Chapter 7
Approximation Algorithms

Algorithm Theory
WS 2014/15

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Approximation Algorithms

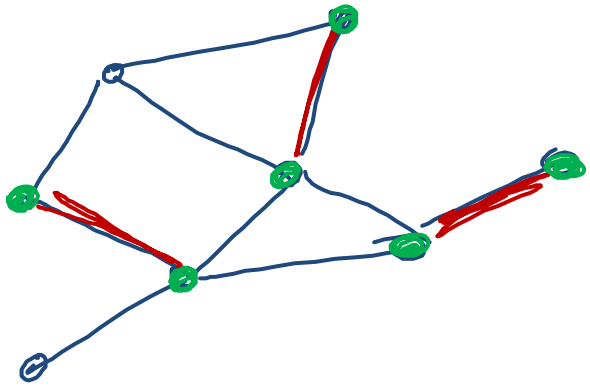
- Optimization appears everywhere in computer science
- We have seen many examples, e.g.:
  - scheduling jobs
  - traveling salesperson
  - maximum flow, maximum matching
  - minimum spanning tree
  - minimum vertex cover
  - …
- Many discrete optimization problems are NP-hard
- They are however still important and we need to solve them
- As algorithm designers, we prefer algorithms that produce solutions which are provably good, even if we can't compute an optimal solution.

Approximation Algorithms: Examples

We have already seen two approximation algorithms

- **Metric TSP:** If distances are positive and satisfy the triangle inequality, the greedy tour is only by a log-factor longer than an optimal tour

- **Maximum Matching and Vertex Cover:** A maximal matching gives solutions that are within a factor of 2 for both problems.



Approximation Ratio

An approximation algorithm is an algorithm that computes a solution for an optimization with an objective value that is provably within a bounded factor of the optimal objective value.

Formally:

- $\mathrm{OPT} \geq 0$: optimal objective value
  $\mathrm{ALG} \geq 0$: objective value achieved by the algorithm

- **Approximation Ratio α:**

$$\textbf{Minimization: } \alpha := \max_{\text{input instances}} \frac{\mathbf{ALG}}{\mathbf{OPT}} \geq 1$$

$\mathbf{OPT}$ ← min possible value

$$\textbf{Maximization: } \alpha := \max_{\text{input instances}} \frac{\mathbf{OPT}}{\mathbf{ALG}} \geq 1$$

$\mathbf{OPT}$ ← max possible value

Example: Load Balancing

We are given:

- m machines $M_1, \ldots, M_m$
- n jobs, processing time of job i is t_i

Goal:

- Assign each job to a machine such that the makespan is minimized

  makespan: largest total processing time of any machine

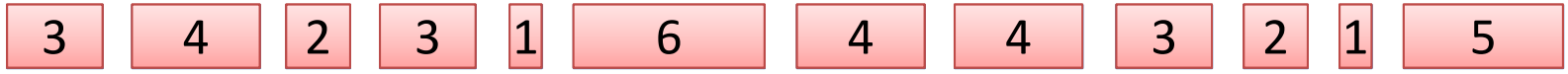
The above load balancing problem is NP-hard and we therefore want to get a good approximation for the problem.

Greedy Algorithm

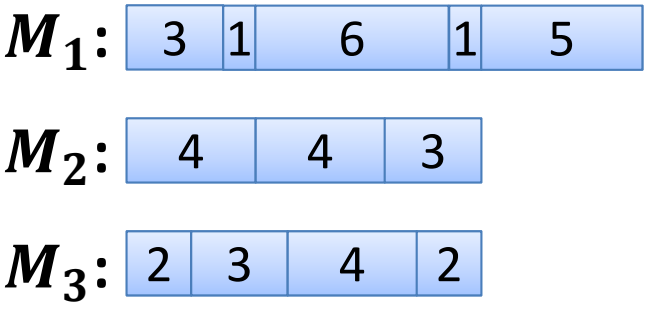
There is a simple greedy algorithm:

- Go through the jobs in an arbitrary order
- When considering job i, assign the job to the machine that currently has the smallest load.

