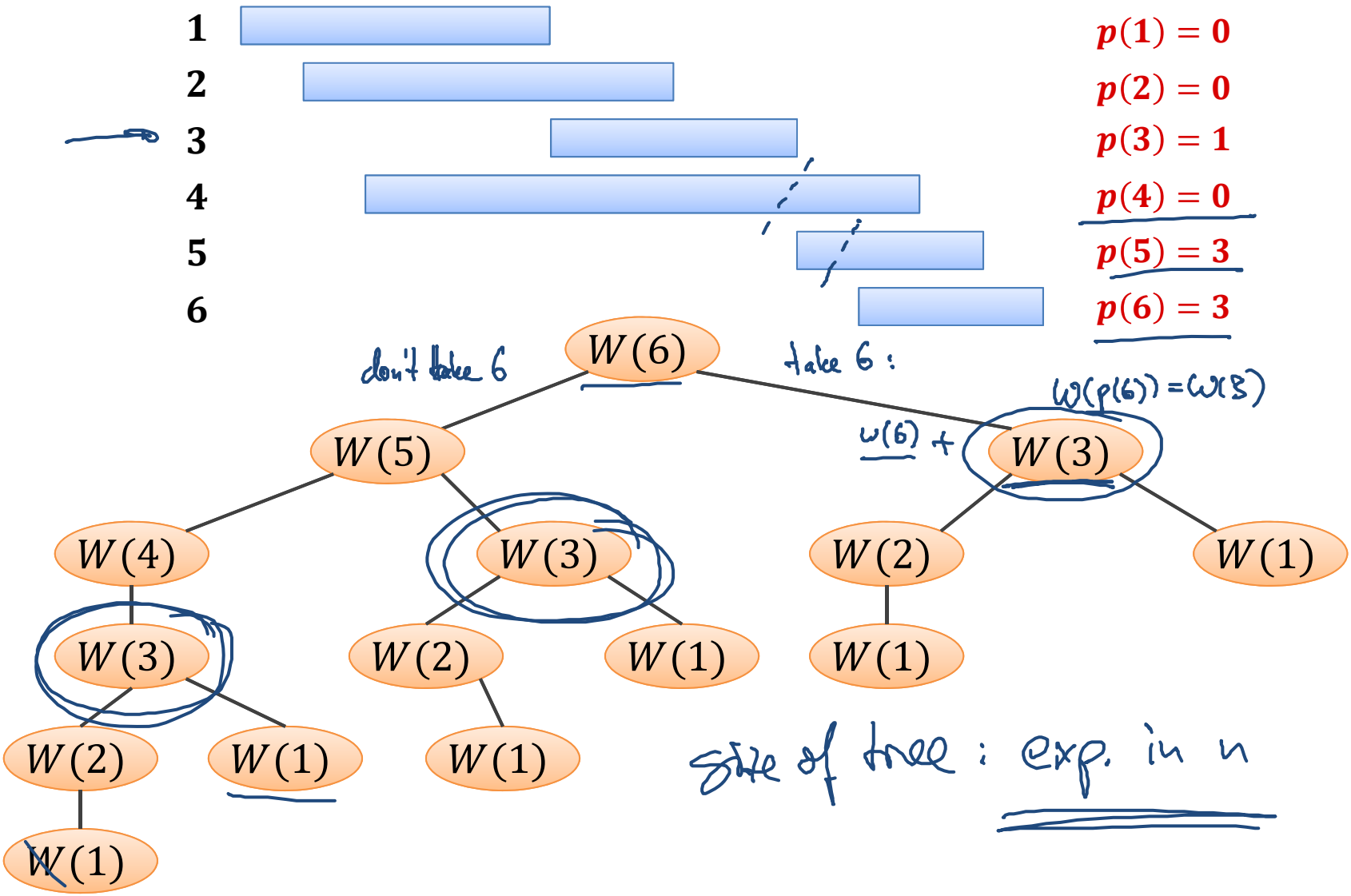
- Recursive definition of optimal weight:

$$\forall k > 1\colon\ W(k) = \max\{W(k-1), w(k) + W(p(k))\}$$

$$W(1) = w(1)$$

- Immediately gives a simple, recursive algorithm

Running Time of Recursive Algorithm



Memoizing the Recursion

- Running time of recursive algorithm: exponential!
- But, alg. only solves n different sub-problems: $W(1), \dots, W(n)$
- There is no need to compute them multiple times

Memoization:

