Chapter 4
Data Structures

Algorithm Theory
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Priority Queue / Heap

- Stores (*key,data*) pairs (like dictionary)
- But, different set of operations:

- **Initialize-Heap**: creates new empty heap
- **Is-Empty**: returns true if heap is empty
- **Insert**(*key,data*): inserts (*key,data*)-pair, returns pointer to entry
- **Get-Min**: returns (*key,data*)-pair with minimum *key*
- **Delete-Min**: deletes minimum (*key,data*)-pair
- **Decrease-Key**(*entry,newkey*): decreases *key* of *entry* to *newkey*
- **Merge**: merges two heaps into one

Implementation of Dijkstra's Algorithm

Store nodes in a priority queue, use $d(s, v)$ as keys:

1. Initialize $d(s, s) = 0$ and $d(s, v) = \infty$ for all $v \neq s$
2. All nodes $v \neq s$ are unmarked

3. Get unmarked node u which minimizes $d(s, u)$:

4. mark node u

5. For all $e = \{u, v\} \in E$, $d(s, v) = \min\{d(s, v), d(s, u) + w(e)\}$

6. Until all nodes are marked

Analysis

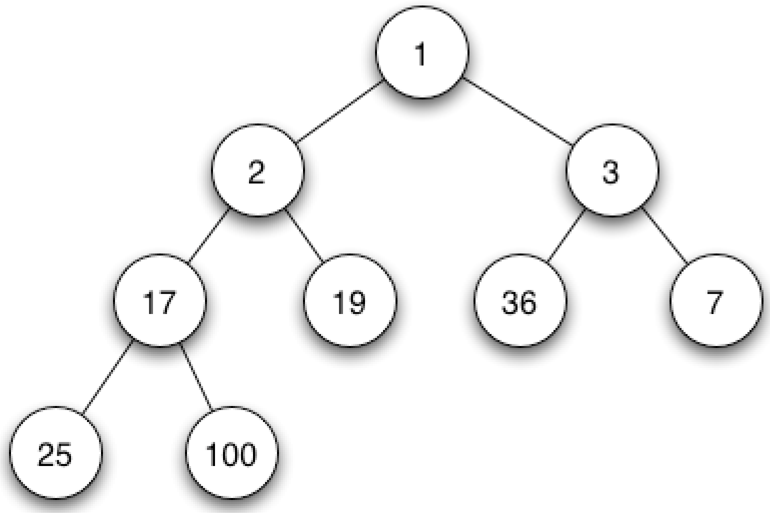
Number of priority queue operations for Dijkstra:

- **Initialize-Heap**: $\mathbf{1}$
- **Is-Empty**: $|\boldsymbol{V}|$
- **Insert**: $|\boldsymbol{V}|$
- **Get-Min**: $|\boldsymbol{V}|$
- **Delete-Min**: $|\boldsymbol{V}|$
- **Decrease-Key**: $|\boldsymbol{E}|$
- **Merge**: $\mathbf{0}$

Priority Queue Implementation

Implementation as min-heap:

→ complete binary tree,
e.g., stored in an array



- **Initialize-Heap**: $\boldsymbol{O(1)}$
- **Is-Empty**: $\boldsymbol{O(1)}$
- **Insert**: $\boldsymbol{O(\log n)}$
- **Get-Min**: $\boldsymbol{O(1)}$
- **Delete-Min**: $\boldsymbol{O(\log n)}$
- **Decrease-Key**: $\boldsymbol{O(\log n)}$
- **Merge** (heaps of size m and n, $m \leq n$): $\boldsymbol{O(m \log n)}$

Better Implementation

- Can we do better?