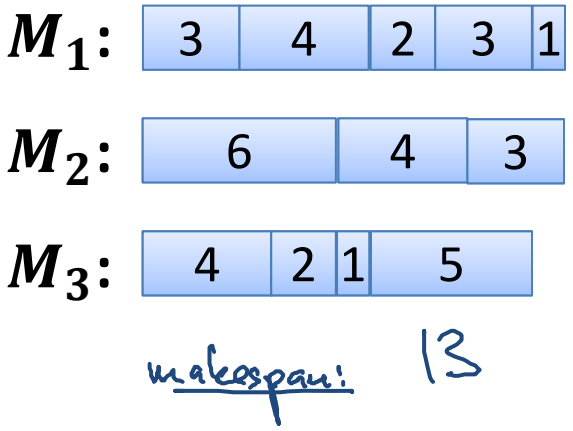
Example: 3 machines, 12 jobs



Greedy Assignment:



Optimal Assignment:



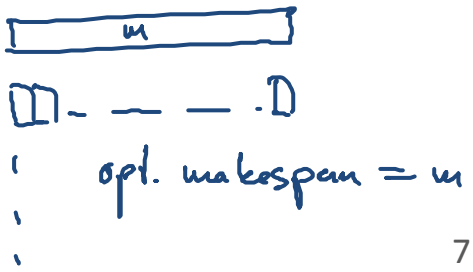
Greedy Analysis

- We will show that greedy gives a 2-approximation
- To show this, we need to compare the solution of greedy with an optimal solution (that we can't compute)
- Lower bound on the optimal makespan T^*:

$$T^* \geq \frac{1}{m} \cdot \sum_{i=1}^{n} t_i =: T_1$$

- Lower bound can be far from T^*:
  - m machines, m jobs of size 1, 1 job of size m

$$T^* = m, \qquad \frac{1}{m} \cdot \sum_{i=1}^{n} t_i = 2$$



Greedy Analysis

- We will show that greedy gives a 2-approximation
- To show this, we need to compare the solution of greedy with an optimal solution (that we can't compute)
- Lower bound on the optimal makespan T^*:

$$T^* \geq \frac{1}{m} \cdot \sum_{i=1}^{n} t_i =: \overline{T}$$

- Second lower bound on optimal makespan T^*:

$$T^* \geq \max_{1 \leq i \leq n} t_i =: \hat{T}$$

$$T^* \geq \max\{\overline{T}, \hat{T}\}$$

Greedy Analysis

Theorem: The greedy algorithm has approximation ratio ≤ 2, i.e., for the makespan T of the greedy solution, we have $T \leq 2T^*$.

Proof:

- For machine k, let T_k be the time used by machine k
- Consider some machine M_i for which $T_i = T$
- Assume that job j is the last one schedule on M_i:

- When job j is scheduled, M_i has the minimum load
  - $\forall k \in \{1, \dots, m\}: T_k \geq T - t_j$

Greedy Analysis

Theorem: The greedy algorithm has approximation ratio ≤ 2, i.e., for the makespan T of the greedy solution, we have $T \leq 2T^*$.

Proof:

- For all machines M_k: load $T_k \geq T - t_j$

$$\left.\begin{aligned} \sum_{k=1}^{m} T_k &\geq m(T - t_j) \\ \sum_{i=1}^{n} t_i &= m \cdot \bar{T} \end{aligned}\right\} \quad \bar{T} \geq T - t_j, \quad \bar{T} \leq T^* \Rightarrow T - t_j \leq T^*$$

