Chapter 5
Graph Algorithms

Algorithm Theory
WS 2014/15

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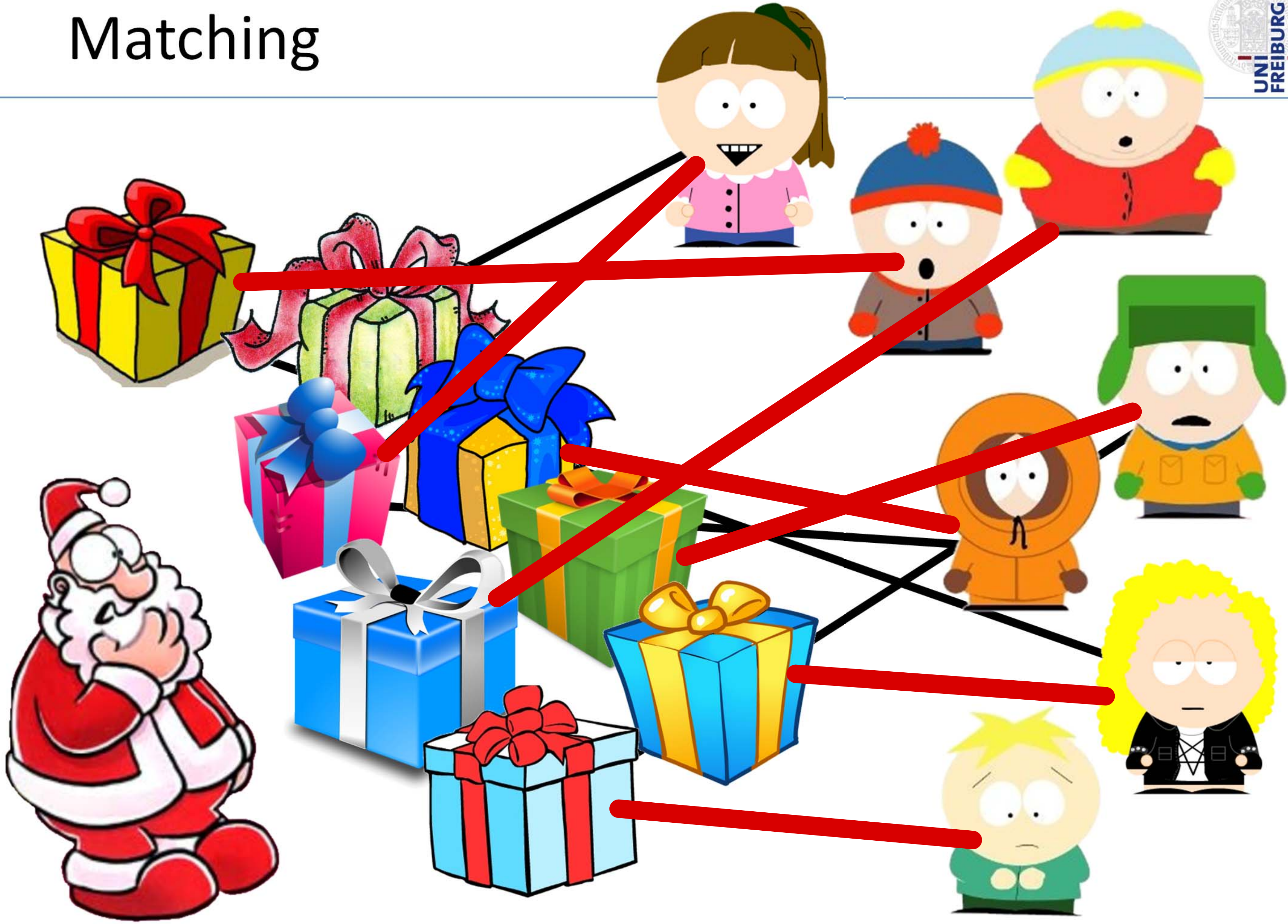
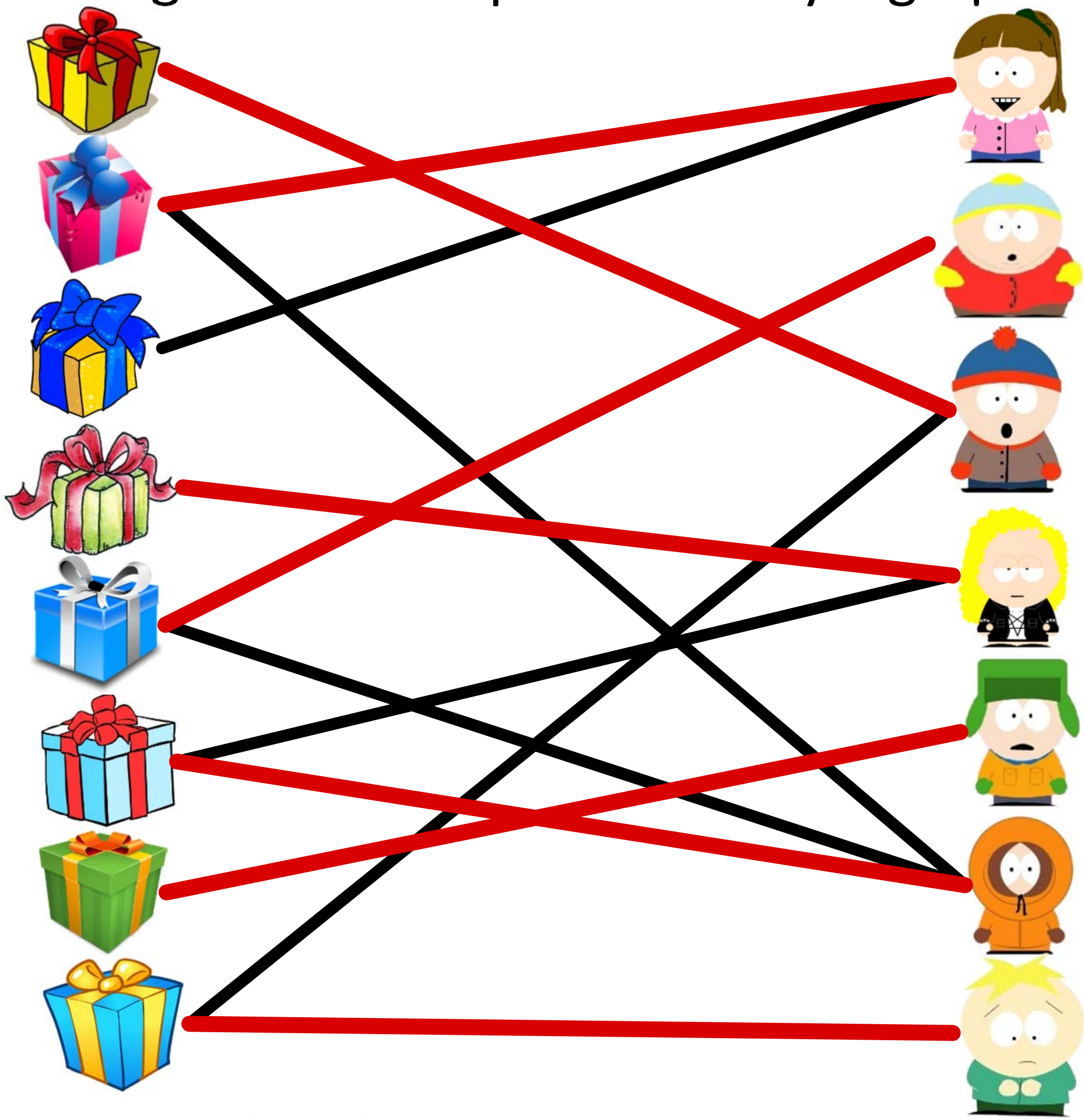
Matching



