Chapter 3
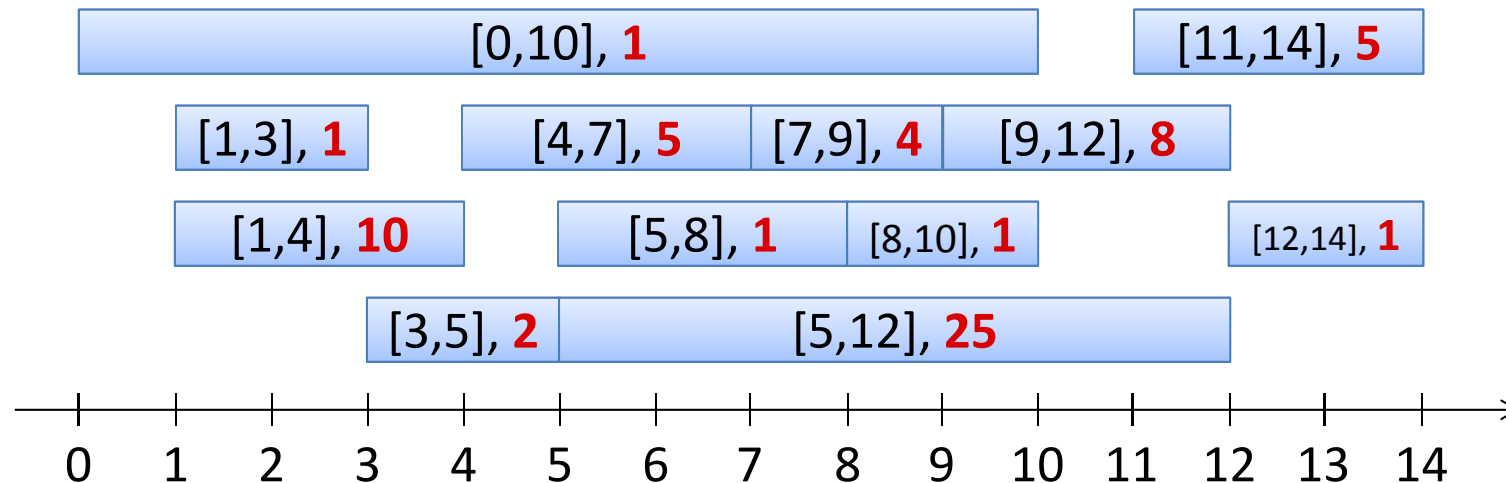# Dynamic Programming

Algorithm Theory
WS 2014/15

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Weighted Interval Scheduling

- **Given:** Set of intervals, e.g.
  [0,10],[1,3],[1,4],[3,5],[4,7],[5,8],[5,12],[7,9],[9,12],[8,10],[11,14],[12,14]

- Each interval has a **weight w**

[0,10], **1**		[11,14], **5**
[1,3], **1**	[4,7], **5**	[7,9], **4**	[9,12], **8**
[1,4], **10**	[5,8], **1**	[8,10], **1**	[12,14], **1**
[3,5], **2**	[5,12], **25**

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

- **Goal:** Non-overlapping set of intervals of largest possible weight
  - Overlap at boundary ok, i.e., [4,7] and [7,9] are non-overlapping