also: $T^* \geq t_j$

$$T = \underbrace{T - t_j}_{\leq T^*} + \underbrace{t_j}_{\leq T^*} \leq 2T^*$$

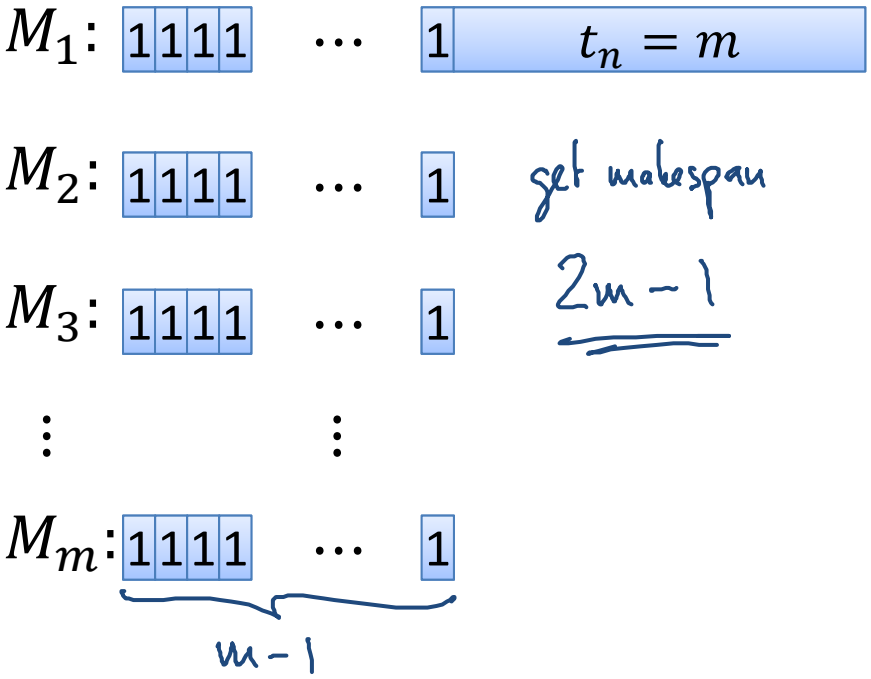
□

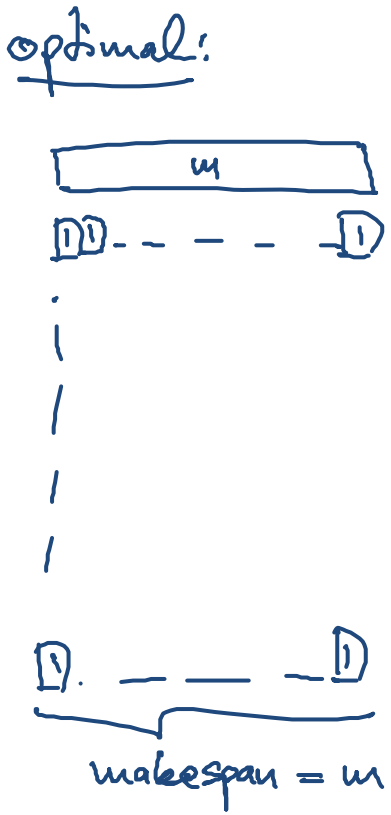
Can We Do Better?

The analysis of the greedy algorithm is almost tight:

- Example with $n = m(m-1) + 1$ jobs
- Jobs $1, \dots, n-1 = m(m-1)$ have $t_i = 1$, job n has $t_n = m$

Greedy Schedule:





Improving Greedy

Bad case for the greedy algorithm:
One large job in the end can destroy everything

Idea: assign large jobs first

Modified Greedy Algorithm:

1. Sort jobs by decreasing length s.t. $t_1 \geq t_2 \geq \cdots \geq t_n$
2. Apply the greedy algorithm as before (in the sorted order)

Lemma: If $n > m$: $T^* \geq t_m + t_{m+1} \geq 2t_{m+1}$

$t_m \geq t_{m+1}$

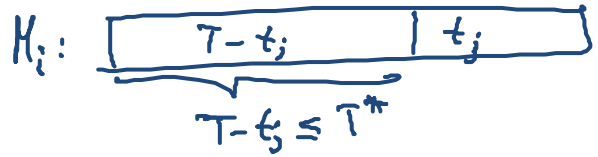
Proof:

- Two of the first $m+1$ jobs need to be scheduled on the same machine
- Jobs m and $m+1$ are the shortest of these jobs

Analysis of the Modified Greedy Alg.

Theorem: The modified algorithm has approximation ratio $\leq {}^3/_2$.

Proof:



- We need to show that $T \leq {}^3/_2 \cdot T^*$
- As before, we consider the machine M_i with $T_i = T$
- Job j (of length t_j) is the last one scheduled on machine M_i
- If j is the only job on M_i, we have $T = T^*$ $(T^* \geq t_j)$
- Otherwise, we have $j \geq m + 1$
  - The first m jobs are assigned to m distinct machines

$\Rightarrow j \geq m+1 \Rightarrow t_j \leq t_{m+1}$

$T^* \geq 2 \cdot t_{m+1} \Rightarrow t_j \leq \frac{T^*}{2}$

$T = \underbrace{T - t_j}_{\leq T^*} + \underbrace{t_j}_{\leq \frac{1}{2} \cdot T^*} \leq \frac{3}{2} \cdot T^*.$

