Chapter 1
Divide and Conquer

Polynomial Multiplication

Algorithm Theory
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Representation of Polynomials

Coefficient representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree n is given by its $n+1$ coefficients $a_0, \dots, a_n$:

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

- Example:

$$p(x) = 3x^3 - 15x^2 + 18x$$

- The most typical (and probably most natural) representation of polynomials

Representation of Polynomials

Point-value representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree n is given by $n+1$ point-value pairs:

$$p = \{(x_0, p(x_0)), (x_1, p(x_1)), \dots, (x_n, p(x_n))\}$$

where $x_i \neq x_j$ for $i \neq j$.

- Example: The polynomial

$$p(x) = 3x(x-2)(x-3)$$

is uniquely defined by the four point-value pairs $(0,0), (1,6), (2,0), (3,0)$.

Operations: Coefficient Representation

Deg.-n polynomials $p(x) = a_n x^n + \cdots + a_0$, $q(x) = b_n x^n + \cdots + b_0$

Addition:

$$p(x) + q(x) = (a_n + b_n)x^n + \cdots + (a_0 + b_0)$$

- Time: $O(n)$

Multiplication:

$$p(x) \cdot q(x) = c_{2n}x^{2n} + \cdots + c_0, \qquad \text{where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

- Naive solution: Need to compute product $a_i b_j$ for all $0 \leq i, j \leq n$
- Time: $O(n^2)$ $\rightsquigarrow$ $O(n^{\log_2 3})$

Operations Point-Value Representation

Degree-n polynomials

$$p = \{(x_0, p(x_0)), \dots, (x_n, p(x_n))\}, q = \{(x_0, q(x_0)), \dots, (x_n, q(x_n))\}$$

- Note: we use the same points $x_0, \dots, x_n$ for both polynomials

Addition:

$$p + q = \{(x_0, p(x_0) + q(x_0)), \dots, (x_n, p(x_n) + q(x_n))\}$$

- Time: $O(n)$

Multiplication:

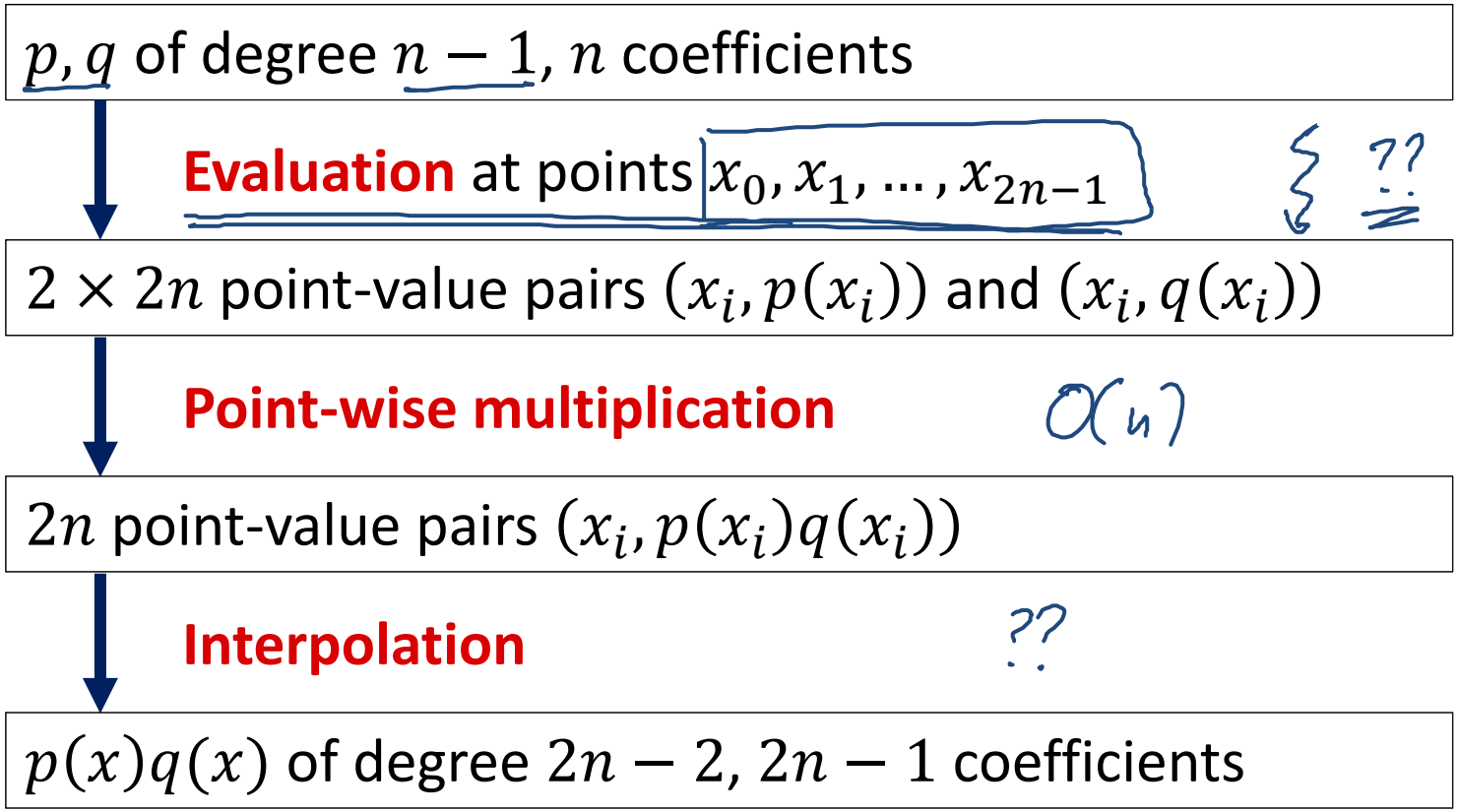
$$p \cdot q = \{(x_0, p(x_0) \cdot q(x_0)), \dots, (x_n, p(x_n) \cdot q(x_n))\}$$