- **Example:** Intervals are room requests of different importance

Greedy Algorithms

Choose available request with earliest finishing time:
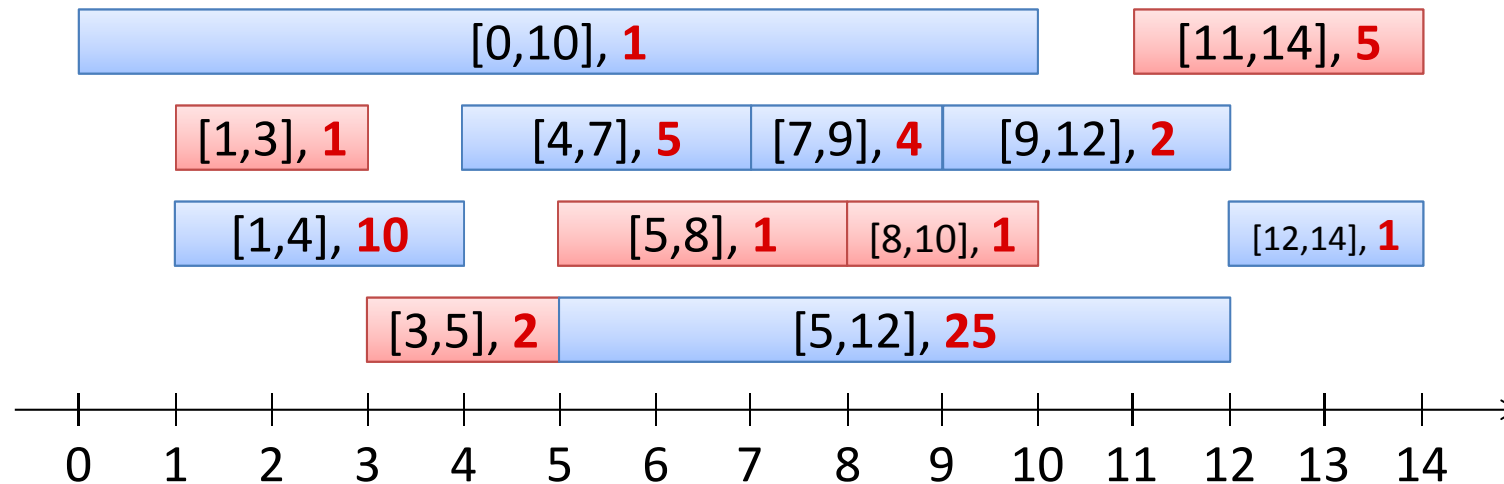
- Algorithm is not optimal any more
  - It can even be arbitrarily bad…

- No greedy algorithm known that works

Solving Weighted Interval Scheduling

- Interval i: start time $s(i)$, finishing time: $f(i)$, weight: $w(i)$

- Assume intervals $1, \ldots, n$ are sorted by increasing $f(i)$
  - $0 < f(1) \leq f(2) \leq \cdots \leq f(n)$, for convenience: $f(0) = 0$

- Simple observation:
  Opt. solution contains interval n or it doesn't contain interval n
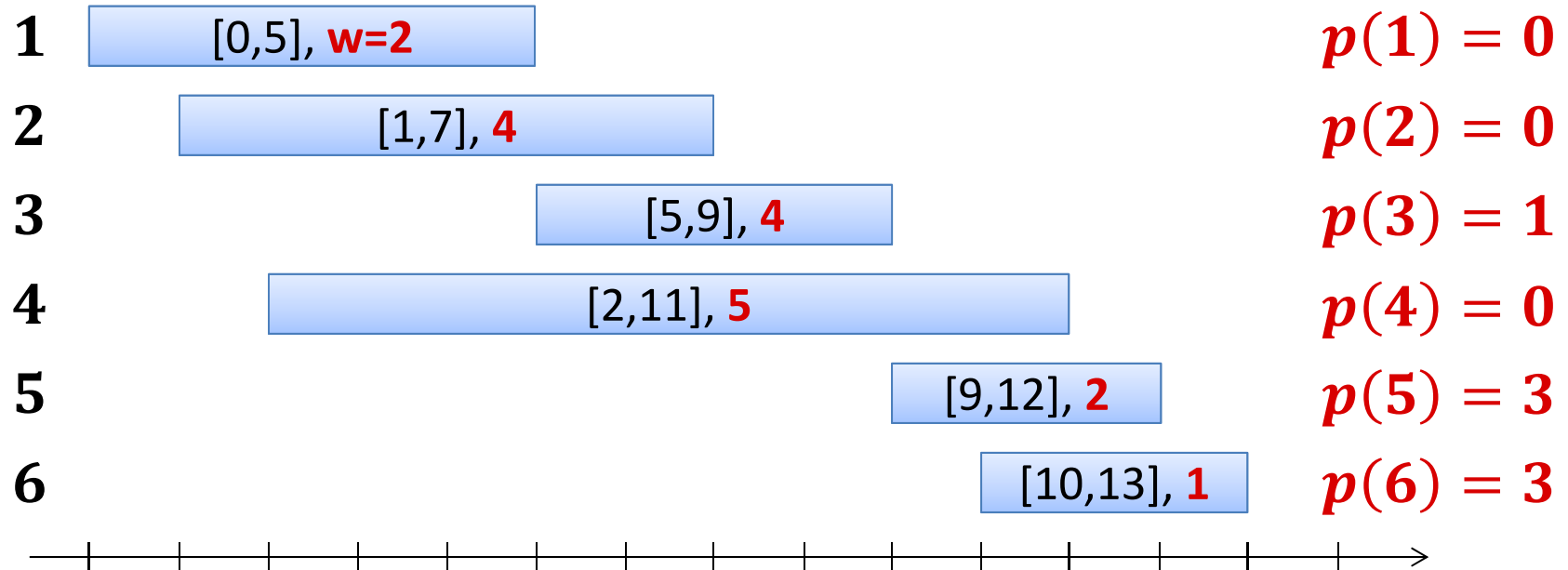
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- Weight of optimal solution for only intervals $1, \dots, k$: $W(k)$
  Define $p(k) := \max\{i \in \{0, \dots, k-1\} : f(i) \leq s(k)\}$