$1, 1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}$

$1 \quad \frac{2}{3} \quad \frac{2}{3}$

$1 \quad \frac{2}{3}$

Metric TSP

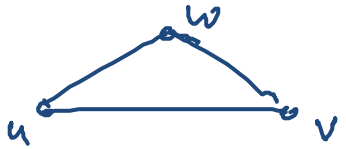
Input:

- Set V of n nodes (points, cities, locations, sites)
- Distance function $d: V \times V \to \mathbb{R}$, i.e., $d(u,v)$: dist. from u to v
- Distance define a metric on V:

$$d(u,v) = d(v,u) \geq 0, \qquad d(u,v) = 0 \iff u = v$$

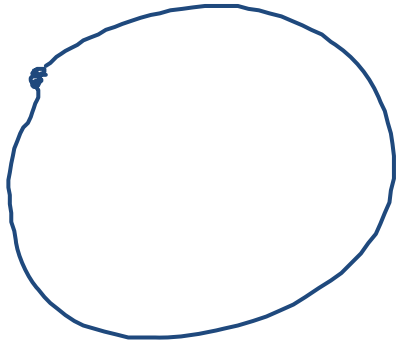
$$d(u,v) \leq d(u,w) + d(v,w)$$

Δ-inequality



Solution:

- Ordering/permutation $v_1, v_2, \ldots, v_n$ of vertices
- Length of TSP path: $\sum_{i=1}^{n-1} d(v_i, v_{i+1})$
- Length of TSP tour: $d(v_n, v_1) + \sum_{i=1}^{n-1} d(v_i, v_{i+1})$



Goal:

- Minimize length of TSP path or TSP tour

Metric TSP

- The problem is NP-hard
- We have seen that the greedy algorithm (always going to the nearest unvisited node) gives an $O(\log n)$-approximation
- Can we get a constant approximation ratio?
- We will see that we can…

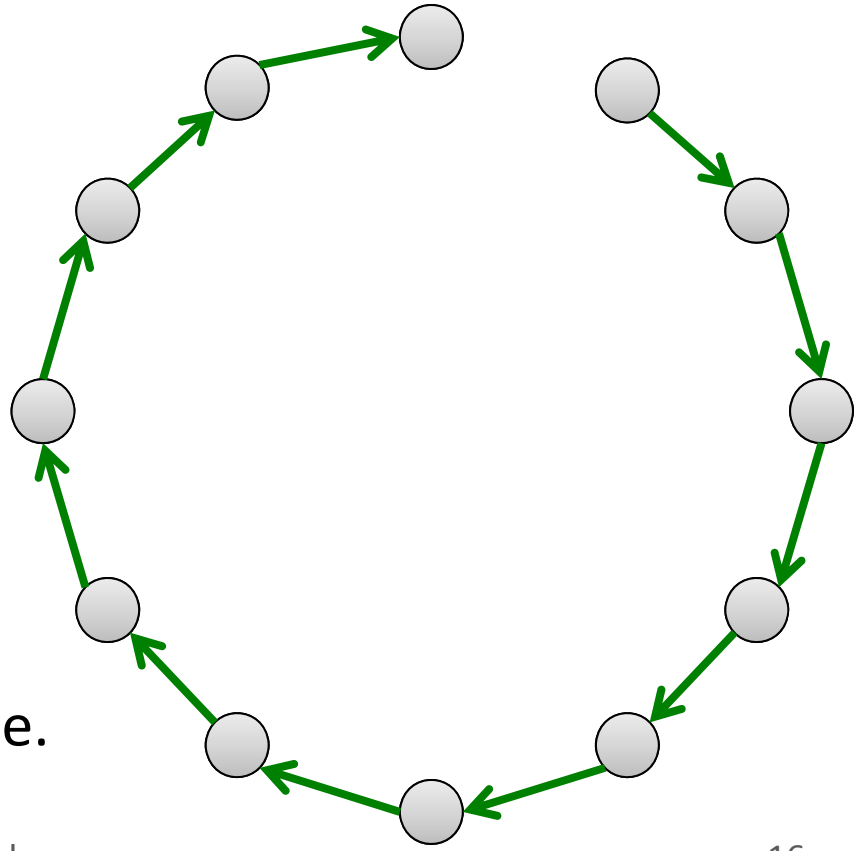
TSP and MST

Claim: The length of an optimal TSP path is lower bounded by the weight of a minimum spanning tree

Proof:

- A TSP path is a spanning tree, it's length is the weight of the tree

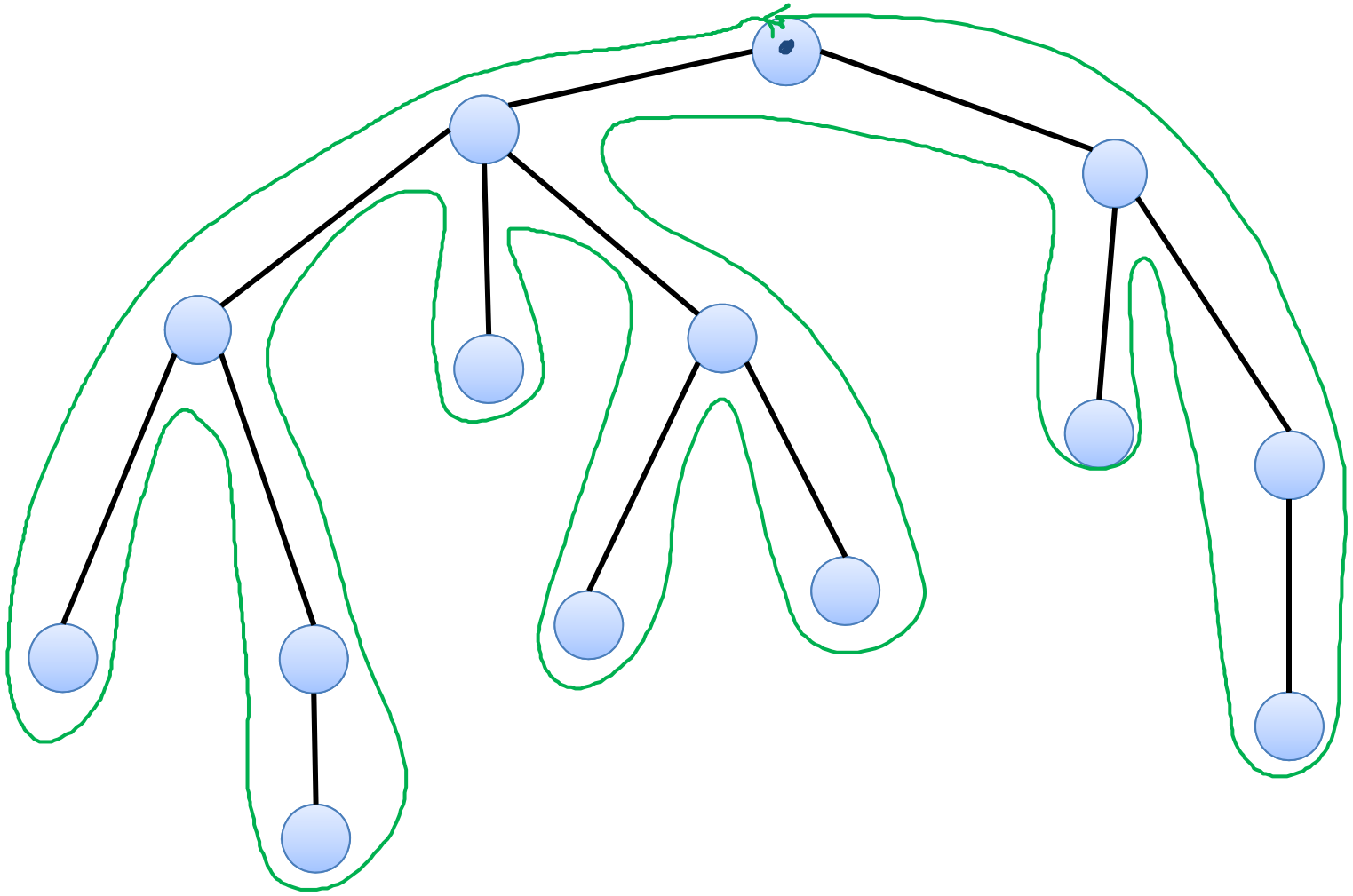
TSP tour $\geq$ TSP path $\geq$ MST



Corollary: Since an optimal TSP tour is longer than an optimal TSP path, the length of an optimal TSP tour is also lower bounded by the weight of a minimum spanning tree.