- Time: $O(n)$

Faster Polynomial Multiplication?

Multiplication is fast when using the point-value representation

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):



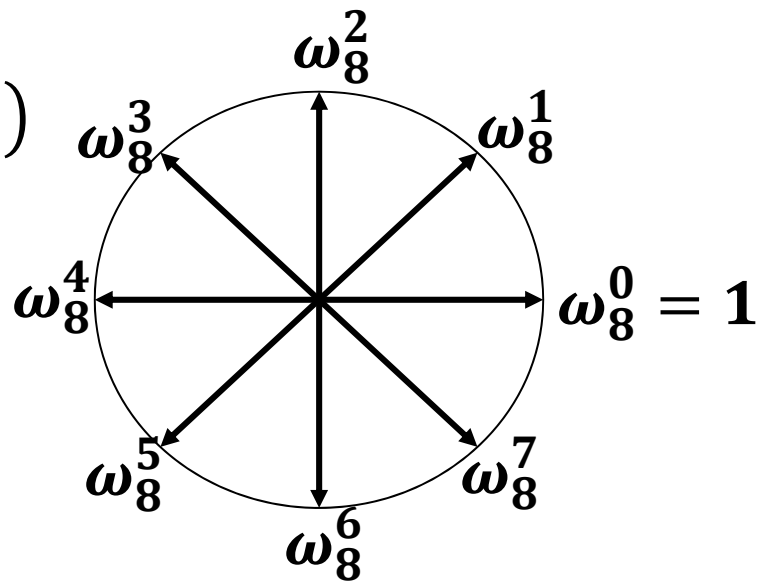
Point-Value Representation of p, q

- Select points $x_0, x_1, \dots, x_{N-1}$ to evaluate p and q in a clever way

Consider the N powers of the principle Nth root of unity:

Principle root of unity: $\omega_N = e^{2\pi i/N}$

$$\left(i = \sqrt{-1}, \qquad e^{2\pi i} = 1\right)$$



Powers of ω_n (roots of unity):

$$1 = \omega_N^0, \omega_N^1, \dots, \omega_N^{N-1}$$

Note: $\omega_N^k = e^{2\pi ik/N} = \cos\frac{2\pi k}{N} + i \cdot \sin\frac{2\pi k}{N}$

Discrete Fourier Transform

- The values $p\left(\omega_N^i\right)$ for $i = 0, \dots, N-1$ uniquely define a polynomial p of degree $< N$.