- Opt. solution does not contain interval n: $\boldsymbol{W(n) = W(n-1)}$

  Opt. solution contains interval n: $\boldsymbol{W(n) = w(n) + W(p(n))}$

Example

Interval:

1   [0,5], **w=2** $p(1) = 0$

2   [1,7], **4** $p(2) = 0$

3   [5,9], **4** $p(3) = 1$

4   [2,11], **5** $p(4) = 0$

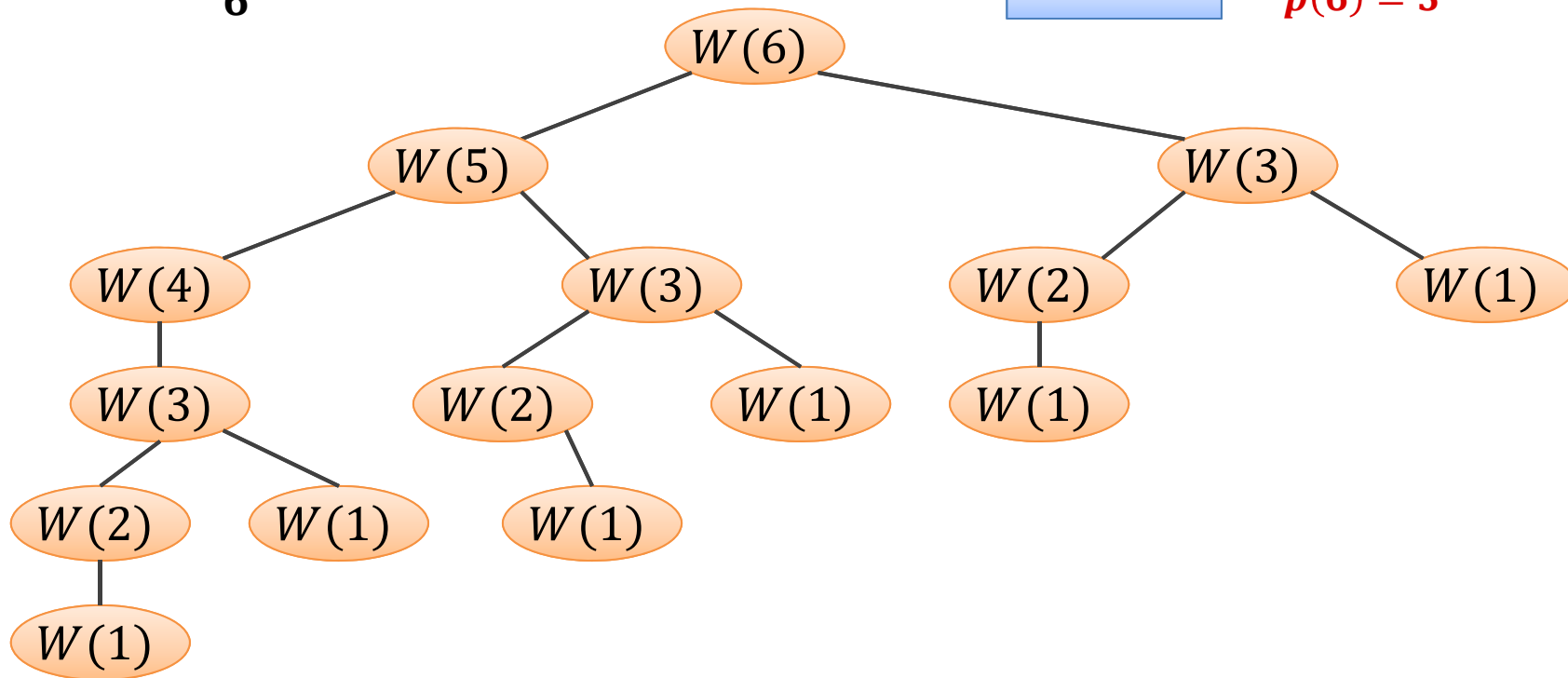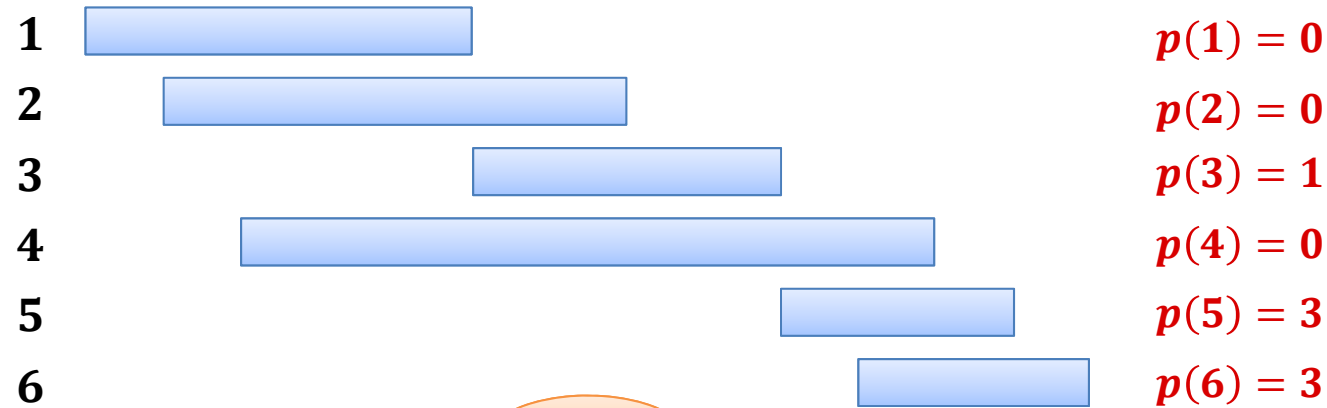5   [9,12], **2** $p(5) = 3$