- Cost of Dijkstra with complete binary min-heap implementation:

$$O(|E|\log|V|)$$

- Can be improved if we can make decrease-key cheaper…

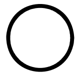
- Cost of merging two heaps is expensive

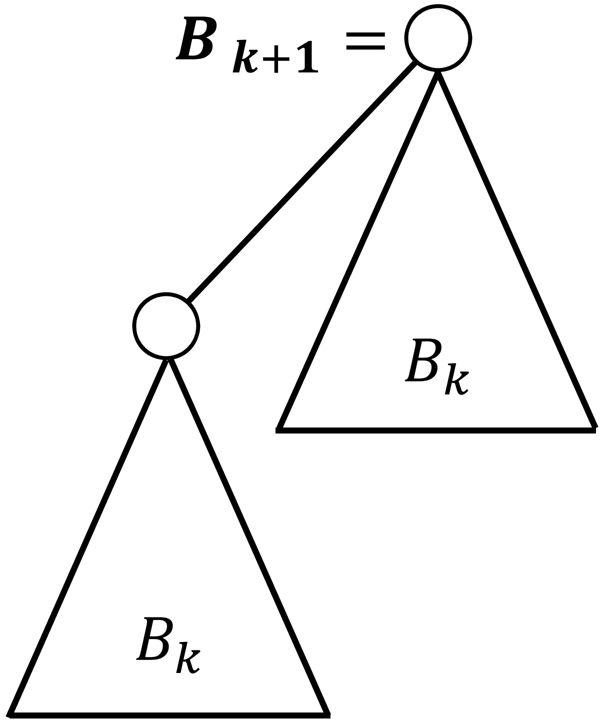
- We will get there in two steps:

  Binomial heap → Fibonacci heap

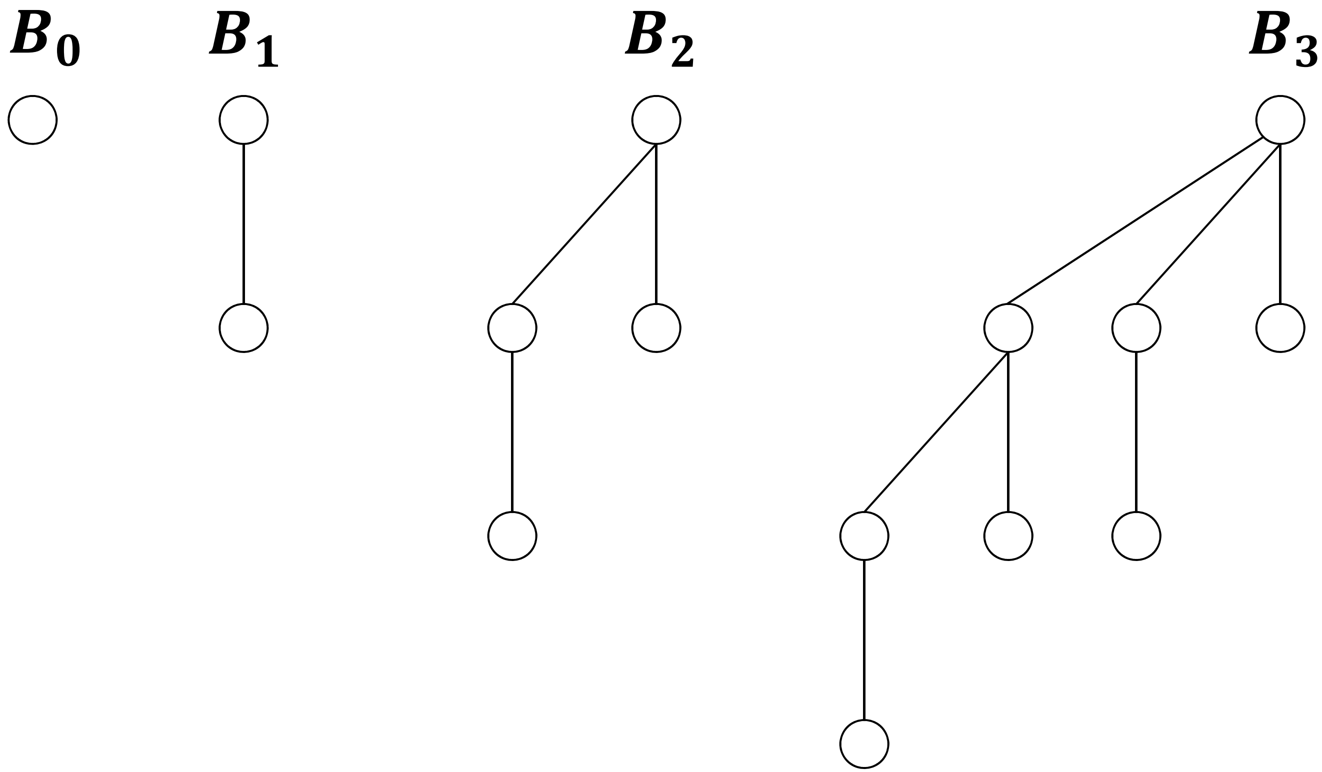
Definition: Binomial Tree

Binomial tree B_k of order k $(n \geq 0)$:

$B_0 =$ 

$B_{k+1} =$ 

Binomial Trees



Properties

1. Tree B_k has 2^k nodes

2. Height of tree B_k is k

3. Root degree of B_k is k

4. In B_k, there are exactly $\binom{k}{i}$ nodes at depth i

Binomial Coefficients

- Binomial coefficient:

$$\binom{k}{i} = \frac{k!}{i!\,(k-i)!} : \#\text{ of size } i \text{ subsets of a set of size } k$$

- Property: $\binom{k}{i} = \binom{k-1}{i-1} + \binom{k-1}{i}$

Pascal triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Number of Nodes at Depth i in B_k

Claim: In B_k, there are exactly $\binom{k}{i}$ nodes at depth i

Binomial Heap

- Keys are stored in nodes of binomial trees of different order

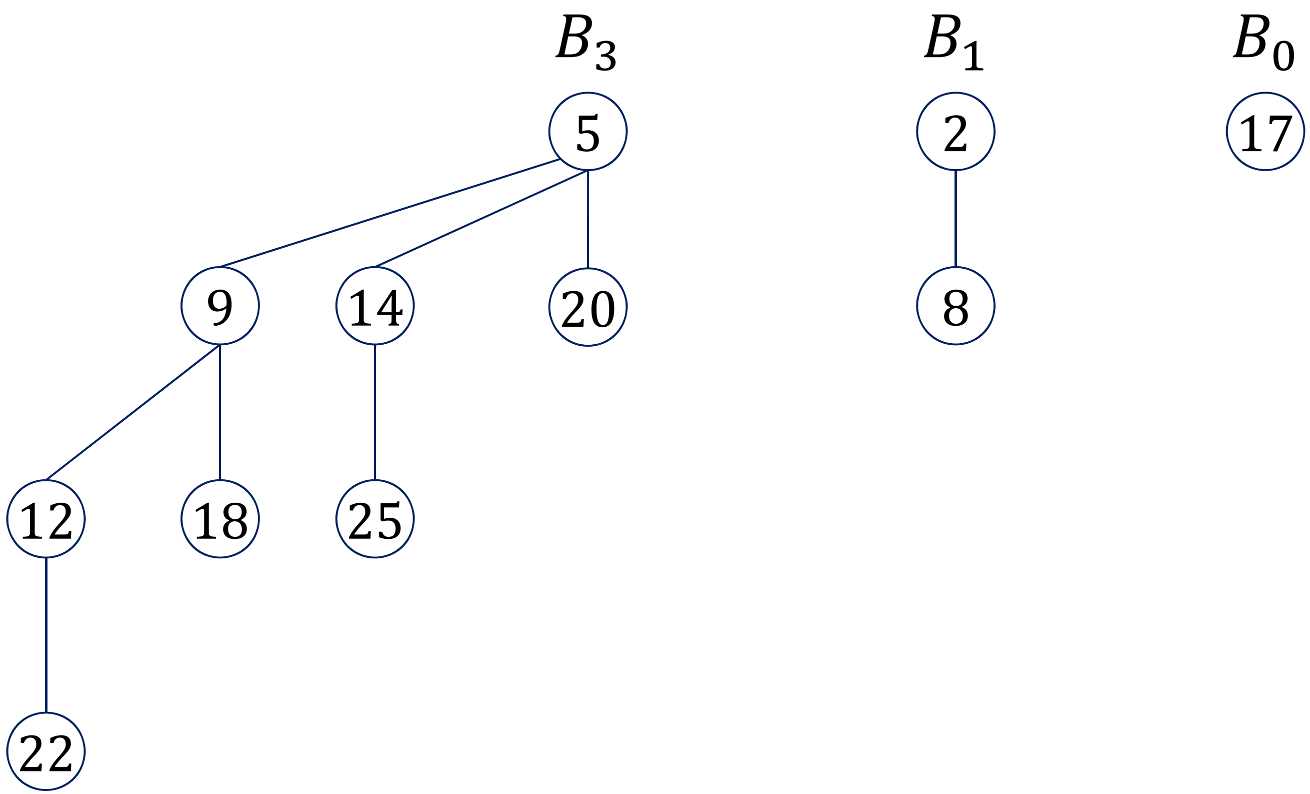
  $\boldsymbol{n}$ **nodes**: there is a binomial tree B_i of order i iff
  bit i of base-2 representation of n is 1.