Discrete Fourier Transform (DFT):

- Assume $a = (a_0, \dots, a_{N-1})$ is the coefficient vector of poly. p

$$(p(x) = a_{N-1}x^{N-1} + \cdots + a_1 x + a_0)$$

$$\mathrm{DFT}_N(a) := \left(p\left(\omega_N^0\right), p(\omega_N^1), \dots, p(\omega_N^{N-1})\right)$$

coeff. vector — values of $p(\cdot)$ at the N^{th} roots of 1.

Faster Polynomial Multiplication?

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

compute DFTs of the coeff. vectors

Evaluation at points $\omega_{2n}^0, \omega_{2n}^1, \dots, \omega_{2n}^{2n-1}$

$2 \times 2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)\right)$ and $\left(\omega_{2n}^k, q\left(\omega_{2n}^k\right)\right)$

Point-wise multiplication

$2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right) q\left(\omega_{2n}^k\right)\right)$

Interpolation

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

Properties of the Roots of Unity

- **Cancellation Lemma:**

  For all integers $n > 0$, $k \geq 0$, and $d > 0$, we have:

$$\omega_{dn}^{dk} = \omega_n^k, \qquad \omega_n^{k+n} = \omega_n^k$$

- **Proof:**

Divide-and-Conquer Approach

- Divide $p(x)$ of degree $N-1$ (N is even) into 2 polynomials of degree $N/2 - 1$ differently than in Karatsuba's algorithm

- $p_0(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{N-2} x^{N/2-1}$ (even coeff.)
  $p_1(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{N-1} x^{N/2-1}$ (odd coeff.)

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{N-1} x^{N-1}$$

$$p(x) = a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_{N-2} x^{N-2}$$
$$+ a_1 x + a_3 x^3 + a_5 x^5 + \ldots + a_{N-1} x^{N-1}$$

$$\boxed{p(x) = p_0(x^2) + x \cdot p_1(x^2)}$$

Discrete Fourier Transform

Evaluation for $k = 0, \dots, N-1$:

$$p(\omega_N^k) = p_0\big((\omega_N^k)^2\big) + \omega_N^k \cdot p_1\big((\omega_N^k)^2\big)$$

$$= \begin{cases} p_0\big(\omega_{N/2}^k\big) + \omega_N^k \cdot p_1\big(\omega_{N/2}^k\big) & \text{if } k < N/2 \\ p_0\big(\omega_{N/2}^{k-N/2}\big) + \omega_N^k \cdot p_1\big(\omega_{N/2}^{k-N/2}\big) & \text{if } k \geq N/2 \end{cases}$$

For the coefficient vector a of $p(x)$:

$$\mathrm{DFT}_N(a) = \Big(p_0\big(\omega_{N/2}^0\big), \dots, p_0\big(\omega_{N/2}^{N/2-1}\big), p_0\big(\omega_{N/2}^0\big), \dots, p_0\big(\omega_{N/2}^{N/2-1}\big)\Big)$$
$$+ \Big(\omega_N^0 p_0\big(\omega_{N/2}^0\big), \dots, \omega_N^{N/2-1} p_0\big(\omega_{N/2}^{N/2-1}\big), \omega_N^{N/2} p_0\big(\omega_{N/2}^0\big), \dots, \omega_N^{N-1} p_0\big(\omega_{N/2}^{N/2-1}\big)\Big)$$

Example

For the coefficient vector a of $p(x)$:

$$\mathrm{DFT}_N(a) = \left(p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right), p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right)\right)$$
$$+\left(\omega_N^0 p_0(\omega_{N/2}^0), \dots, \omega_N^{N/2-1} p_0\left(\omega_{N/2}^{N/2-1}\right), \omega_N^{N/2} p_0(\omega_{N/2}^0), \dots, \omega_N^{N-1} p_0\left(\omega_{N/2}^{N/2-1}\right)\right)$$