6   [10,13], **1** $p(6) = 3$

Recursive Definition of Optimal Solution

- Recall:
  - $W(k)$: weight of optimal solution with intervals $1, \dots, k$
  - $p(k)$: last interval to finish before interval k starts

- Recursive definition of optimal weight:

$$\forall k > 1: W(k) = \max\{W(k-1), w(k) + W(p(k))\}$$

$$W(1) = w(1)$$

- Immediately gives a simple, recursive algorithm

Running Time of Recursive Algorithm

$p(1) = 0$

$p(2) = 0$

$p(3) = 1$

$p(4) = 0$

$p(5) = 3$

$p(6) = 3$

Memoizing the Recursion

- Running time of recursive algorithm: exponential!

- But, alg. only solves n different sub-problems: $W(1), \ldots, W(n)$

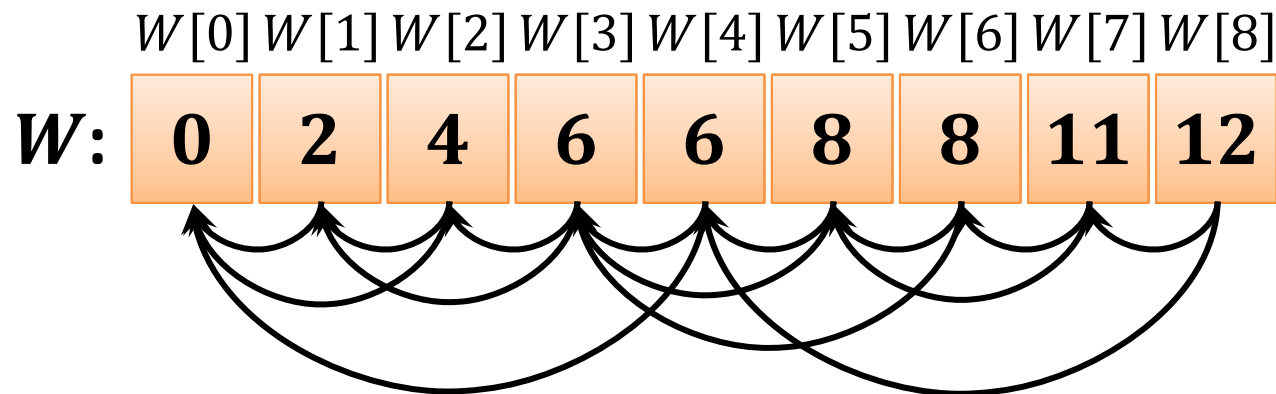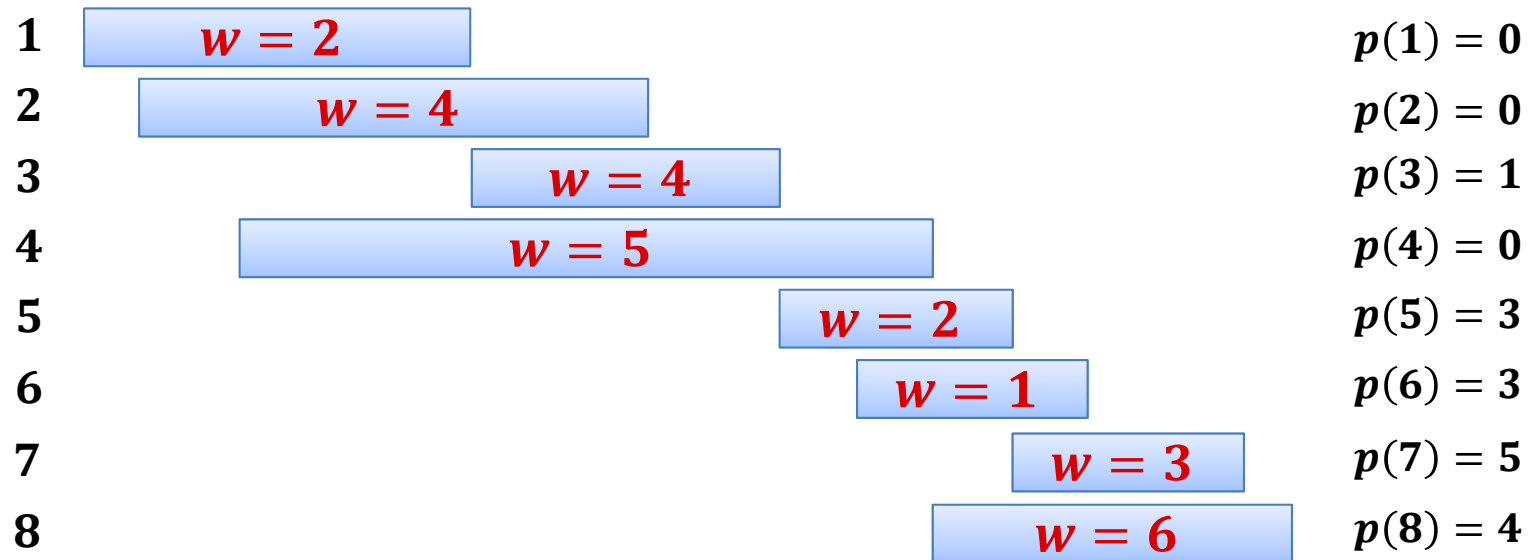- There is no need to compute them multiple times

Memoization:

- **Store already computed values** for future use (recursive calls)

Efficient algorithm:

1. $W[0] := 0$; compute values $p(i)$

2. **for** $i := 1$ **to** n **do**

3. $\quad W[i] := \max\{W[i-1], w(i) + W[p(i)]\}$

4. **end**

Example

1     $w = 2$ $p(1) = 0$

2     $w = 4$ $p(2) = 0$

3     $w = 4$ $p(3) = 1$

4     $w = 5$ $p(4) = 0$

5     $w = 2$ $p(5) = 3$

6     $w = 1$ $p(6) = 3$

7     $w = 3$ $p(7) = 5$

8     $w = 6$ $p(8) = 4$

$W[0]\ W[1]\ W[2]\ W[3]\ W[4]\ W[5]\ W[6]\ W[7]\ W[8]$

W: | 0 | 2 | 4 | 6 | 6 | 8 | 8 | 11 | 12 |

Computing the schedule: **store where you come from!**

Matrix-chain multiplication

Given: sequence (chain) $\langle A_1, A_2, \ldots, A_n \rangle$ of matrices

Goal: compute the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$

Problem: Parenthesize the product in a way that minimizes the number of scalar multiplications.