The MST Tour

Walk around the MST…

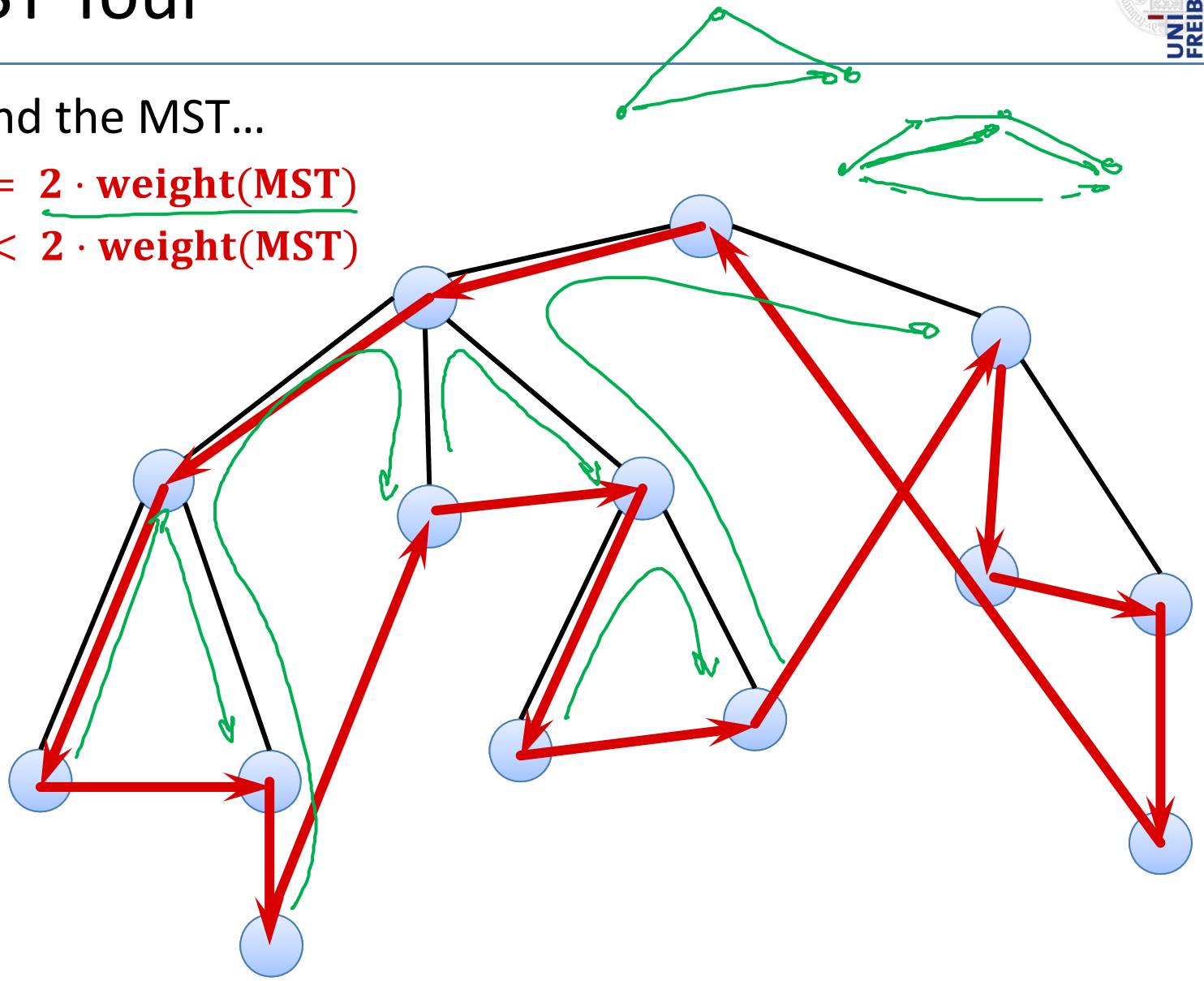


The MST Tour

Walk around the MST…

Cost (walk) $= 2 \cdot \mathbf{weight(MST)}$

Cost (tour) $< 2 \cdot \mathbf{weight(MST)}$



Approximation Ratio of MST Tour

Theorem: The MST TSP tour gives a 2-approximation for the metric TSP problem.

Proof:

- Triangle inequality → length of tour is at most $2 \cdot \text{weight}(\text{MST})$
- We have seen that $\text{weight}(\text{MST}) < \text{opt. tour length}$

Can we do even better?