$N = 4$:

$$p(\omega_4^0) = p_0(\omega_2^0) + \omega_4^0 p_1(\omega_2^0)$$
$$p(\omega_4^1) = p_0(\omega_2^1) + \omega_4^1 p_1(\omega_2^1)$$
$$p(\omega_4^2) = p_0(\omega_2^0) + \omega_4^2 p_1(\omega_2^0)$$
$$p(\omega_4^3) = p_0(\omega_2^1) + \omega_4^3 p_1(\omega_2^1)$$

Need: $(p_0(\omega_2^0), p_0(\omega_2^1))$ and $(p_1(\omega_2^0), p_1(\omega_2^1))$

(DFTs of coefficient vectors of p_0 and p_1)

Recursive Structure

For simplicity, we abuse notation in the following:

- Poly. $p(x) = a_{N-1}x^{N-1} + \cdots + a_0$ with coefficient vector a

  Let $\mathrm{DFT}_N(p) := \mathrm{DFT}_N(a)$

N is a power of 2

Recursive structure:

- For $N = 4$:

$$\left(\mathrm{DFT}_4(p)\right)_k = p\left(\omega_4^k\right)$$
$$= \left(\mathrm{DFT}_2(p_0)\right)_{k \bmod 2} + \omega_4^k \cdot \left(\mathrm{DFT}_2(p_1)\right)_{k \bmod 2}$$

- General N (assume N is even):

$$\left(\mathrm{DFT}_N(p)\right)_k = p\left(\omega_N^k\right)$$
$$= \left(\mathrm{DFT}_{N/2}(p_0)\right)_{k \bmod N/2} + \omega_N^k \cdot \left(\mathrm{DFT}_{N/2}(p_1)\right)_{k \bmod N/2}$$

for specific k, this can be comp. in $O(1)$ time.

Computation of DFT_N

- Divide-and-conquer algorithm for $\mathrm{DFT}_N(p)$:

1. Divide

$N \leq 1$: $\mathrm{DFT}_1(p) = a_0$

$N > 1$: Divide p into p_0 (even coeff.) and p_1 (odd coeff).

} $O(N)$ time

2. Conquer

Solve $\mathrm{DFT}_{N/2}(p_0)$ and $\mathrm{DFT}_{N/2}(p_1)$ recursively

3. Combine

Compute $\mathrm{DFT}_N(p)$ based on $\mathrm{DFT}_{N/2}(p_0)$ and $\mathrm{DFT}_{N/2}(p_1)$ } $O(N)$ time

Analysis

- $T(N)$: time to compute $\mathrm{DFT}_N(p)$:

$$T(N) = 2T(N/2) + O(N), \qquad T(1) = O(1)$$

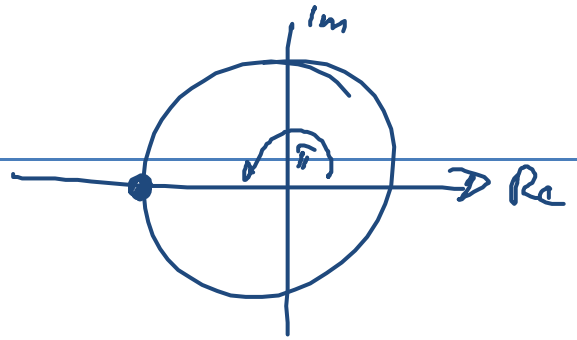
- As for mergesort, comparing orders, closest pair of points:

$$T(N) = O(N \cdot \log N)$$

$\omega_N^{2k} = \omega_{N/2}^{k}$

convert coeff. repr. to point-value repr. requires $O(N \log N)$ time when eval. at points ω_N^k

Small Improvement



Polynomial p of degree $N-1$:

$$p(\omega_N^k) = \begin{cases} p_0(\omega_{N/2}^k) + \omega_N^k \cdot p_1(\omega_{N/2}^k) & \text{if } k < N/2 \\ p_0\left(\omega_{N/2}^{k-N/2}\right) + \omega_N^k \cdot p_1\left(\omega_{N/2}^{k-N/2}\right) & \text{if } k \geq N/2 \end{cases}$$

$$= \begin{cases} p_0(\omega_{N/2}^k) + \omega_N^k \cdot p_1(\omega_{N/2}^k) & \text{if } k < N/2 \\ p_0\left(\omega_{N/2}^{k-N/2}\right) - \omega_N^{k-N/2} \cdot p_1\left(\omega_{N/2}^{k-N/2}\right) & \text{if } k \geq N/2 \end{cases}$$