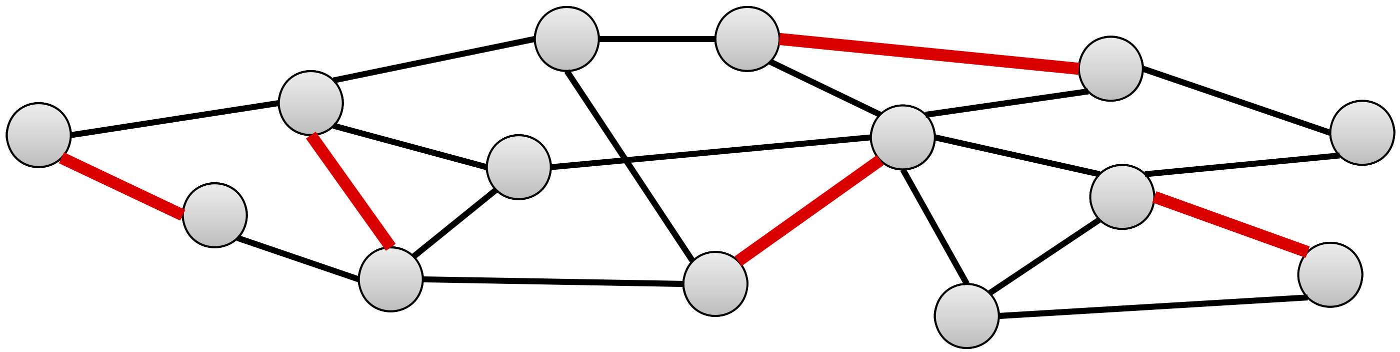
Gifts-Children Graph

- Which child likes which gift can be represented by a graph



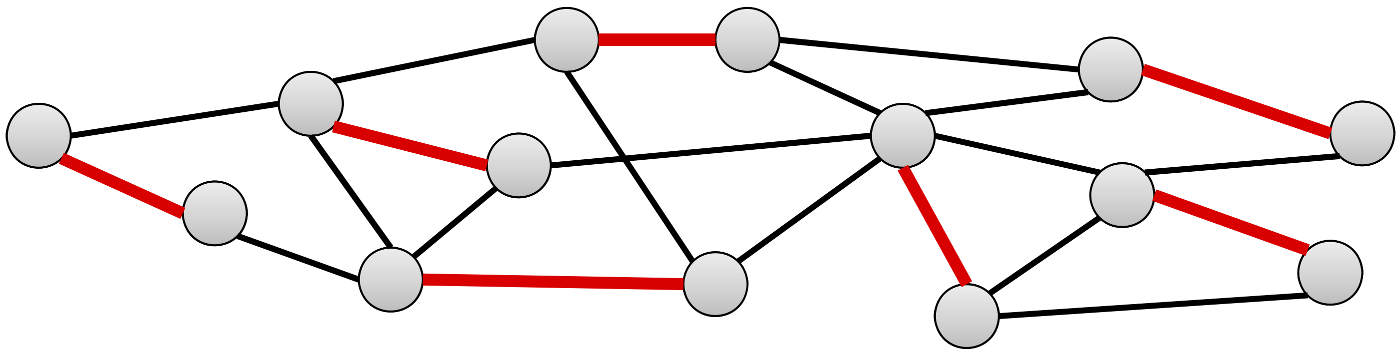
Matching

Matching: Set of pairwise non-incident edges



Maximal Matching: A matching s.t. no more edges can be added

Maximum Matching: A matching of maximum possible size



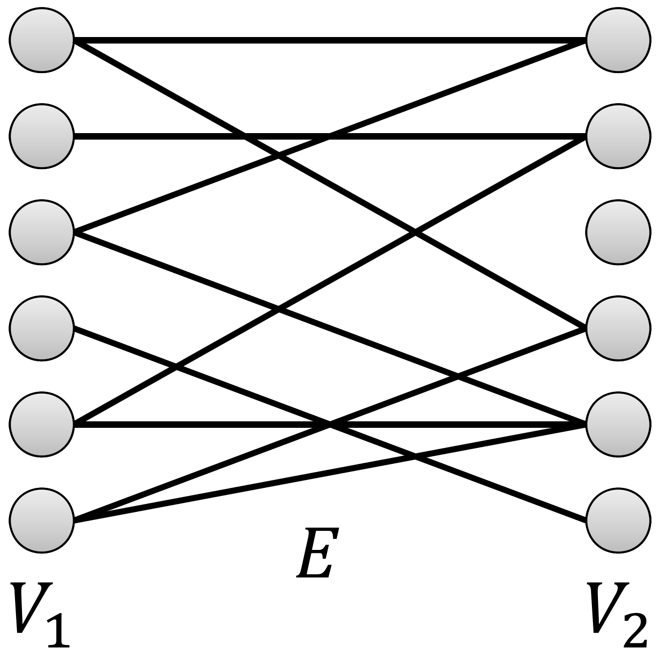
Perfect Matching: Matching of size $n/2$ (every node is matched)

Bipartite Graph

Definition: A graph $G = (V, E)$ is called bipartite iff its node set can be partitioned into two parts $V = V_1 \cup V_2$ such that for each edge $\{u, v\} \in E$,