Definition: A product of matrices is *fully parenthesized* if it is

- a single matrix
- or the product of two fully parenthesized matrix products, surrounded by parentheses.

Example

All possible fully parenthesized matrix products of the chain $\langle A_1, A_2, A_3, A_4 \rangle$:

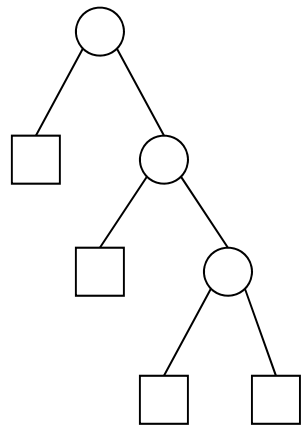$$(A_1 (A_2 (A_3 A_4)))$$

$$(A_1 ((A_2 A_3) A_4))$$

$$((A_1 A_2)(A_3 A_4))$$
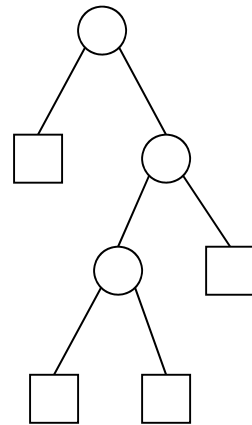
$$((A_1 (A_2 A_3)) A_4)$$

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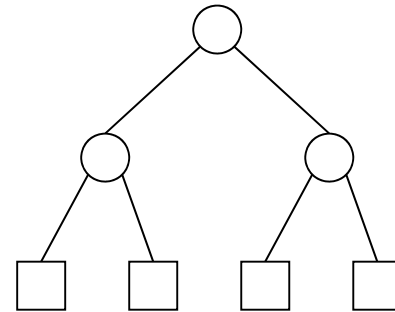
Different parenthesizations

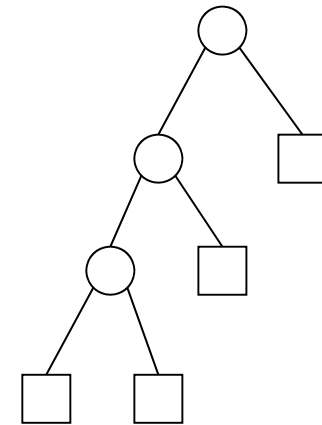Different parenthesizations correspond to different trees:

$(A_1(A_2(A_3A_4)))$

$(A_1((A_2A_3)A_4))$

$((A_1A_2)(A_3A_4))$

$(((A_1A_2)A_3)A_4)$

Number of different parenthesizations

- Let $P(n)$ be the number of alternative parenthesizations of the product $A_1 \cdot \ldots \cdot A_n$:

$$P(1) = 1$$

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k), \qquad \text{for } n \geq 2$$

$$P(n+1) = \frac{1}{n+1}\binom{2n}{n} \approx \frac{4^n}{n\sqrt{\pi n}} + O\left(\frac{4^n}{\sqrt{n^5}}\right)$$

$$P(n+1) = C_n \qquad (n^{th} \text{ Catalan number})$$

- Thus: Exhaustive search needs exponential time!

Multiplying Two Matrices

$$A = \left(a_{ij}\right)_{p\times q}, \qquad B = \left(b_{ij}\right)_{q\times r}, \qquad A \cdot B = C = \left(c_{ij}\right)_{p\times r}$$

$$c_{ij} = \sum_{k=1}^{q} a_{ik}b_{kj}$$

Remark:

Using this algorithm, multiplying two $(n \times n)$ matrices requires n^3 multiplications. This can also be done using $O(n^{2.376})$ multiplications.

Algorithm *Matrix-Mult*

Input: $(p \times q)$ matrix A, $(q \times r)$ matrix B

Output: $(p \times r)$ matrix $C = A \cdot B$

1 **for** $i := 1$ **to** p **do**

2   **for** $j := 1$ **to** r **do**

3        $C[i,j] := 0;$

4   **for** $k := 1$ **to** q **do**

5           $C[i,j] := C[i,j] + A[i,k] \cdot B[k,j]$

Number of multiplications and additions: $\boldsymbol{p \cdot q \cdot r}$

Matrix-chain multiplication: Example

Computation of the product $A_1 A_2 A_3$, where

A_1 : (50×5) matrix

A_2 : (5×100) matrix

A_3 : (100×10) matrix

a) Parenthesization $((A_1 A_2)A_3)$ and $(A_1(A_2 A_3))$ require:

$A' = (A_1 A_2)$: $A'' = (A_2 A_3)$:

$A' A_3$: $A_1 A''$:

Sum:

Structure of an Optimal Parenthesization

- $(A_{\ell\dots r})$: optimal parenthesization of $A_\ell \cdot \dots \cdot A_r$

  For some $1 \leq k < n$: $(A_{1\dots n}) = ((A_{1\dots k}) \cdot (A_{k+1\dots n}))$

- Any optimal solution contains optimal solutions for sub-problems

- Assume matrix A_i is a $(d_{i-1} \times d_i)$-matrix

- Cost to solve sub-problem $A_\ell \cdot \dots \cdot A_r, \ell \leq r$ optimally: $C(\ell, r)$
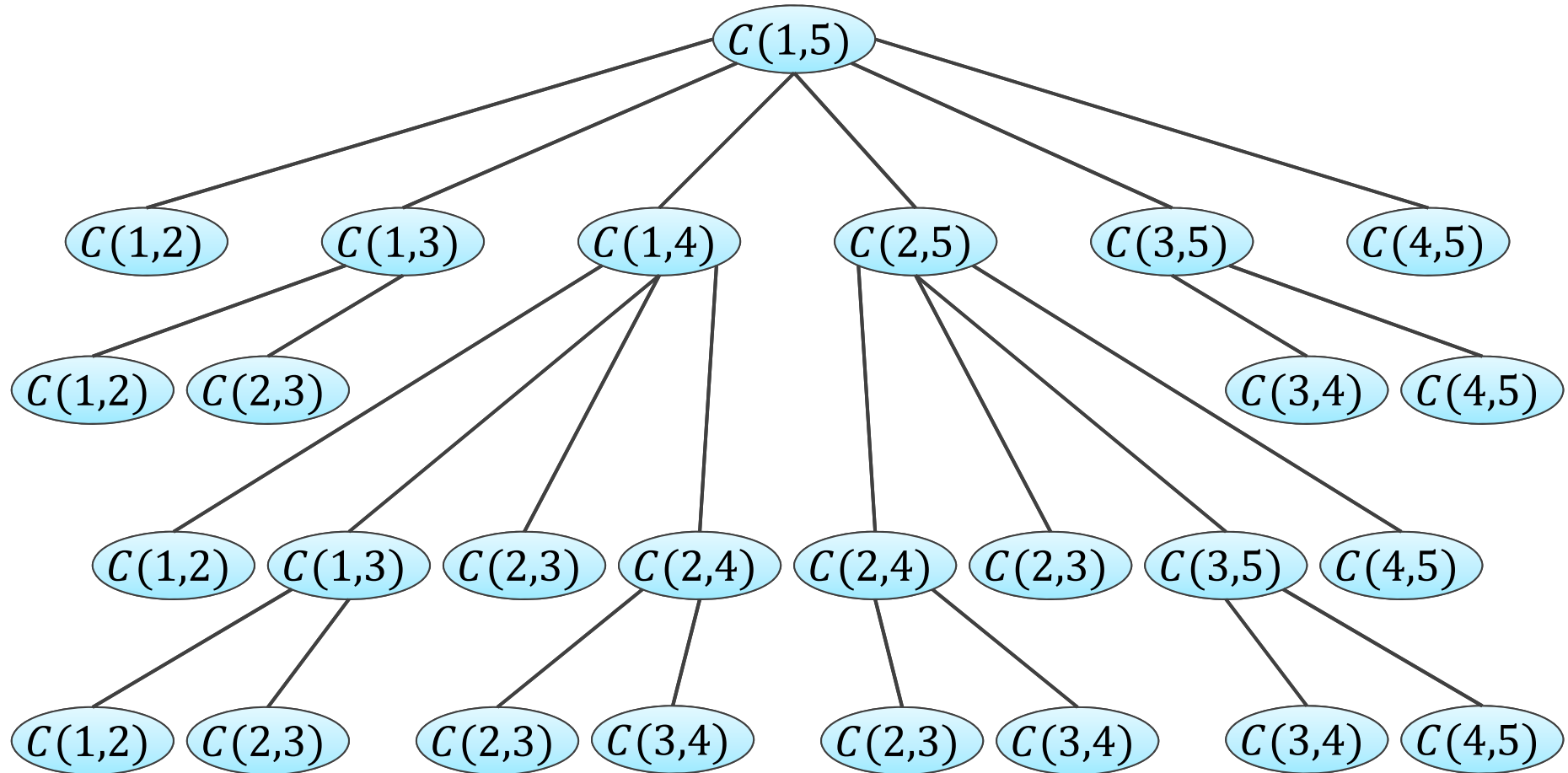
- Then:
  $$C(a, b) = \min_{a \leq k < b} C(a, k) + C(k+1, b) + d_{a-1} d_k d_b$$