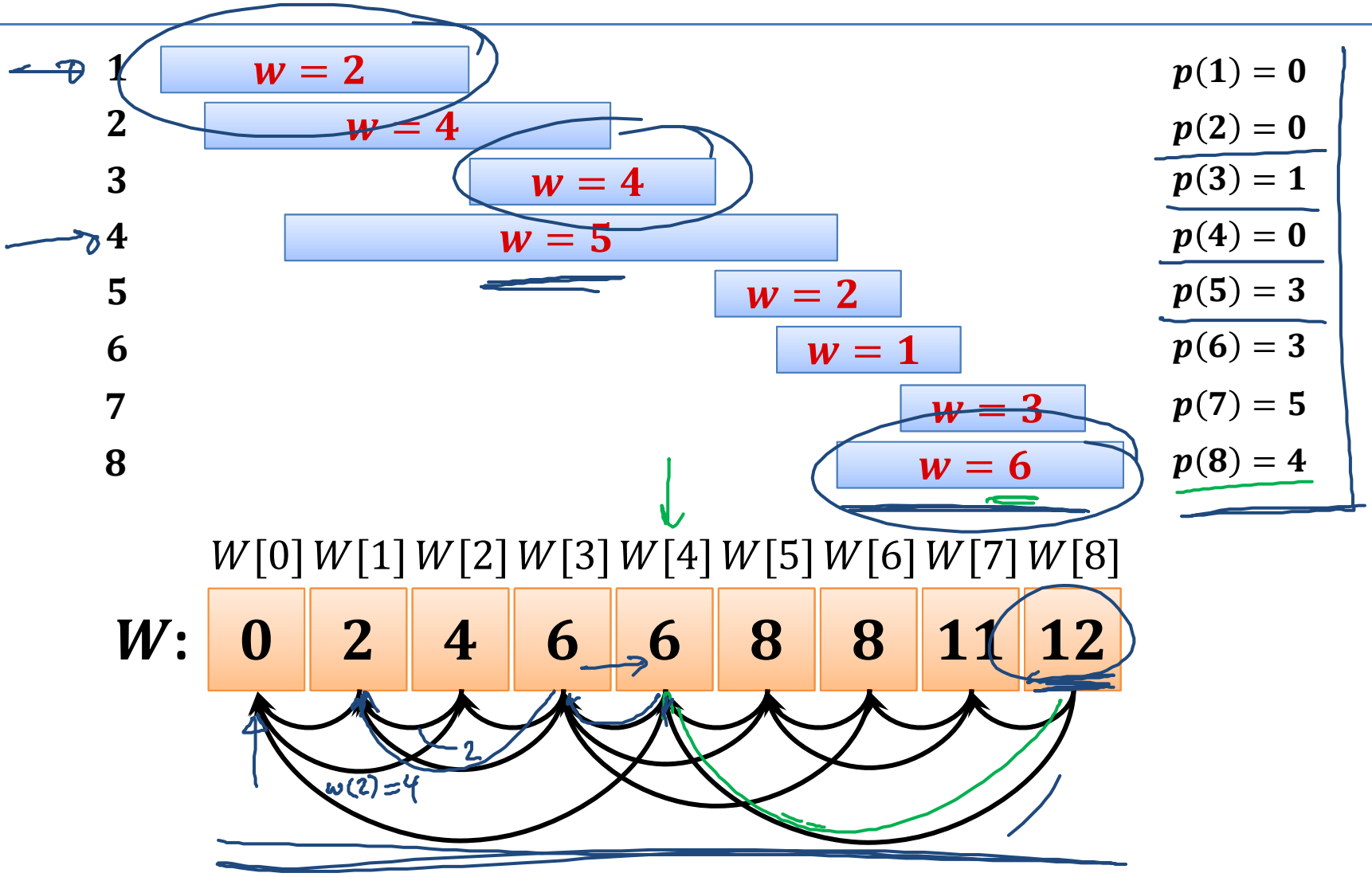
- **Store already computed values** for future use (recursive calls)

Efficient algorithm:

1. $W[0] := 0$; compute values $p(i)$
2. **for** $i := 1$ **to** n **do**
3. $\quad W[i] := \max\{W[i-1], w(i) + W[p(i)]\}$
4. **end**

$O(n)$ time

Example



Computing the schedule: **store where you come from!**

Matrix-chain multiplication

Given: sequence (chain) $\langle A_1, A_2, \dots, A_n \rangle$ of matrices

Goal: compute the product $A_1 \cdot A_2 \cdot \dots \cdot A_n$

Problem: Parenthesize the product in a way that minimizes the number of scalar multiplications.

Definition: A product of matrices is *fully parenthesized* if it is

- a single matrix
- or the product of two fully parenthesized matrix products, surrounded by parentheses.