$$\omega_N^k = e^{\frac{2\pi i}{N}\cdot k} = e^{\frac{2\pi i}{N}\left(k-\frac{N}{2}+\frac{N}{2}\right)} = \omega_N^{k-\frac{N}{2}} \cdot e^{\frac{2\pi i}{N}\cdot\frac{N}{2}} = \omega_N^{k-\frac{N}{2}} \underbrace{e^{\pi i}}_{=-1} = -\omega_N^{k-\frac{N}{2}}$$

Need to compute $p_0(\omega_{N/2}^k)$ and $\omega_N^k \cdot p_1(\omega_{N/2}^k)$ for $0 \leq k < N/2$.

Example

$$p(\omega_4^0) = p_0(\omega_2^0) + \omega_4^0 \cdot p_1(\omega_2^0)$$
$$p(\omega_4^1) = p_0(\omega_2^1) + \omega_4^1 \cdot p_1(\omega_2^1)$$
$$p(\omega_4^2) = p_0(\omega_2^0) - \omega_4^0 \cdot p_1(\omega_2^0)$$
$$p(\omega_4^3) = p_0(\omega_2^1) - \omega_4^1 \cdot p_1(\omega_2^1)$$

Fast Fourier Transform (FFT) Algorithm

Algorithm FFT(a)

- Input: Array a of length N, where N is a power of 2
- Output: $\mathrm{DFT}_N(a)$

if $N = 1$ **then return** a_0; // $a = [a_0]$

$d^{[0]} := \mathrm{FFT}([a_0, a_2, \dots, a_{N-2}])$;

$d^{[1]} := \mathrm{FFT}([a_1, a_3, \dots, a_{N-1}])$;

$\omega_N := e^{2\pi i/N}$; $\omega := 1$;

for $k = 0$ **to** $N/2 - 1$ **do** // $\omega = \omega_N^k$

$x := \omega \cdot d_k^{[1]}$;

$d_k := d_k^{[0]} + x$; $d_{k+N/2} := d_k^{[0]} - x$;