$$|\{u, v\} \cap V_1| = 1.$$

- Thus, edges are only between the two parts



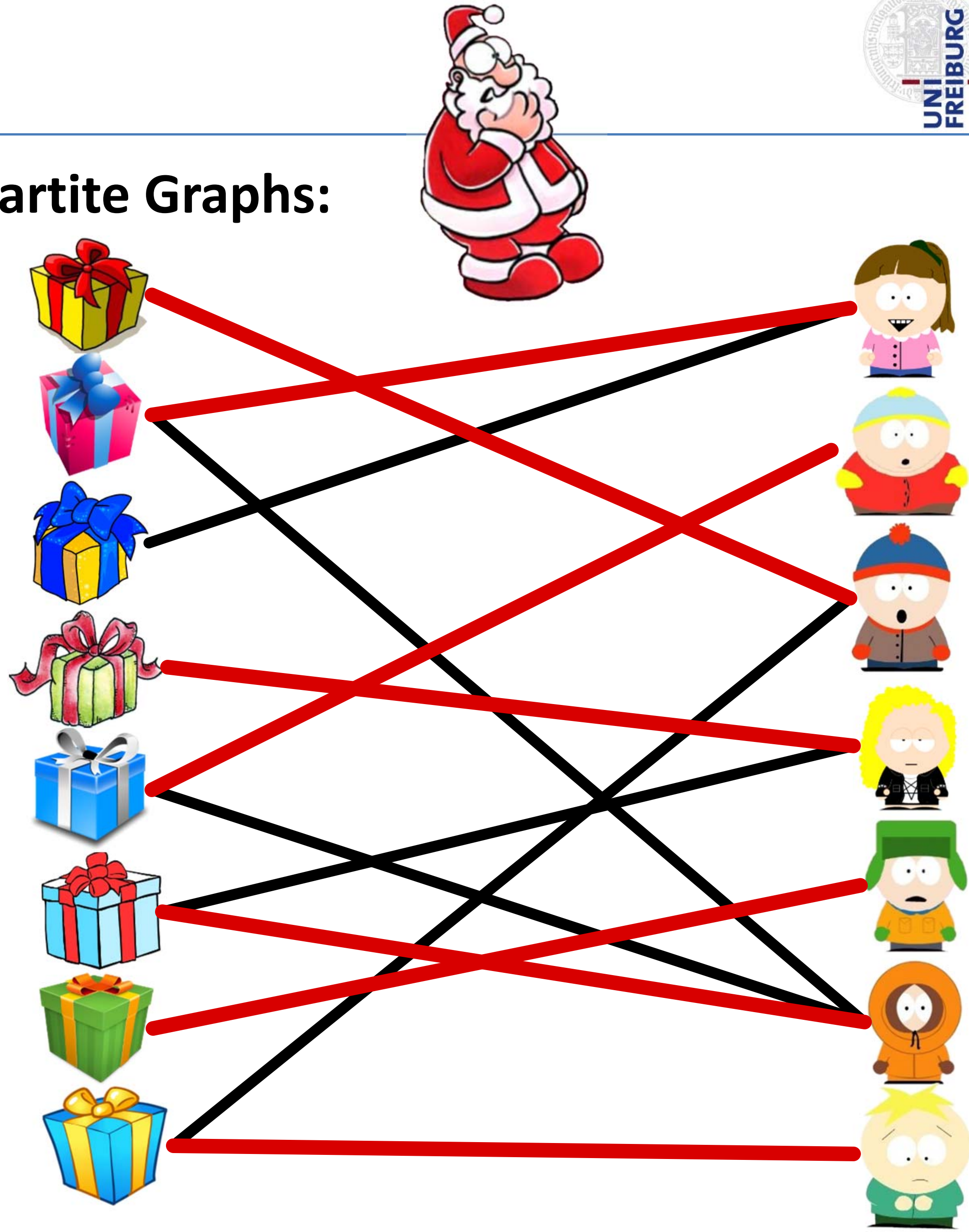
Santa's Problem

Maximum Matching in Bipartite Graphs:

Every child can get a gift iff there is a matching of size #children

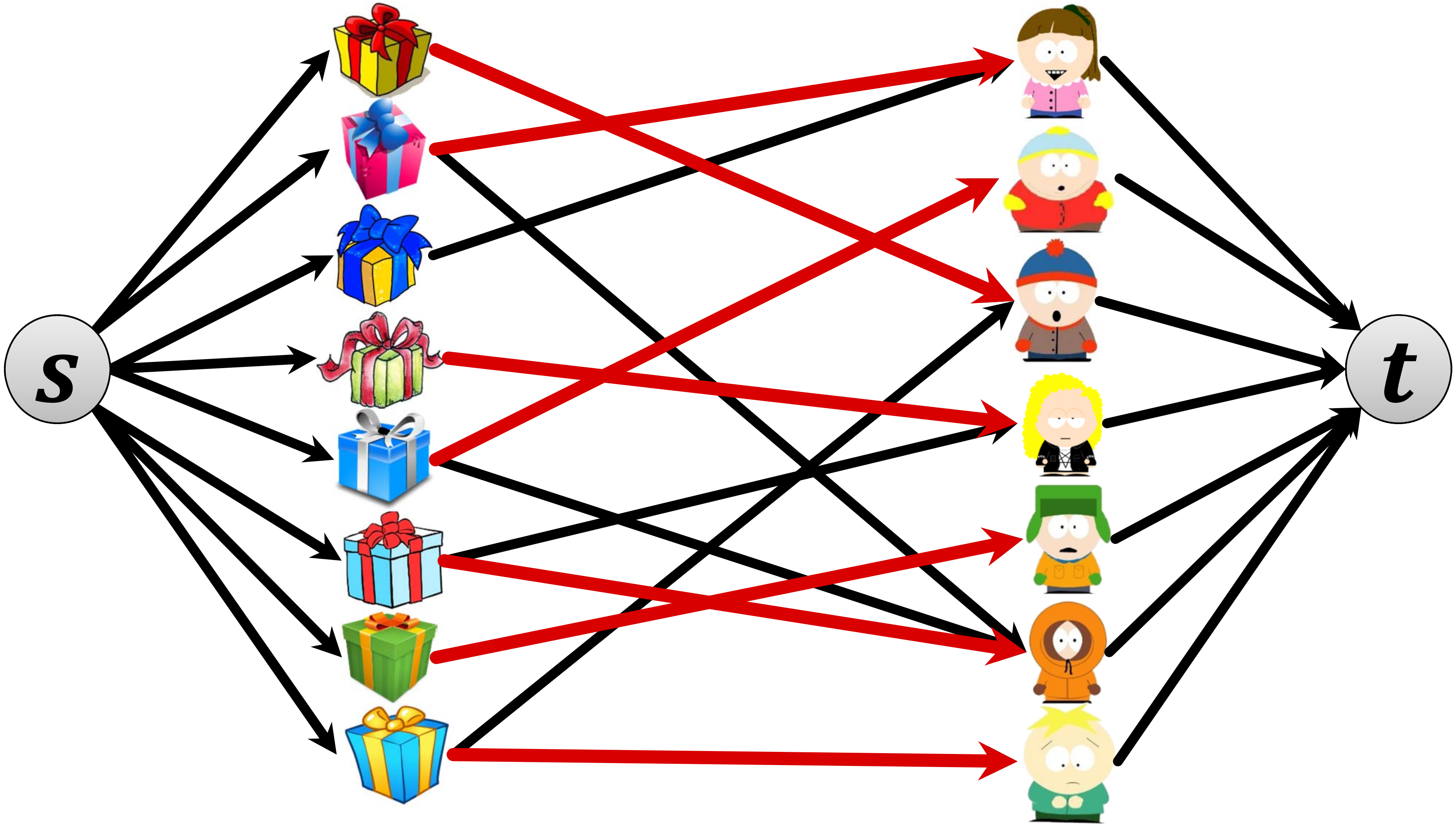
Clearly, every matching is at most as big

If #children = #gifts, there is a solution iff there is a perfect matching



Reducing to Maximum Flow

- Like edge-disjoint paths…



all capacities are 1

Reducing to Maximum Flow

Theorem: Every integer solution to the max flow problem on the constructed graph induces a maximum bipartite matching of G.

Proof:

1. An integer flow f of value $|f|$ induces a matching of size $|f|$
   - Left nodes (gifts) have incoming capacity 1
   - Right nodes (children) have outgoing capacity 1
   - Left and right nodes are incident to ≤ 1 edge e of G with $f(e) = 1$