- **Min-Heap Property:**

  Key of node $v \leq$ keys of all nodes in sub-tree of v

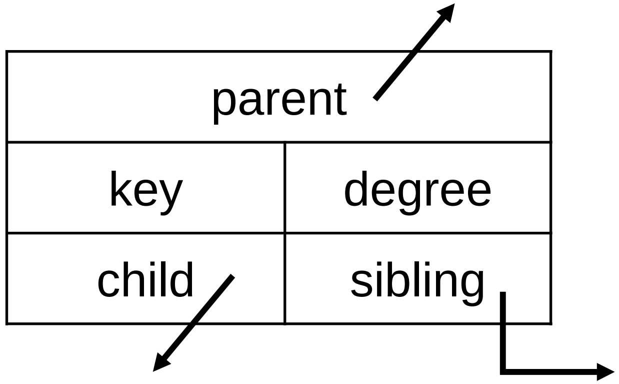
Example

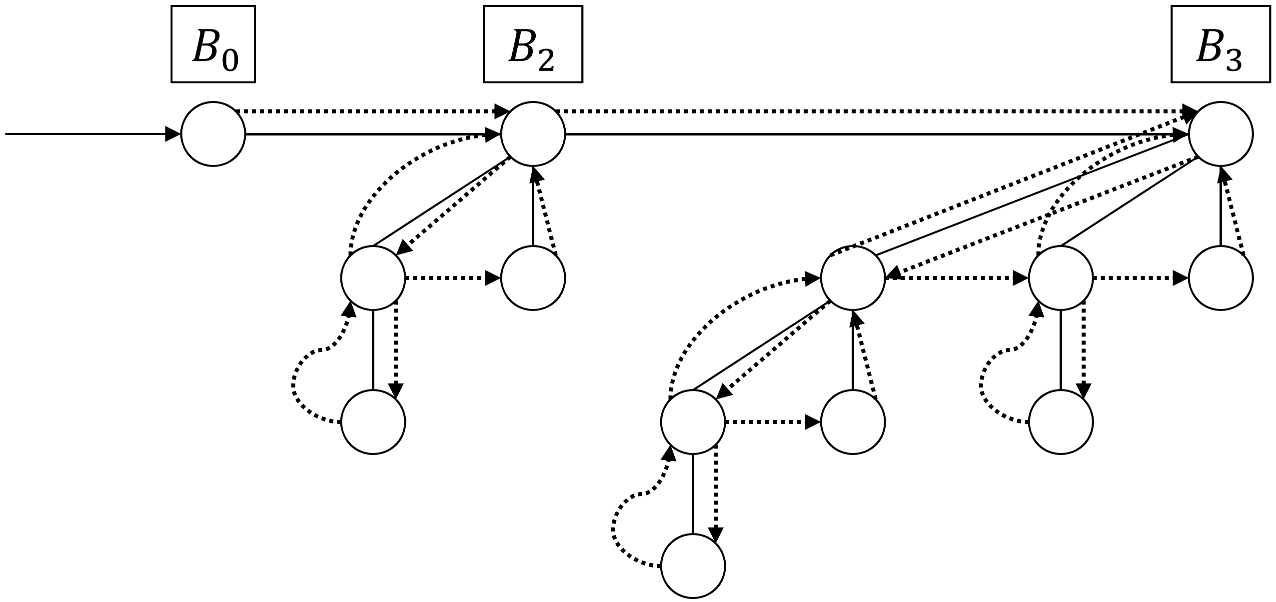
- 11 keys: $\{2, 5, 8, 9, 12, 14, 17, 18, 20, 22, 25\}$
- Binary representation of n: $(11)_2 = 1011$
→ trees B_0, B_1, and B_3 present



Child-Sibling Representation

Structure of a node:



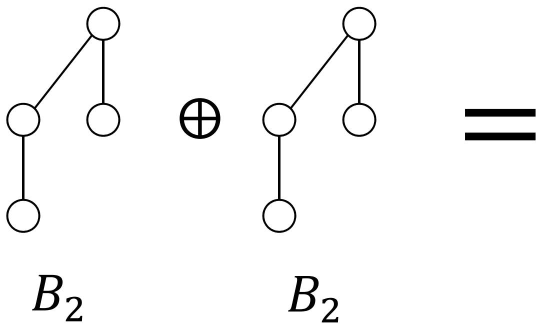


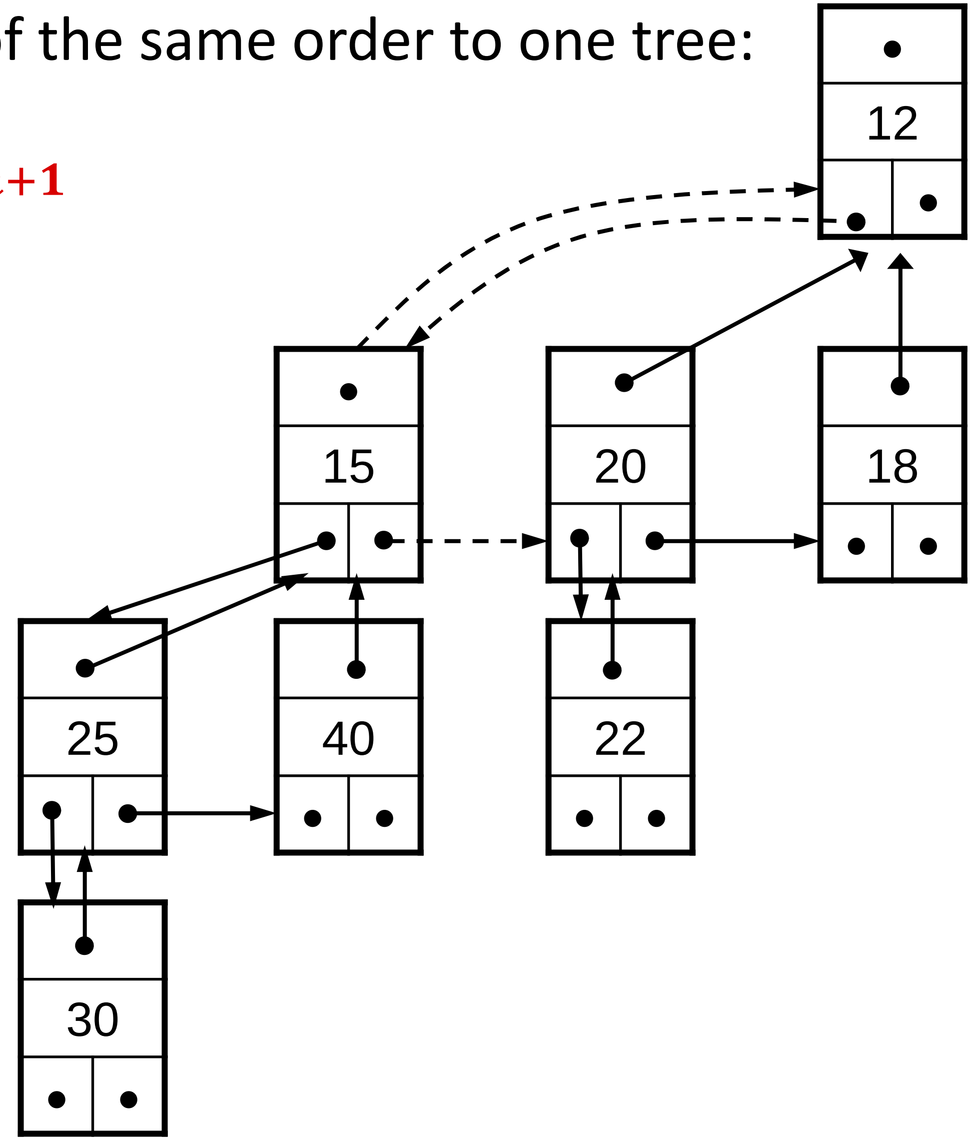
Link Operation

- Unite two binomial trees of the same order to one tree:

$$B_n \oplus B_n \Rightarrow B_{n+1}$$

- Time: $O(1)$

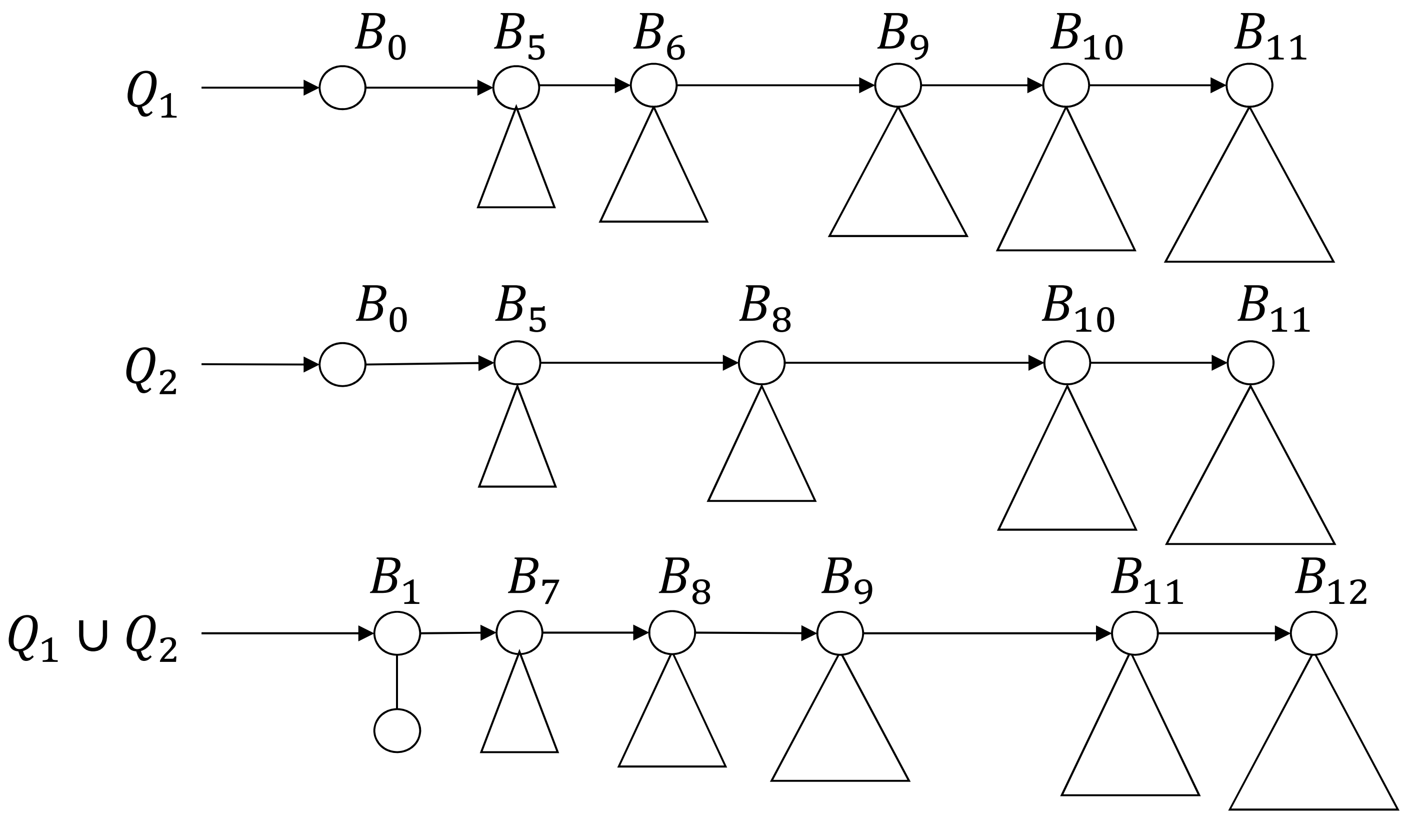




Merge Operation

Merging two binomial heaps:

- **For $i = 0, 1, \dots, \log n$:**
  If there are 2 or 3 binomial trees B_i: apply link operation to merge 2 trees into one binomial tree B_{i+1}



Time: $O(\log n)$