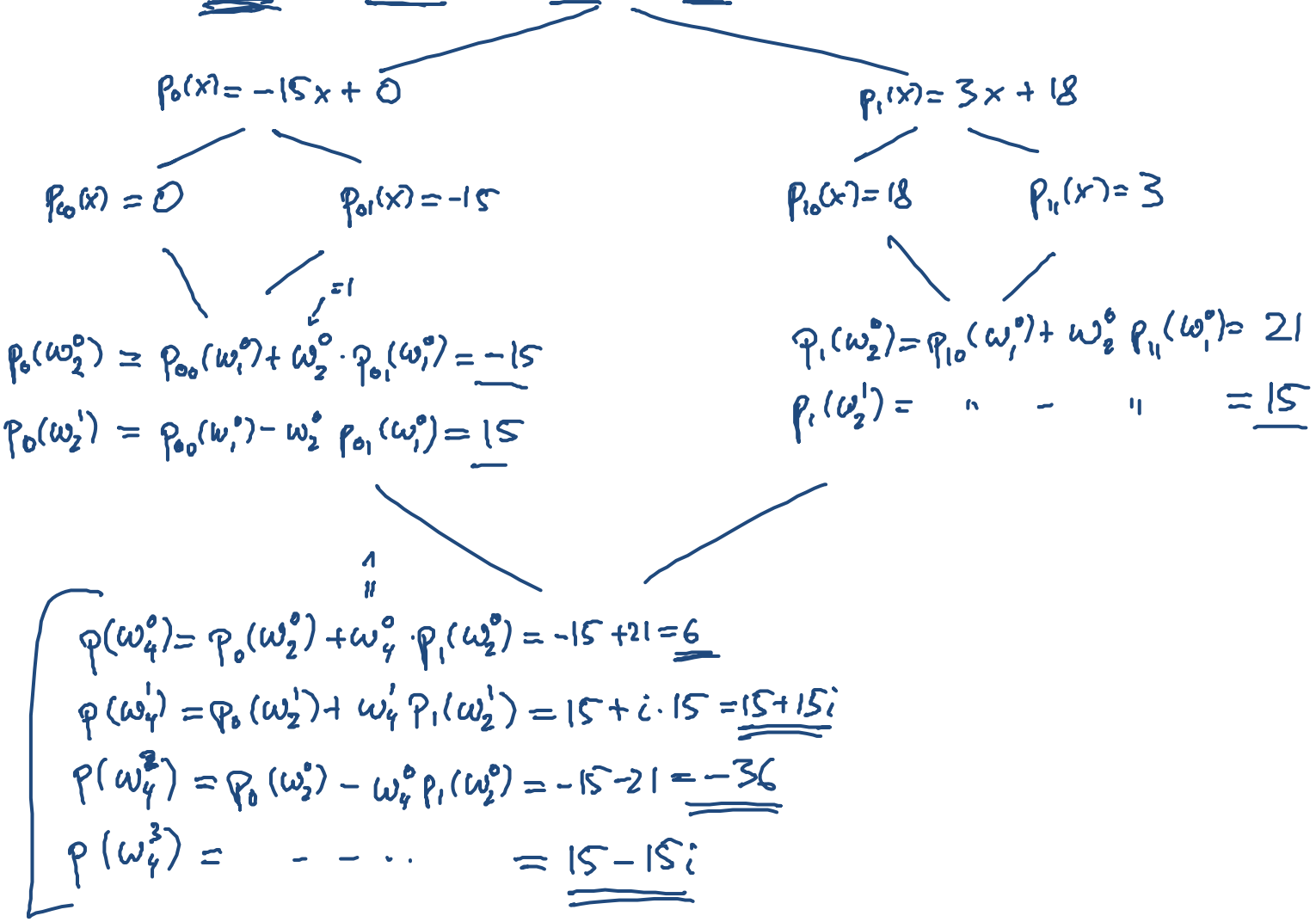
$\omega := \omega \cdot \omega_N$

end;

return $d = [d_0, d_1, \dots, d_{N-1}]$;

Example

- $p(x) = 3x^3 - 15x^2 + 18x + 0$, $a = [0,18,-15,3]$



Faster Polynomial Multiplication?

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

↓ **Evaluation** at $\omega_{2n}^0, \omega_{2n}^1, \dots, \omega_{2n}^{2n-1}$ using **FFT** $N=2n$, $O(n \log n)$

$2 \times 2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)\right)$ and $\left(\omega_{2n}^k, q\left(\omega_{2n}^k\right)\right)$

↓ **Point-wise multiplication** $O(n)$

$2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right) q\left(\omega_{2n}^k\right)\right)$

↓ **Interpolation** ?

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

Interpolation

Convert point-value representation into coefficient representation

Input: $(x_0, y_0), \dots, (x_{n-1}, y_{n-1})$ with $x_i \neq x_j$ for $i \neq j$

Output:

Degree-$(n-1)$ polynomial with coefficients $a_0, \dots, a_{n-1}$ such that

$$\begin{aligned}
p(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_{n-1} x_0^{n-1} = y_0 \\
p(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1} = y_1 \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
p(x_{n-1}) &= a_0 + a_1 x_{n-1} + a_2 x_{n-1}^2 + \cdots + a_{n-1} x_{n-1}^{n-1} = y_{n-1}
\end{aligned}$$

→ linear system of equations for $a_0, \dots, a_{n-1}$

Interpolation

Matrix Notation:

$$\begin{pmatrix} 1 & x_0 & \cdots & x_0^{n-1} \\ 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & \cdots & x_{n-1}^{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

- System of equations solvable iff $x_i \neq x_j$ for all $i \neq j$

Special Case $x_i = \omega_n^i$:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

Interpolation

- Linear system:

$$W \cdot \boldsymbol{a} = \boldsymbol{y} \quad \Longrightarrow \quad \boldsymbol{a} = W^{-1} \cdot \boldsymbol{y}$$

$$W_{i,j} = \omega_n^{ij}, \qquad \boldsymbol{a} = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