2. A matching of size k implies a flow f of value $|f| = k$
   - For each edge $\{u, v\}$ of the matching:

   $$f\big((s,u)\big) = f\big((u,v)\big) = f\big((v,t)\big) = 1$$

   - All other flow values are 0

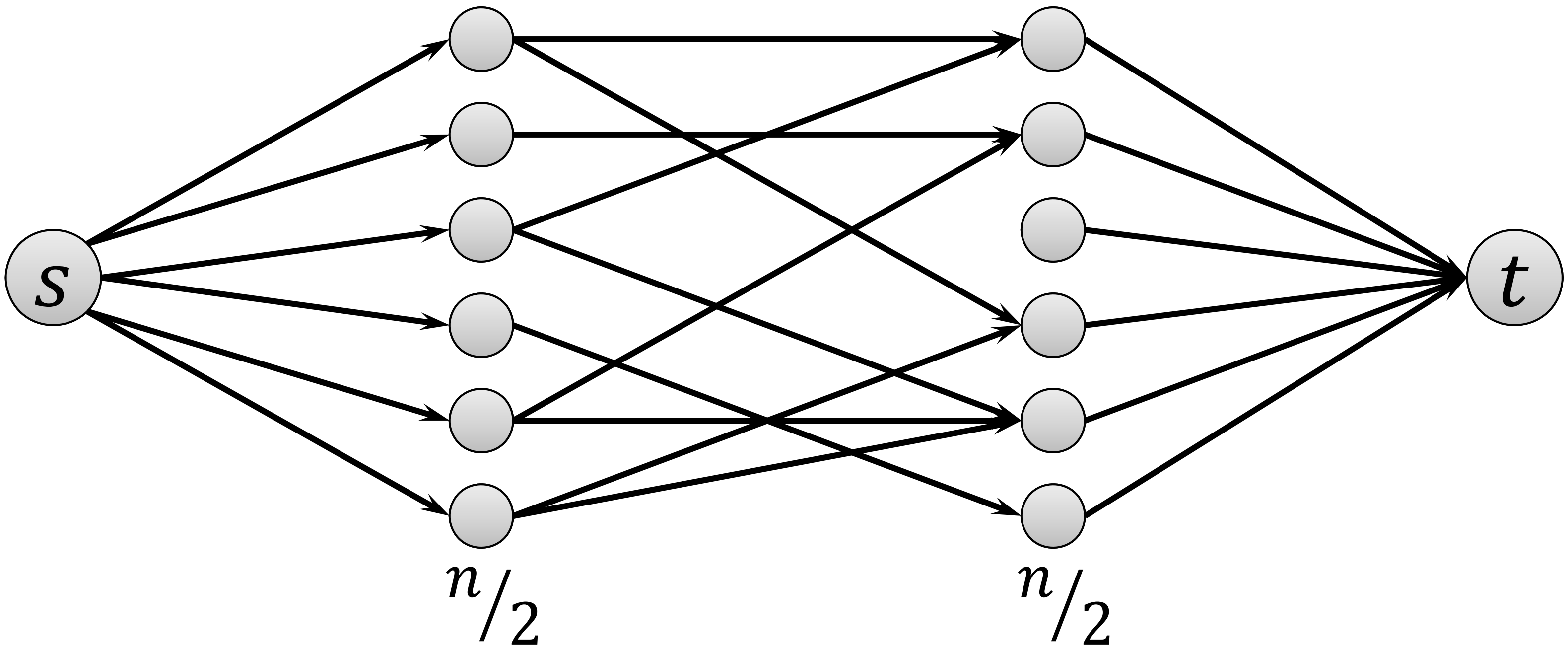
Running Time of Max. Bipartite Matching

Theorem: A maximum matching of a bipartite graph can be computed in time $O(m \cdot n)$.

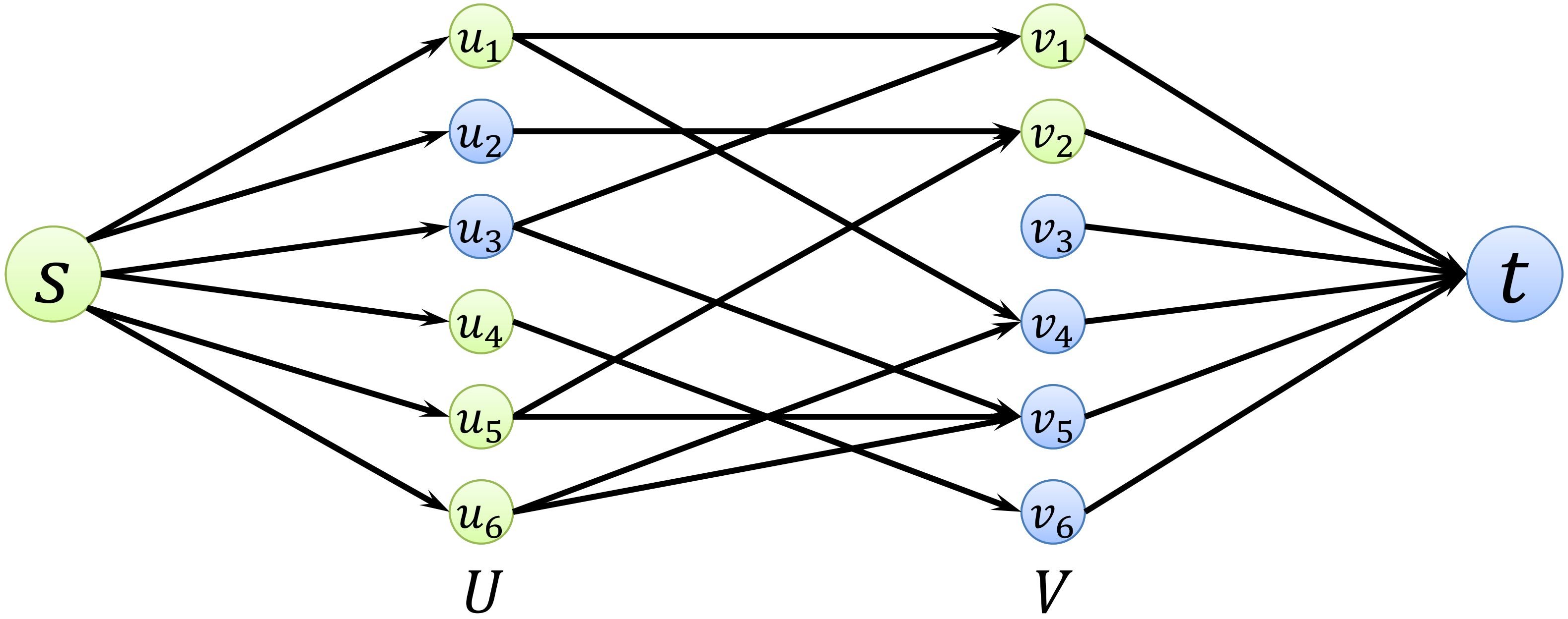
Perfect Matching?

- There can only be a perfect matching if both sides of the partition have size $n/2$.

- There is no perfect matching, iff there is an s-t cut of size $< n/2$ in the flow network.



s-t Cuts



Partition (A, B) of node set such that $s \in A$ and $t \in B$

- If $v_i \in A$: edge (v_i, t) is in cut (A, B)
- If $u_i \in B$: edge (s, u_i) is in cut (A, B)
- Otherwise (if $u_i \in A$, $v_i \in B$), all edges from u_i to some $v_j \in B$ are in cut (A, B)

Hall's Marriage Theorem

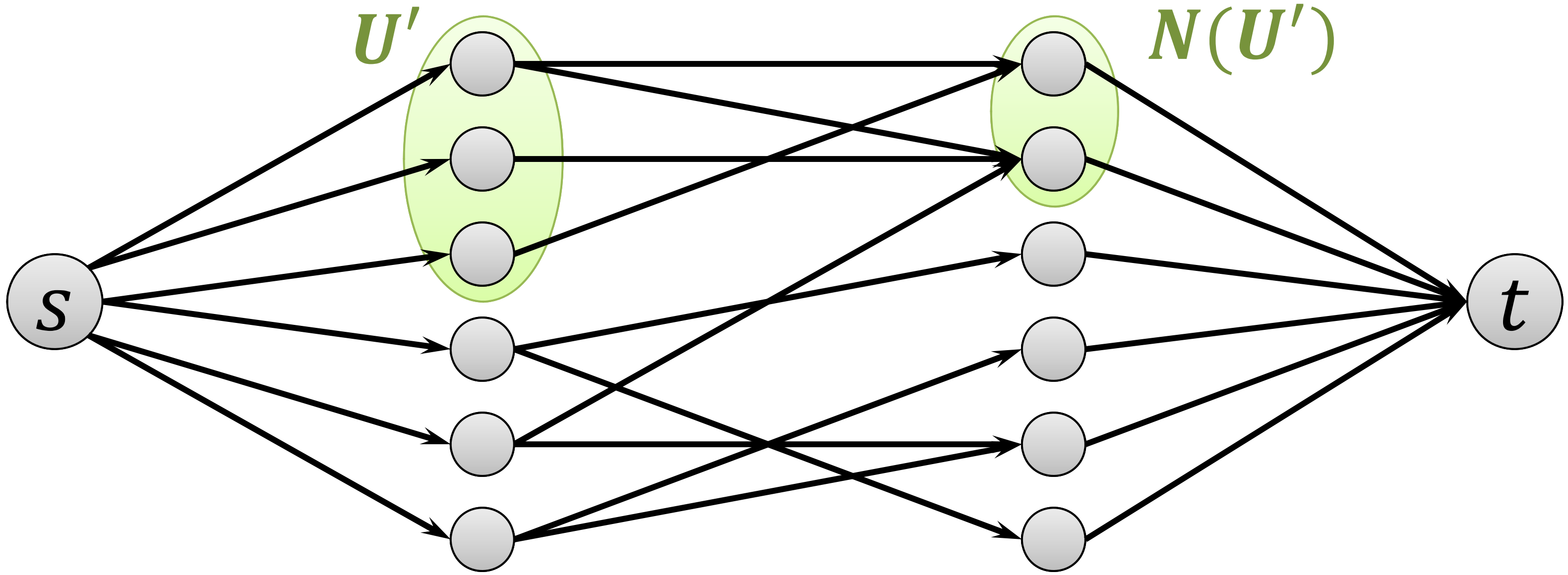
Theorem: A bipartite graph $G = (U \cup V, E)$ for which $|U| = |V|$ has a perfect matching if and only if

