Chapter 2
Greedy Algorithms

Algorithm Theory
WS 2014/15

Fabian Kuhn

Exam:

Feb. 23, 9am - 10:30am

Matroids

- Same, but more abstract…

Matroid: **pair $(\boldsymbol{E}, \boldsymbol{I})$**

- $\boldsymbol{E}$: set, called the **ground set**
- $\boldsymbol{I}$: finite family of finite subsets of E (i.e., $I \subseteq 2^E$), called **independent sets**

(E, I) needs to satisfy 3 properties:

1. Empty set is independent, i.e., $\emptyset \in \boldsymbol{I}$ (implies that $I \neq \emptyset$)
2. **Hereditary property**: For all $A \subseteq E$ and all $A' \subseteq A$,

   if $\boldsymbol{A} \in \boldsymbol{I}$, then also $\boldsymbol{A'} \in \boldsymbol{I}$
3. **Augmentation / Independent set exchange property:** If $A, B \in I$ and $|A| > |B|$, there exists $x \in A \setminus B$ such that

   $$\mathbf{B}' := \boldsymbol{B} \cup \{\boldsymbol{x}\} \in \boldsymbol{I}$$

Matroids and Greedy Algorithms

Weighted matroid: each $e \in E$ has a weight $w(e) > 0$

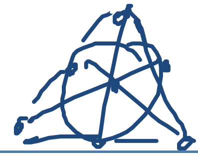
Goal: find maximum weight independent set

Greedy algorithm:

1. Start with $S = \emptyset$
2. Add max. weight $e \in E \setminus S$ to S such that $S \cup \{e\} \in I$

Claim: greedy algorithm computes optimal solution

Matroids: Examples



Forests of a graph $G = (V, E)$:

- forest F: subgraph with no cycles (i.e., $F \subseteq E$)
- $\mathcal{F}$: set of all forests → $(E, \mathcal{F})$ is a matroid
- Greedy algorithm gives maximum weight forest (equivalent to MST problem)

Bicircular matroid of a graph $G = (V, E)$:

- $\mathcal{B}$: set of edges such that every connected subset has ≤ 1 cycle
- $(E, \mathcal{B})$ is a matroid → greedy gives max. weight such subgraph

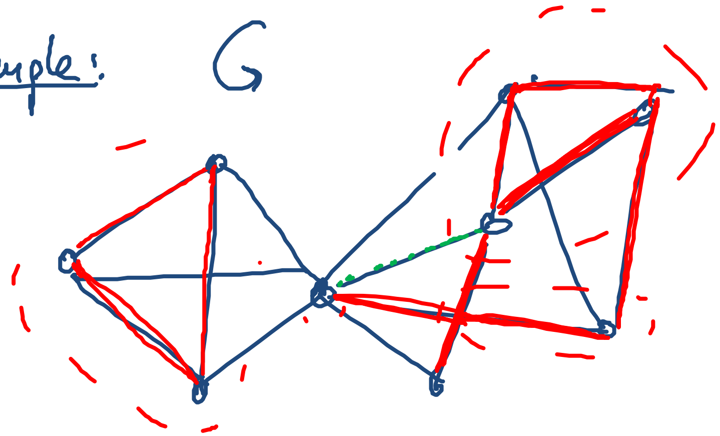
Linearly independent vectors: $\mathbb{R}^n$

- Vector space V, E: finite set of vectors, I: sets of lin. indep. vect.
- Fano matroid can be defined like that

Bicircular Matroid

given graph $G=(V,E)$, define $\mathcal{B}$: $S \subseteq E$ $S \in \mathcal{B}$ if in (V,S), every comp. has ≤ 1 cycle.

example: G



Claim: $(E,\mathcal{B})$ is a matroid

Proof: We need to show that $(E,\mathcal{B})$ satisfies properties 1., 2., 3.

Prop 1. $\emptyset \in \mathcal{B}$ $(V,\emptyset)$ has no cycles ✓

Prop 2. $A \in \mathcal{B} \rightarrow A' \subseteq A : A' \in \mathcal{B}$ ✓

Exchange prop (3.): edge set $A \in \mathcal{B}$, $C \in \mathcal{B}$ $\underbrace{(V,A),(V,C)}_{\text{every comp. has } \leq 1 \text{ cycle}}$

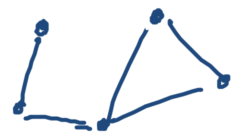
$|C| > |A| \rightarrow \exists e \in C \setminus A$

s.t. $A \cup \{e\} \in \mathcal{B}$

Bicircular Matroid

Components with ≤ 1 cycle

Assume, component has k nodes



$\rightarrow$ comp. has $\leq k$ edges

no cycles : exactly $k-1$ edges
(from connectivity)

cycle : exactly k edges

(V,S) : $S \in \mathcal{B}$

$|S| \leq n$

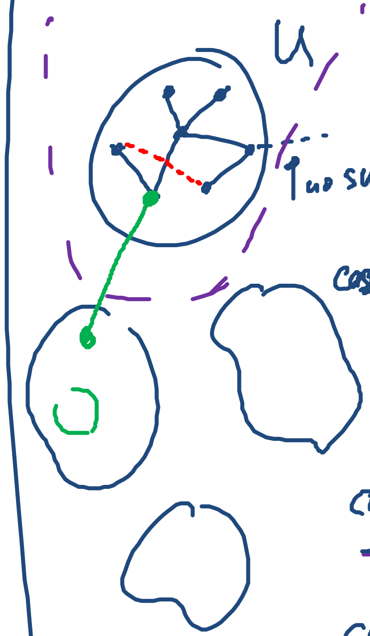
$|S| = n \iff$ all comp. have a cycle

(V,A) , (V,C) $\quad (A, C \in \mathcal{B})$

$|A| < |C| \implies |A| \leq n-1$

(V,A)

$\hookrightarrow$ there is a component $U \subseteq V$ with no cycle



$\implies$ comp. has $|U|-1$ edges

case 0: If C has an edge f connecting U to $V \setminus U$ then f can be added

If not:

case 1) $C \setminus A$ contains edge f between to nodes in $U \rightarrow$ add f

case 2) consider matroid defined by $V \setminus U$

Greedoid

- Matroids can be generalized even more
- Relax hereditary property:

  Replace $\quad A' \subseteq A \subseteq I \quad \Longrightarrow \quad A' \in I$

  by $\quad \emptyset \neq A \subseteq I \quad \Longrightarrow \quad \exists a \in A,\ \text{s.t. } A \setminus \{a\} \in I$

- Exchange property holds as before
- Under certain conditions on the weights, greedy is optimal for computing the max. weight $A \in I$ of a greedoid.
  - Additional conditions automatically satisfied by hereditary property
- More general than matroids