Claim:

$$\boldsymbol{W_{ij}^{-1} = \frac{\omega_n^{-ij}}{n}}$$

Proof: Need to show that $W^{-1}W = I_n$

DFT Matrix Inverse

$$(W^{-1})_{i,j} = \frac{1}{n} \cdot \omega_n^{-ij}$$

$$W^{-1}W = \begin{pmatrix} \frac{1}{n} & \frac{\omega_n^{-i}}{n} & \cdots & \frac{\omega_n^{-(n-1)i}}{n} \\ & & \vdots & \\ & & \cdots & \end{pmatrix} \cdot \begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & \omega_n^{j} & \cdots \\ \cdots & \omega_n^{2j} & \cdots \\ & \vdots & \\ \cdots & \omega_n^{(n-1)j} & \cdots \end{pmatrix}$$

$$(W^{-1}W)_{i,j} = \frac{1}{n} \sum_{\ell=0}^{n-1} \omega_n^{-i\cdot\ell} \cdot \omega_n^{j\cdot\ell} = \frac{1}{n} \cdot \sum_{\ell=0}^{n-1} \omega^{\ell(j-i)}$$

DFT Matrix Inverse

$$(W^{-1}W)_{i,j} = \sum_{\ell=0}^{n-1} \frac{\omega_n^{\ell(j-i)}}{n}$$

Need to show $(W^{-1}W)_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Case $i = j$:

$$(W^{-1}W)_{i,i} = \sum_{\ell=0}^{n-1} \frac{1}{n} \cdot \omega_n^{\ell \cdot 0} = n \cdot \frac{1}{n} = 1$$

DFT Matrix Inverse

$$(W^{-1}W)_{i,j} = \sum_{\ell=0}^{n-1} \frac{\omega_n^{\ell(j-i)}}{n}$$

Case $i \neq j$:

$i, j \in \{0, \dots, n-1\}$

$$(W^{-1}W)_{i,j} = \frac{1}{n}\sum_{\ell=0}^{n-1} \left(\omega_n^{j-i}\right)^{\ell} = \frac{1}{n}\,\frac{\omega_n^{(j-i)n} - 1}{\omega_n^{j-i} - 1} = \frac{1}{n}\,\frac{0}{\neq 0} = 0$$

fixed

geometric series

$\omega_n^{j-i} - 1 \neq 0$

$$\left(\omega_n^k\right)^n = 1$$

$x^n = 1$

$$\sum_{\ell=0}^{n-1} q^{\ell} = \frac{q^n - 1}{q - 1}$$

Inverse DFT

- $W^{-1} = \begin{pmatrix} \frac{1}{n} & \frac{\omega_n^{-k}}{n} & \cdots & \frac{\omega_n^{-(n-1)k}}{n} \\ & & \vdots & \\ & & \cdots & \end{pmatrix}$
- We get $\boldsymbol{a} = W^{-1} \cdot \boldsymbol{y}$ and therefore

$$a_k = \left(\frac{1}{n} \quad \frac{\omega_n^{-k}}{n} \quad \cdots \quad \frac{\omega_n^{-(n-1)k}}{n}\right) \cdot \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

$$= \frac{1}{n} \cdot \sum_{j=0}^{n-1} \omega_n^{-kj} \cdot y_j$$

ω_n^{-k}

DFT and Inverse DFT

Inverse DFT:

$$a_k = \frac{1}{n} \cdot \sum_{j=0}^{n-1} \omega_n^{-kj} \cdot y_j$$

- Define polynomial $q(x) = y_0 + y_1 x + \cdots + y_{n-1} x^{n-1}$:

$$a_k = \frac{1}{n} \cdot q(\omega_n^{-k})$$

DFT:

- Polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$:

$$y_k = p(\omega_n^k)$$

DFT and Inverse DFT

$$q(x) = y_0 + y_1 x + \cdots + y_{n-1} x^{n-1}, \qquad a_k = \frac{1}{n} \cdot q\left(\omega_n^{-k}\right):$$

- Therefore:

$$(a_0, a_1, \dots, a_{n-1})$$

$$= \frac{1}{n} \cdot \left(q(\omega_n^{-0}), q(\omega_n^{-1}), q(\omega_n^{-2}), \dots, q\left(\omega_n^{-(n-1)}\right)\right)$$

$$= \frac{1}{n} \cdot \left(q(\omega_n^{0}), q(\omega_n^{n-1}), q(\omega_n^{n-2}), \dots, q(\omega_n^{1})\right)$$