$$\forall \boldsymbol{U'} \subseteq \boldsymbol{U} : |\boldsymbol{N}(\boldsymbol{U'})| \geq |\boldsymbol{U'}|,$$

where $N(U') \subseteq V$ is the set of neighbors of nodes in U'.

Proof: No perfect matching $\Leftrightarrow$ some s-t cut has capacity $< n/2$

1. Assume there is U' for which $|N(U')| < |\mathrm{U}'|$:



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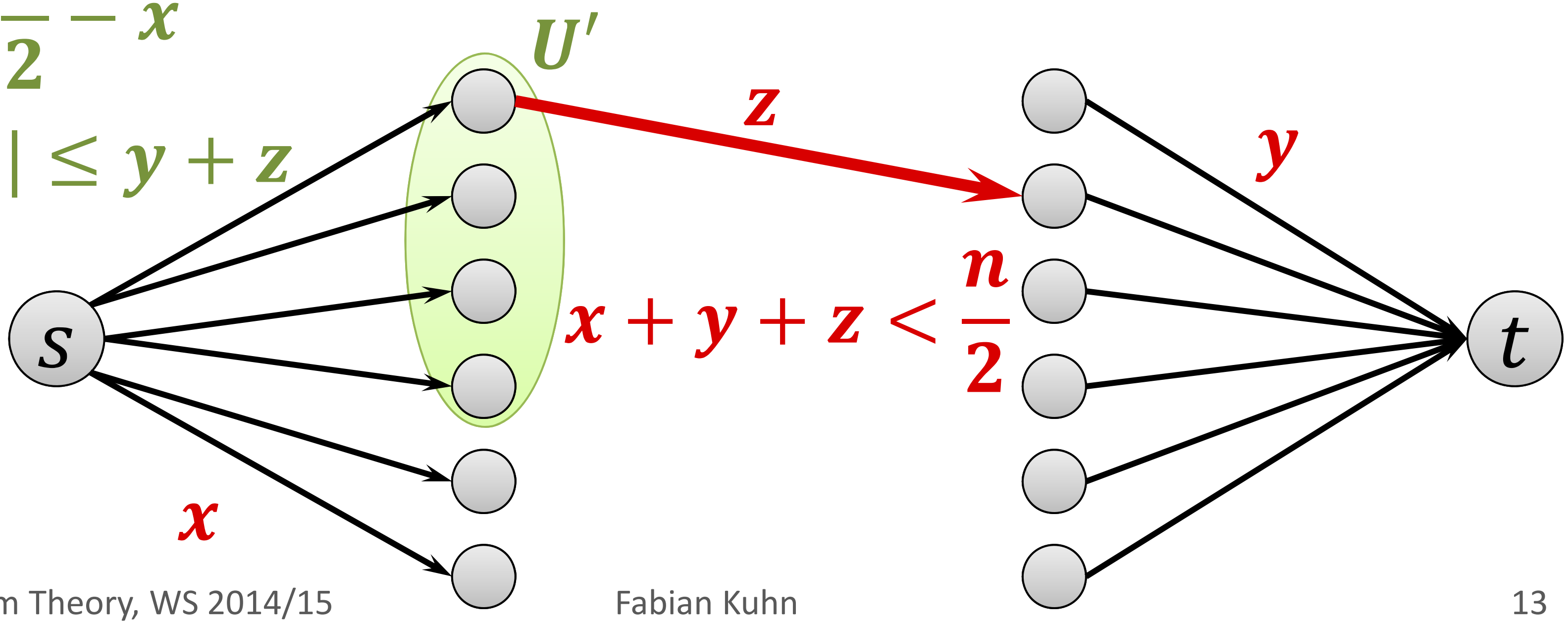
Proof: No perfect matching $\Leftrightarrow$ some s-t cut has capacity $< n/2$

2. Assume that there is a cut (A, B) of capacity $< n/2$

$$|U'| = \frac{n}{2} - x$$

$$|N(U')| \leq y + z$$



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2. Assume that there is a cut (A, B) of capacity $< n$

$$|\boldsymbol{U'}| = \frac{\boldsymbol{n}}{\boldsymbol{2}} - \boldsymbol{x}$$

$$|\boldsymbol{N}(\boldsymbol{U'})| \leq \boldsymbol{y} + \boldsymbol{z}$$

$$\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} < \frac{\boldsymbol{n}}{\boldsymbol{2}}$$

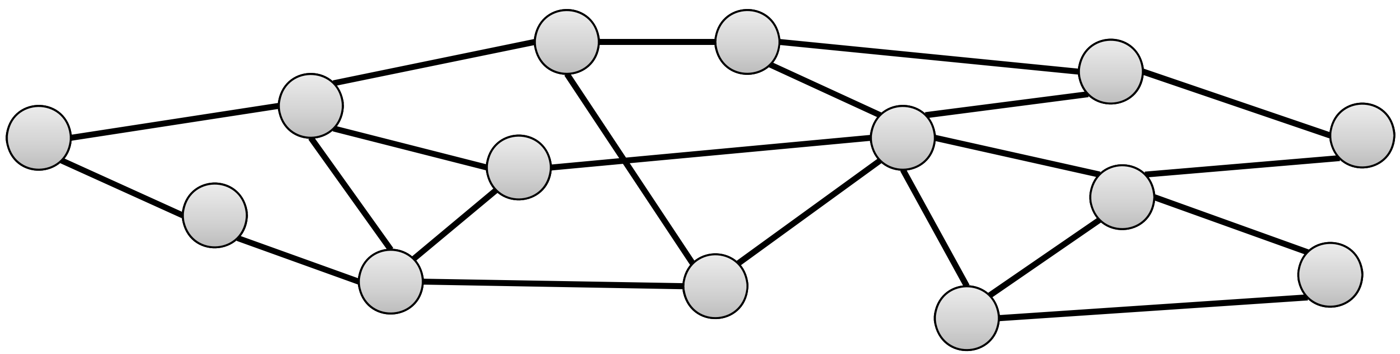
What About General Graphs

- Can we efficiently compute a maximum matching if G is not bipartite?