Example

All possible fully parenthesized matrix products of the chain $\langle A_1, A_2, A_3, A_4 \rangle$:

$$(A_1 (A_2 (A_3 A_4)))$$

$$(A_1 ((A_2 A_3) A_4))$$

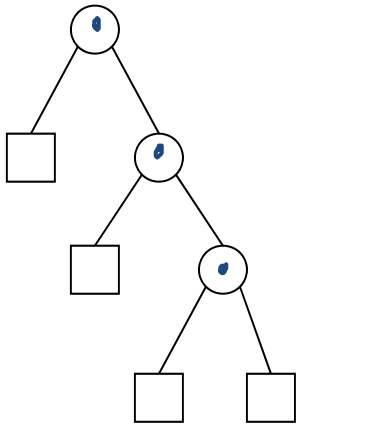
$$((A_1 A_2) (A_3 A_4))$$

$$((A_1 (A_2 A_3)) A_4)$$

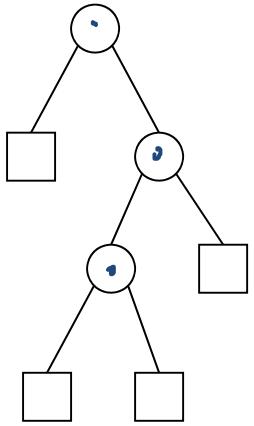
$$(((A_1 A_2) A_3) A_4)$$

Different parenthesizations

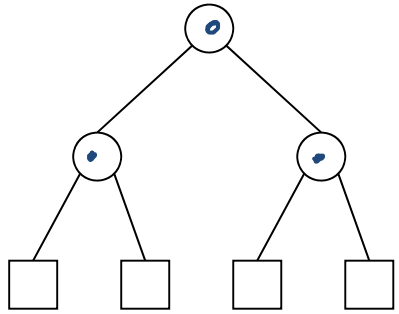
Different parenthesizations correspond to different trees:



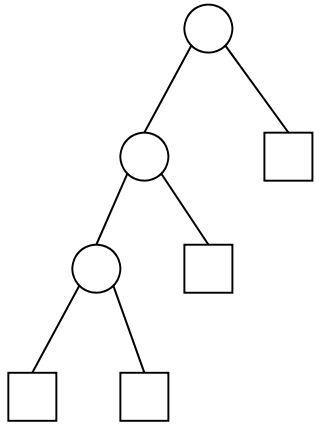
$(A_1(A_2(A_3A_4)))$



$\big(A_1\big((A_2A_3)A_4\big)\big)$



$\big((A_1A_2)(A_3A_4)\big)$



$\Big(\big((A_1A_2)A_3\big)A_4\Big)$

Number of different parenthesizations

- Let $P(n)$ be the number of alternative parenthesizations of the product $A_1 \cdot \ldots \cdot A_n$:

$$P(1) = 1$$

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k), \qquad \text{for } n \geq 2$$

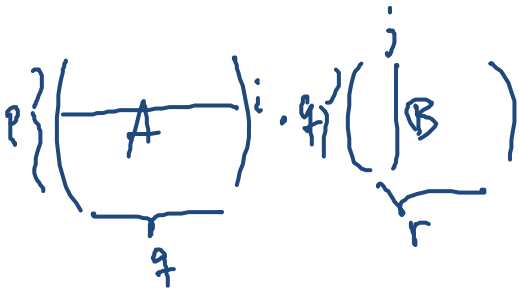
$$P(n+1) = \frac{1}{n+1}\binom{2n}{n} \approx \frac{4^n}{n\sqrt{\pi n}} + O\left(\frac{4^n}{\sqrt{n^5}}\right)$$

$$P(n+1) = C_n \qquad (n^{th} \text{ Catalan number})$$

- Thus: Exhaustive search needs exponential time!

Multiplying Two Matrices

$$A = (a_{ij})_{p\times q}, \qquad B = (b_{ij})_{q\times r}, \qquad A \cdot B = C = (c_{ij})_{p\times r}$$



$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}$$

Algorithm *Matrix-Mult*

Input: $(p \times q)$ matrix A, $(q \times r)$ matrix B

Output: $(p \times r)$ matrix $C = A \cdot B$

```
1 for i := 1 to p do
2     for j := 1 to r do
3         C[i,j] := 0;
4         for k := 1 to q do
5             C[i,j] := C[i,j] + A[i,k] · B[k,j]
```

Remark:

Using this algorithm, multiplying two $(n \times n)$ matrices requires n^3 multiplications. This can also be done using $O(n^{2.376})$ multiplications.