Metric TSP Subproblems

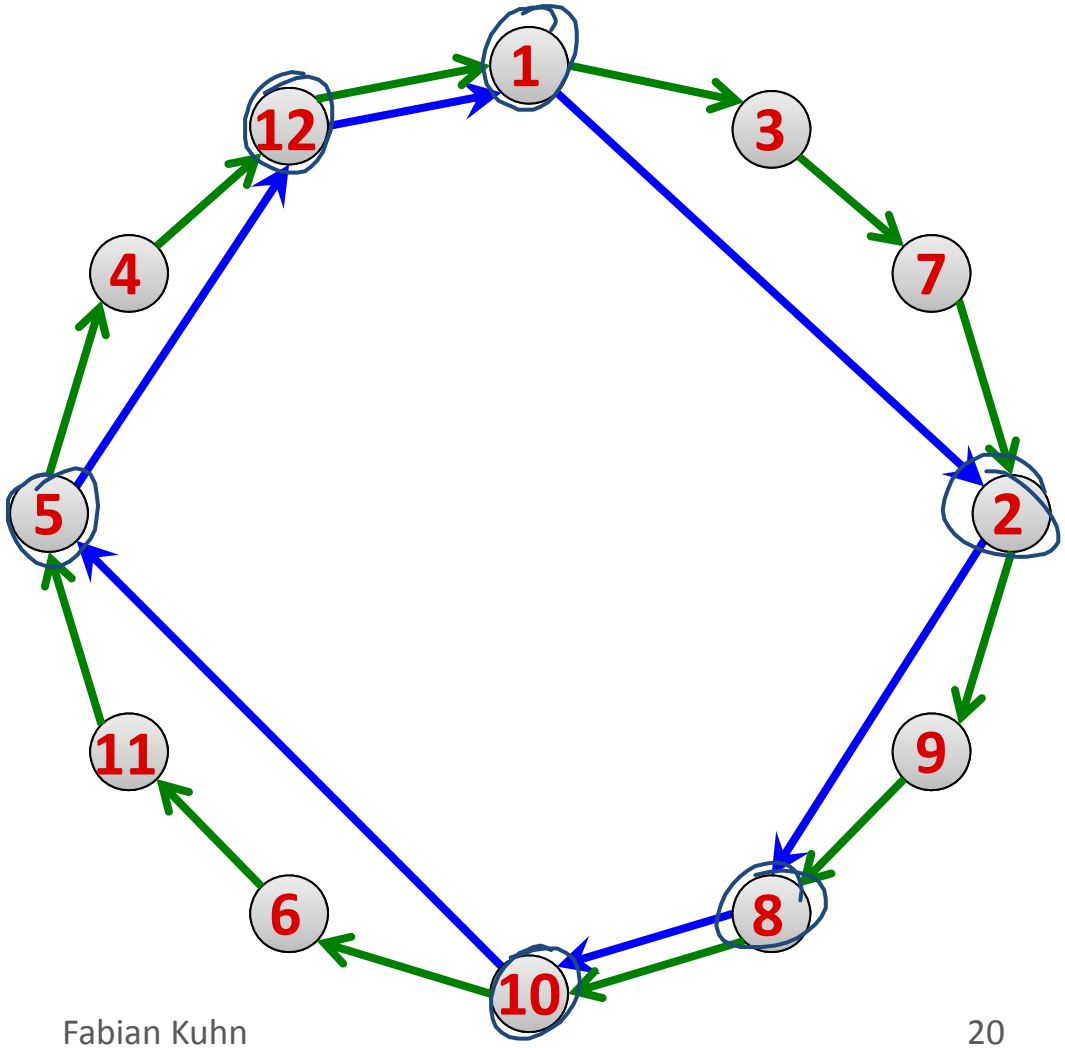
Claim: Given a metric (V, d) and (V', d) for $V' \subseteq V$, the optimal TSP path/tour of (V', d) is at most as large as the optimal TSP path/tour of (V, d).

Optimal TSP tour of nodes $1, 2, \dots, 12$

Induced TSP tour for nodes $1, 2, 5, 8, 10, 12$

blue tour $\leq$ green tour

triangle inequality



TSP and Matching

- Consider a metric TSP instance (V, d) with an even number of nodes $|V|$

- Recall that a perfect matching is a matching $M \subseteq V \times V$ such that every node of V is incident to an edge of M.

- Because $|V|$ is even and because in a metric TSP, there is an edge between any two nodes $u, v \in V$, any partition of V into $|V|/2$ pairs is a perfect matching.