- Recall:

$$\mathrm{DFT}_n(\boldsymbol{y}) = \left(q(\omega_n^{0}), q(\omega_n^{1}), q(\omega_n^{2}), \dots, q(\omega_n^{n-1})\right)$$

$$= n \cdot (a_0, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$$

DFT and Inverse DFT

- We have $\mathrm{DFT}_n(\boldsymbol{y}) = n \cdot (a_0, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$:

$$a_i = \begin{cases} \frac{1}{n} \cdot (\mathrm{DFT}_n(\boldsymbol{y}))_0 & \text{if } i = 0 \\ \frac{1}{n} \cdot (\mathrm{DFT}_n(\boldsymbol{y}))_{n-i} & \text{if } i \neq 0 \end{cases}$$

- DFT and inverse DFT can both be computed using FFT algorithm in $O(n \log n)$ time.

- 2 polynomials of degr. $< n$ can be multiplied in time $O(n \log n)$.

DFT_{2n}

Faster Polynomial Multiplication?

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

↓ **Evaluation** at $\omega_{2n}^0, \omega_{2n}^1, \dots, \omega_{2n}^{2n-1}$ using **FFT** $O(n \log n)$

$2 \times 2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)\right)$ and $\left(\omega_{2n}^k, q\left(\omega_{2n}^k\right)\right)$

↓ **Point-wise multiplication** $O(n)$

$2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right) q\left(\omega_{2n}^k\right)\right)$

↓ **Interpolation** using **FFT** $O(n \log n)$

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

$(p(x))^5$ — degree $n-1$

$(\omega_N^k, p^5(\omega_N^k))$, $N \geq 5n-4$

Convolution

- More generally, the polynomial multiplication algorithm computes the convolution of two vectors:

$$\boldsymbol{a} = (a_0, a_1, \dots, a_{m-1})$$
$$\boldsymbol{b} = (b_0, b_1, \dots, b_{n-1})$$

$$\boldsymbol{a} * \boldsymbol{b} = (c_0, c_1, \dots, c_{m+n-2}),$$
$$\text{where } c_k = \sum_{\substack{(i,j): i+j=k \\ i<m, j<n}} a_i b_j$$

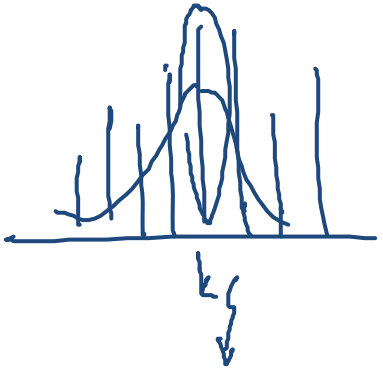
- c_k is exactly the coefficient of x^k in the product polynomial of the polynomials defined by the coefficient vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

More Applications of Convolutions

Signal Processing Example:

- Assume $\boldsymbol{a} = (a_0, \dots, a_{n-1})$ represents a sequence of measurements over time
- Measurements might be noise and have to be smoothed out
- Replace a_i by weighted average of nearby last m and next m measurements (e.g., Gaussian smoothing):

$$a_i' = \frac{1}{Z} \cdot \sum_{j=i-m}^{i+m} a_j e^{-(i-j)^2}$$



- New vector $\boldsymbol{a}'$ is the convolution of $\boldsymbol{a}$ and the weight vector

$$\frac{1}{Z} \cdot \left(e^{-m^2}, e^{-(m-1)^2}, \dots, e^{-1}, 1, e^{-1}, \dots, e^{-(m-1)^2}, e^{-m^2}\right)$$

- Might need to take care of boundary points…

More Applications of Convolutions

Combining Histograms:

- Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ represent two histograms
- E.g., annual income of all men & annual income of all women

- Goal: Get new histogram $\boldsymbol{c}$ representing combined income of all possible pairs of men and women:

$$\boldsymbol{c = a * b}$$

Also, the DFT (and thus the FFT alg.) has many other applications!