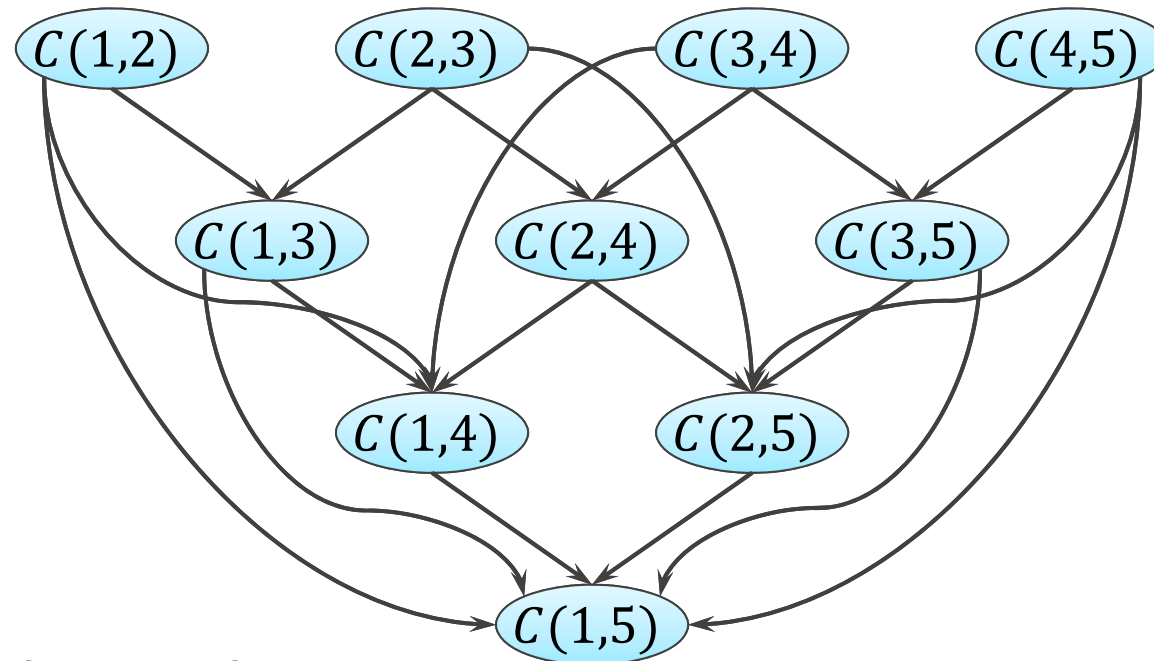 $$C(a, a) = 0$$

Recursive Computation of Opt. Solution

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:

Using Meomization

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:

Compute $A_1 \cdot \ldots \cdot A_n$:

- Each $C(i,j), i < j$ is computed exactly once $\rightarrow$ $O(n^2)$ values

- Each $C(i,j)$ dir. depends on $C(i,k), C(k,j)$ for $i < k < j$

  Cost for each $C(i,j)$: $O(n)$ $\rightarrow$ overall time: $\boldsymbol{O(n^3)}$

Dynamic Programming

„Memoization" for increasing the efficiency of a recursive solution:

- Only the *first time* a sub-problem is encountered, its solution is computed and then stored in a table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned
  (without repeated computation!).

- *Computing the solution*: For each sub-problem, store how the value is obtained (according to which recursive rule).

Dynamic Programming

Dynamic programming / memoization can be applied if

- Optimal solution contains optimal solutions to sub-problems (recursive structure)

- Number of sub-problems that need to be considered is small

Remarks about matrix-chain multiplication

1. There is an algorithm that determines an optimal parenthesization in time

$$O(n \cdot \log n).$$

2. There is a linear time algorithm that determines a parenthesization using at most

$$1.155 \cdot C(1, n)$$

multiplications.