- How good is a maximal matching?
  - A matching that cannot be extended…

- **Vertex Cover:** set $S \subseteq V$ of nodes such that

$$\forall \{u, v\} \in E, \qquad \{u, v\} \cap S \neq \emptyset.$$



- A vertex cover covers all edges by incident nodes

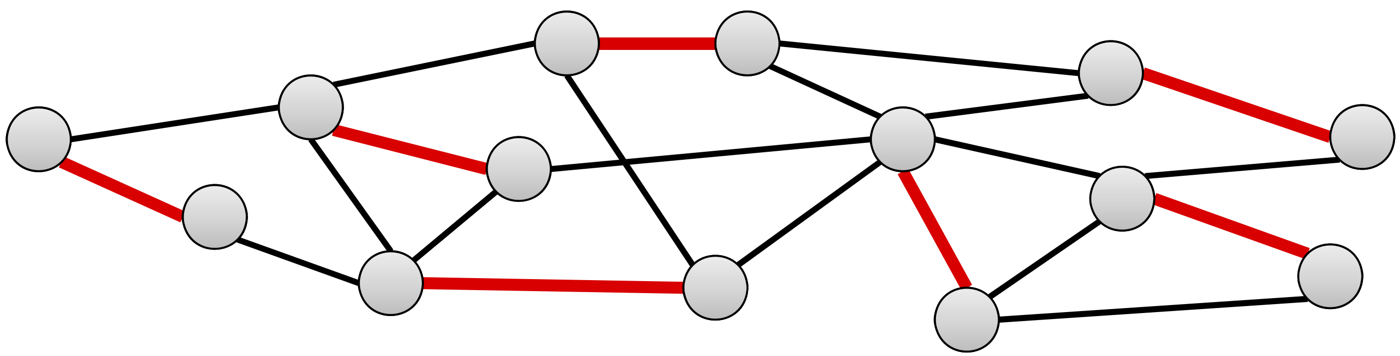
Vertex Cover vs Matching

Consider a matching M and a vertex cover S

Claim: $|M| \leq |S|$

Proof:

- At least one node of every edge $\{u, v\} \in M$ is in S
- Needs to be a different node for different edges from M



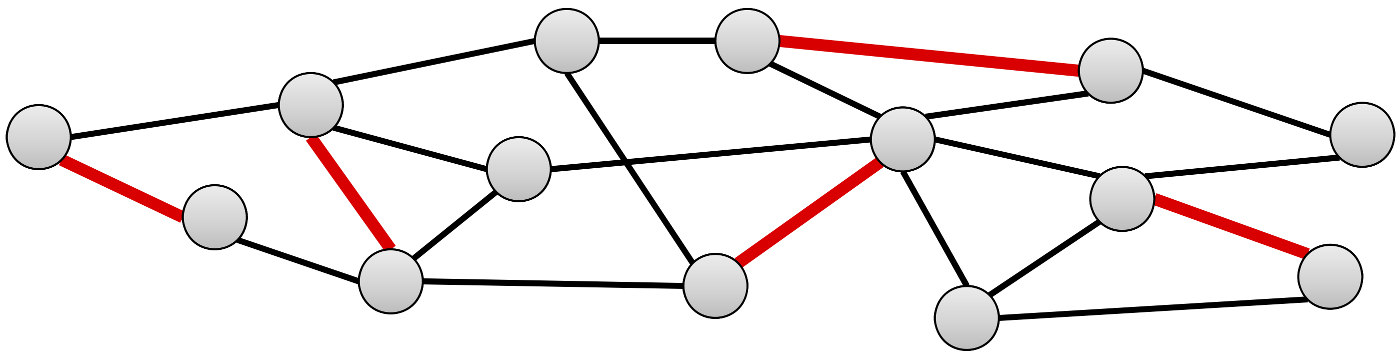
Vertex Cover vs Matching

Consider a matching M and a vertex cover S

Claim: If M is maximal and S is minimum, $|S| \leq 2|M|$

Proof:

- M is maximal: for every edge $\{u, v\} \in E$, either u or v (or both) are matched



- Every edge $e \in E$ is "covered" by at least one matching edge
- Thus, the set of the nodes of all matching edges gives a vertex cover S of size $|S| = 2|M|$.

Maximal Matching Approximation

Theorem: For any maximal matching M and any maximum matching M^*, it holds that

$$|M| \geq \frac{|M^*|}{2}.$$

Proof:

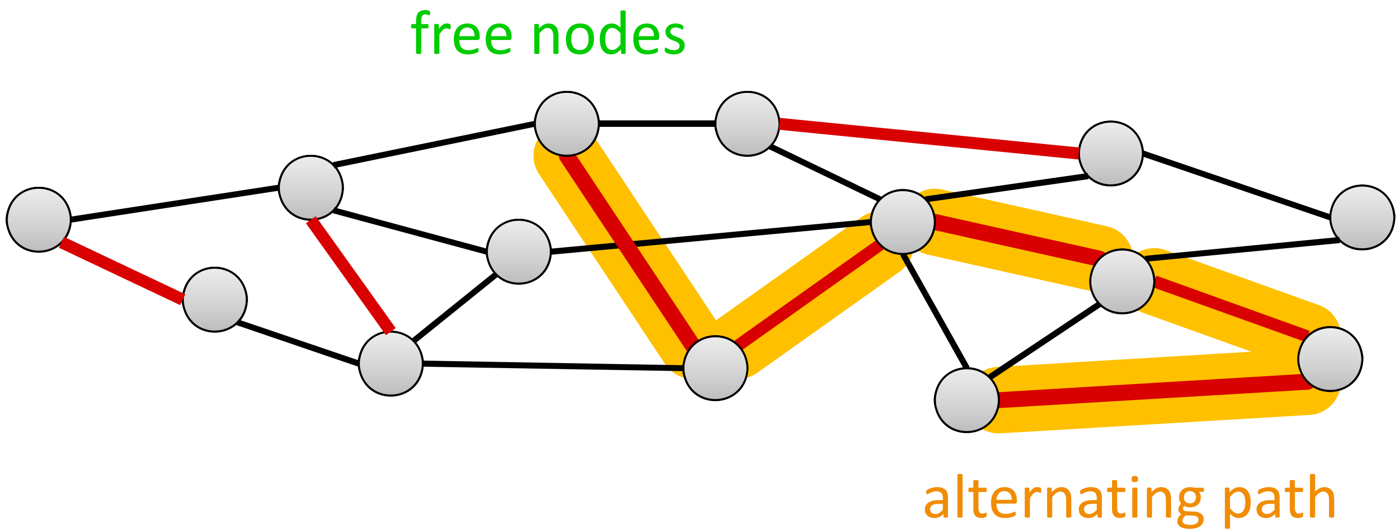
Theorem: The set of all matched nodes of a maximal matching M is a vertex cover of size at most twice the size of a min. vertex cover.

Augmenting Paths

Consider a matching M of a graph $G = (V, E)$:

- A node $v \in V$ is called **free** iff it is not matched

Augmenting Path: A (odd-length) path that starts and ends at a free node and visits edges in $E \setminus M$ and edges in M alternatingly.



- Matching M can be improved using an augmenting path by switching the role of each edge along the path

Augmenting Paths

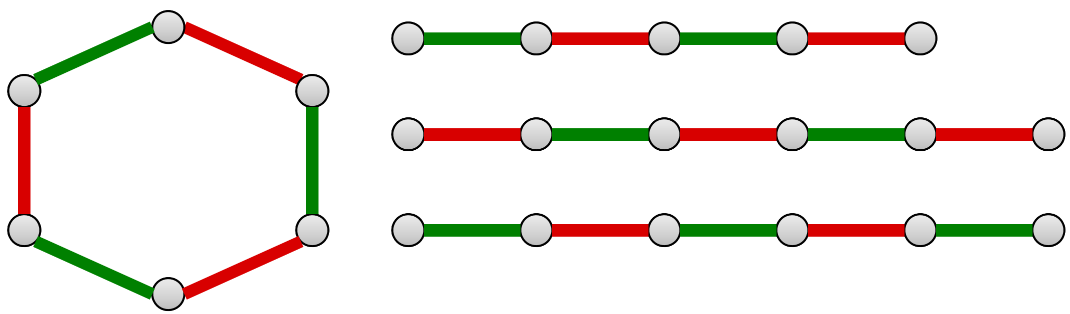
Theorem: A matching M of $G = (V, E)$ is maximum if and only if there is no augmenting path.