Number of multiplications and additions: $\boldsymbol{p \cdot q \cdot r}$

Matrix-chain multiplication: Example

Computation of the product $A_1 A_2 A_3$, where

A_1 : (50×5) matrix

A_2 : (5×100) matrix

A_3 : (100×10) matrix

a) Parenthesization $((A_1 A_2)A_3)$ and $(A_1(A_2 A_3))$ require:

$A' = (A_1 A_2)$: $5 \cdot 50 \cdot 100 = 25000$ (50×100) $A'' = (A_2 A_3)$: $5 \cdot 100 \cdot 10 = 5000$ (5×10)

$A'A_3$: $50 \cdot 100 \cdot 10 = 50000$ $A_1 A''$: $50 \cdot 5 \cdot 10 = 2500$

Sum: 75'000 7'500

Structure of an Optimal Parenthesization

- $(A_{\ell \dots r})$: optimal parenthesization of $A_\ell \cdot \dots \cdot A_r$

  For some $1 \leq k < n$: $\boldsymbol{(A_{1\dots n}) = ((A_{1\dots k}) \cdot (A_{k+1\dots n}))}$

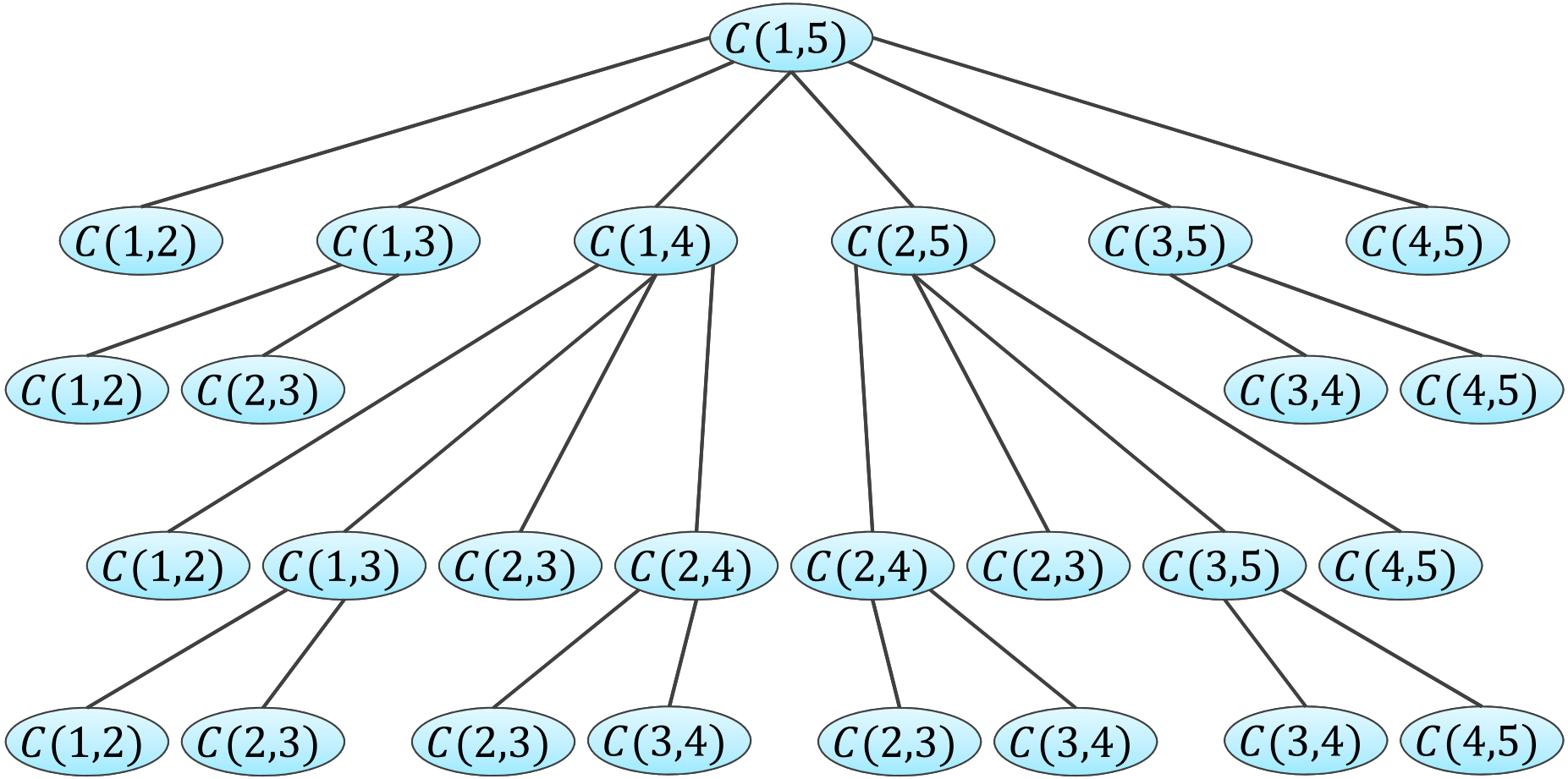
- Any optimal solution contains optimal solutions for sub-problems
- Assume matrix A_i is a $(d_{i-1} \times d_i)$-matrix
- Cost to solve sub-problem $A_\ell \cdot \dots \cdot A_r$, $\ell \leq r$ optimally: $C(\ell, r)$
- Then:

$$\boldsymbol{C(a,b) = \min_{a \leq k < b} C(a,k) + C(k+1,b) + d_{a-1} d_k d_b}$$

$$\boldsymbol{C(a,a) = 0}$$

Recursive Computation of Opt. Solution

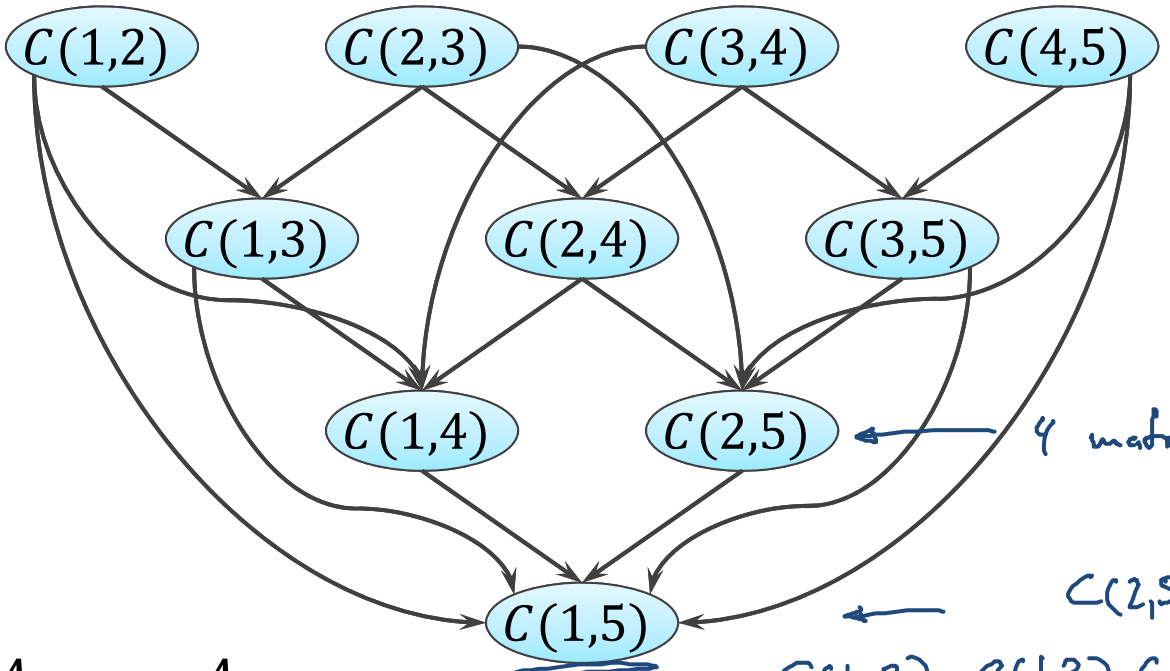
Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:



$C(i,j)$ $\binom{n}{2} = \Theta(n^2)$

Using Meomization

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:



Compute $A_1 \cdot \ldots \cdot A_n$:

- Each $C(i,j)$, $i < j$ is computed exactly once → $O(n^2)$ values
- Each $C(i,j)$ dir. depends on $C(i,k)$, $C(k,j)$ for $i < k < j$

  Cost for each $C(i,j)$: $O(n)$ → overall time: $\boldsymbol{O(n^3)}$

Dynamic Programming

„Memoization“ for increasing the efficiency of a recursive solution:

- Only the *first time* a sub-problem is encountered, its solution is computed and then stored in a table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned
  (without repeated computation!).

- *Computing the solution*: For each sub-problem, store how the value is obtained (according to which recursive rule).

Dynamic Programming

Dynamic programming / memoization can be applied if

- Optimal solution contains optimal solutions to sub-problems (recursive structure)
- Number of sub-problems that need to be considered is small