Proof:

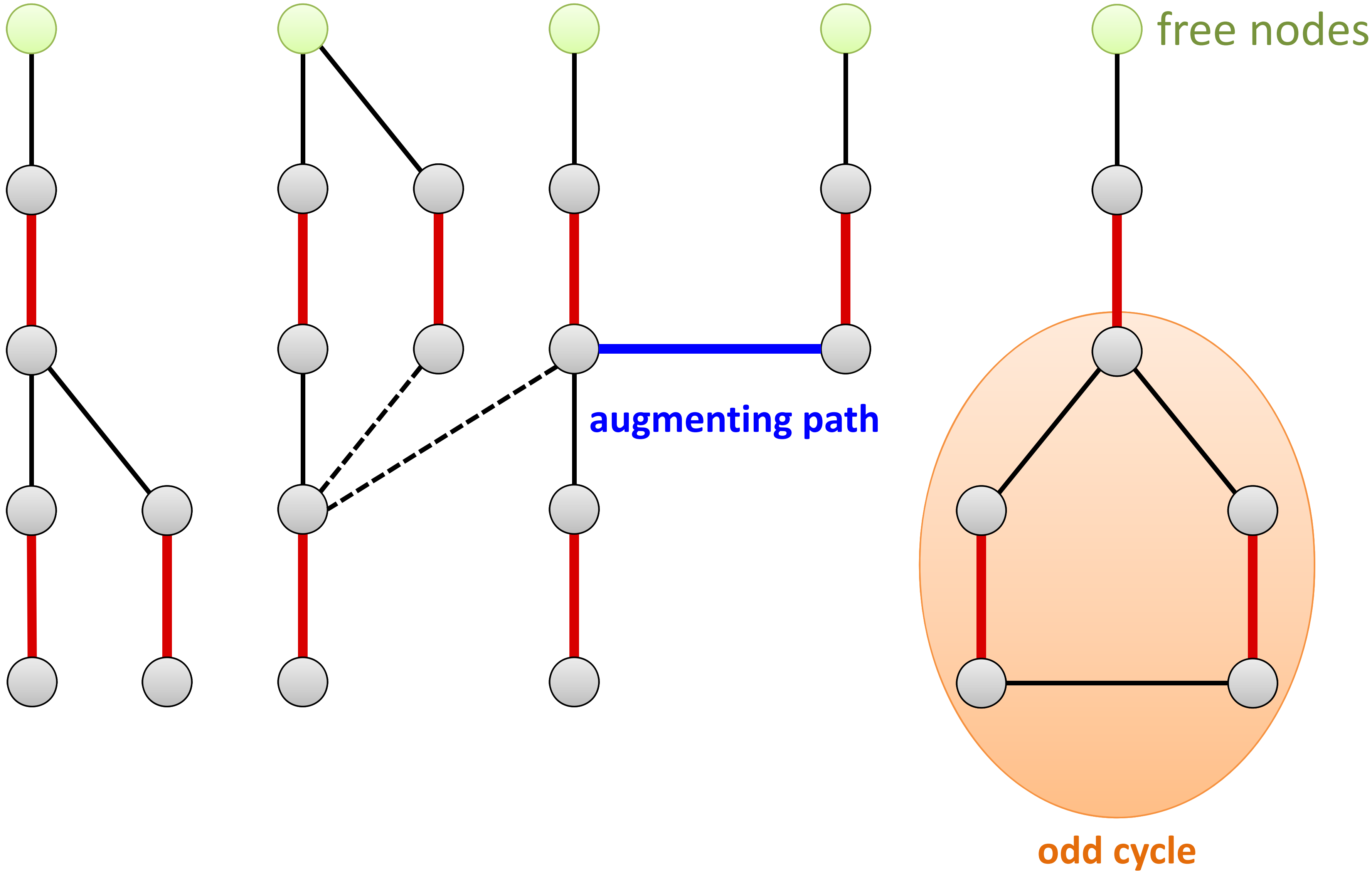
- Consider non-max. matching M and max. matching M^* and define

$$F := M \setminus M^*, \qquad F^* := M^* \setminus M$$

- Note that $F \cap F^* = \emptyset$ and $|F| < |F^*|$
- Each node $v \in V$ is incident to at most one edge in both F and F^*
- $F \cup F^*$ induces even cycles and paths

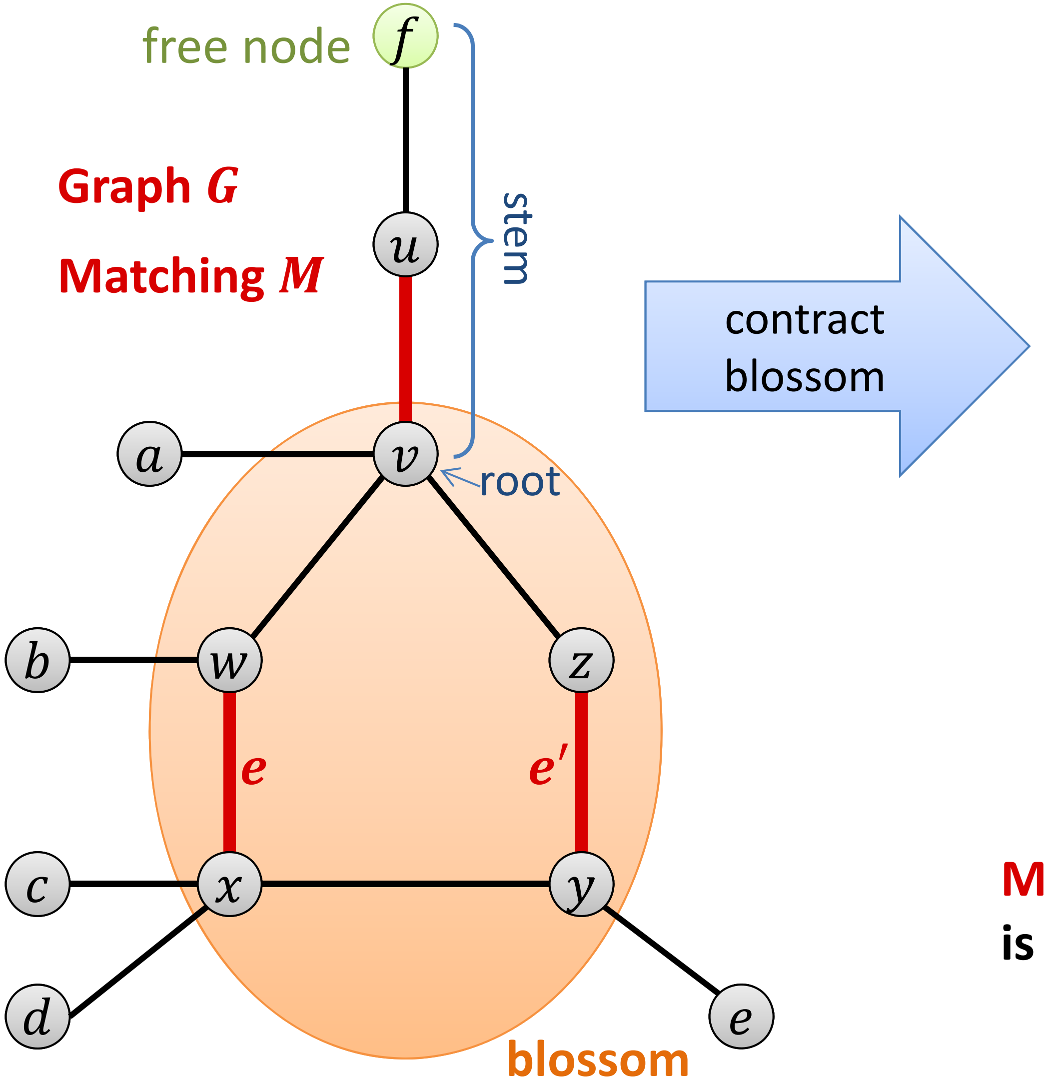


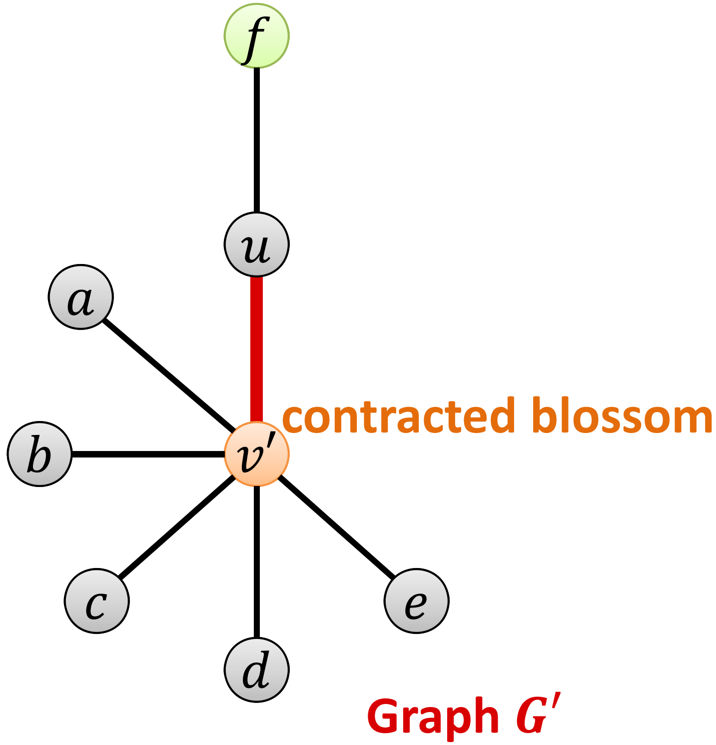
Finding Augmenting Paths



Blossoms

- If we find an odd cycle…

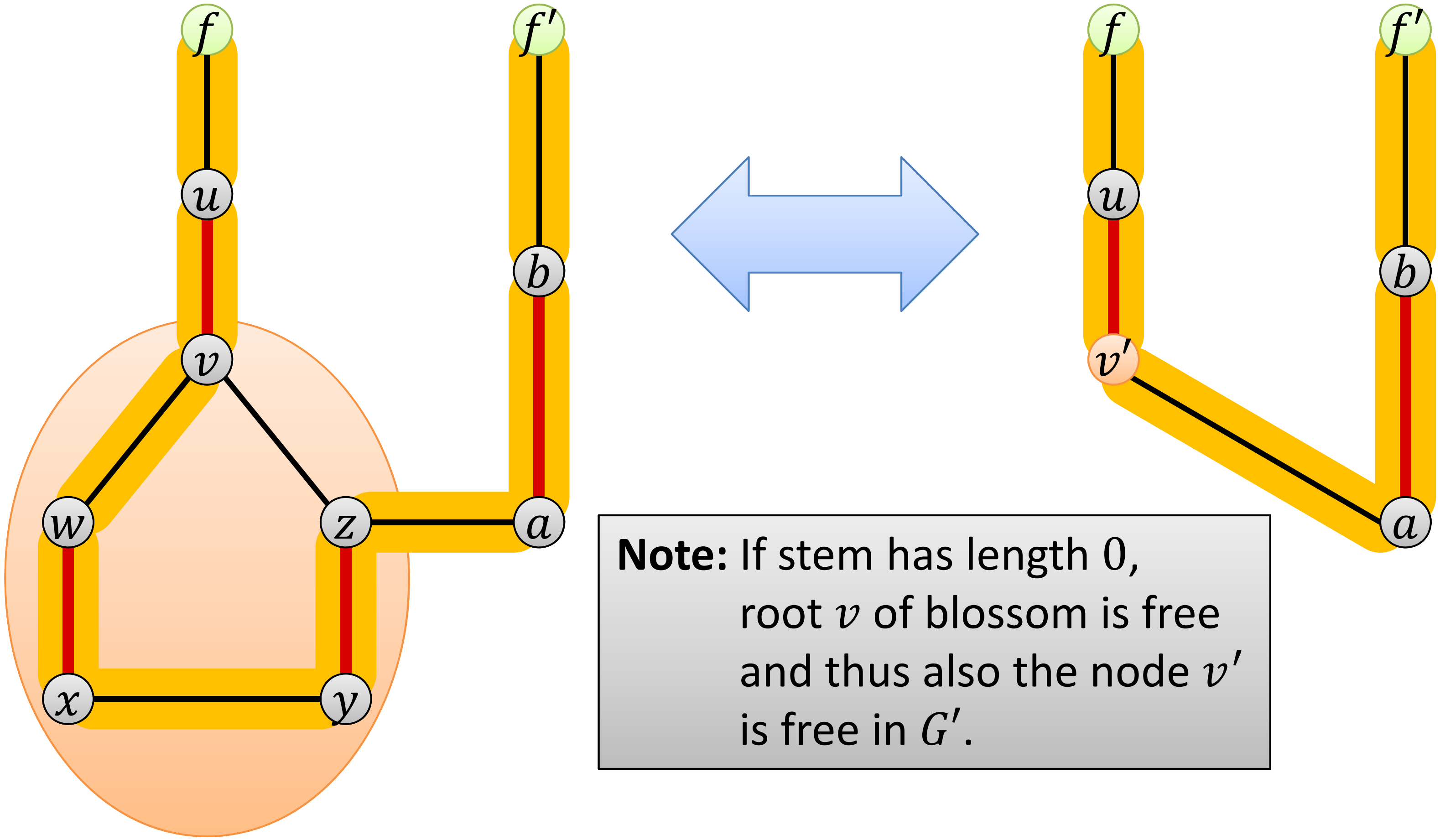




Matching $M' = M \setminus \{e, e'\}$
is a matching of G'.

Contracting Blossoms

Lemma: Graph G has an augmenting path w.r.t. matching M iff G' has an augmenting path w.r.t. matching M'



Note: If stem has length 0, root v of blossom is free and thus also the node v' is free in G'.

Also: The matching M can be computed efficiently from M'.

Edmond's Blossom Algorithm

Algorithm Sketch:

1. Build a tree for each free node
2. Starting from an explored node u at even distance from a free node f in the tree of f, explore some unexplored edge $\{u, v\}$:

   1. If v is an unexplored node, v is matched to some neighbor w:
      add w to the tree (w is now explored)

   2. If v is explored and in the same tree:
      at odd distance from root → ignore and move on
      at even distance from root → **blossom found**

   3. If v is explored and in another tree
      at odd distance from root → ignore and move on
      at even distance from root → **augmenting path found**

Running Time

Finding a Blossom: Repeat on smaller graph

Finding an Augmenting Path: Improve matching

Theorem: The algorithm can be implemented in time $O(mn^2)$.

Matching Algorithms

We have seen:

- $O(mn)$ time alg. to compute a max. matching in *bipartite graphs*
- $O(mn^2)$ time alg. to compute a max. matching in *general graphs*

Better algorithms:

- Best known running time (bipartite and general gr.): $O(m\sqrt{n})$

Weighted matching:

- Edges have weight, find a matching of **maximum total weight**
- *Bipartite graphs*: flow reduction works in the same way
- *General graphs*: can also be solved in polynomial time
  (Edmond's algorithms is used as blackbox)

Happy Holidays!