- The weight of a matching M is the sum of the distances represented by all edges in M:

$$w(M) = \sum_{\{u,v\} \in M} d(u,v)$$

TSP and Matching

$TSP_{opt} \geq 2 \cdot$ "min. weight perf. matching"

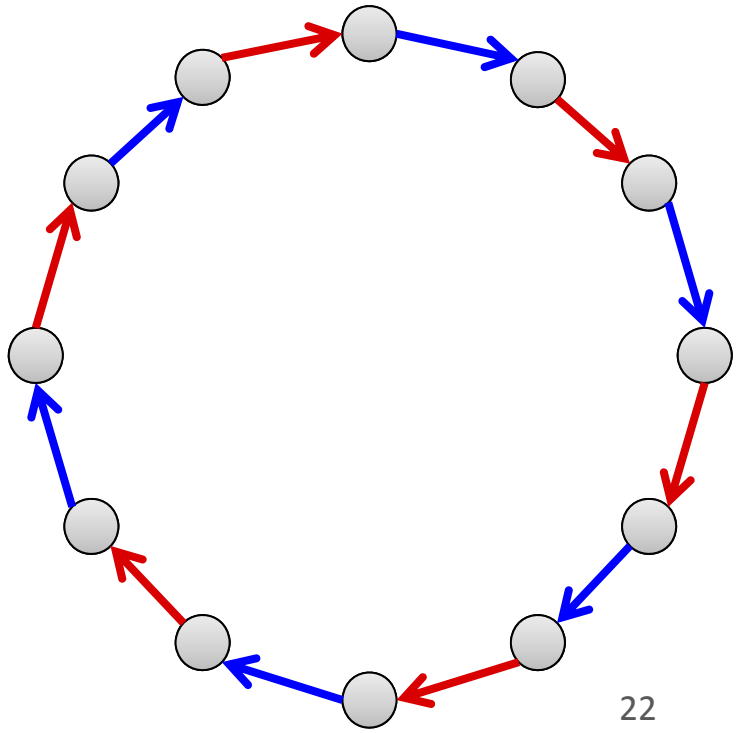
Lemma: Assume we are given a TSP instance (V, d) with an even number of nodes. The length of an optimal TSP tour of (V, d) is at least twice the weight of a minimum weight perfect matching of (V, d).

Proof:

- The edges of a TSP tour can be partitioned into 2 perfect matchings

red: perf. matching
blue: perf. matching

TSP_{OPT} = "red" + "blue"
$\geq$ min. perf. m.



Minimum Weight Perfect Matching

Claim: If $|V|$ is even, a minimum weight perfect matching of (V, d) can be computed in polynomial time

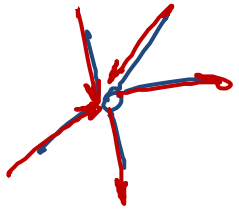
Proof Sketch:

- We have seen that a maximum matching in an unweighted graph can be computed in polynomial time
- With a more complicated algorithm, also a maximum weighted matching can be computed in polynomial time
- In a complete graph, a maximum weighted matching is also a (maximum weight) perfect matching
- Define weight $w(u, v) := D - d(u, v)$
- A maximum weight perfect matching for (V, w) is a minimum weight perfect matching for (V, d)

Algorithm Outline

Problem of MST algorithm:

- Every edge has to be visited twice



Goal:

- Get a graph on which every edge only has to be visited once (and where still the total edge weight is small compared to an optimal TSP tour)

Euler Tours:

- A tour that visits each edge of a graph exactly once is called an Euler tour
- An Euler tour in a (multi-)graph exists if and only if every node of the graph has even degree
- That's definitely not true for a tree, but can we modify our MST suitably?

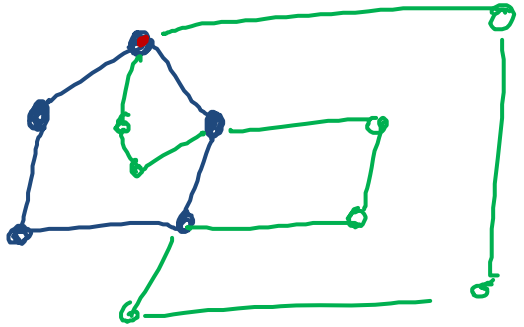
Euler Tour

Theorem: A connected (multi-)graph G has an Euler tour if and only if every node of G has even degree.

Proof:

- If G has an odd degree node, it clearly cannot have an Euler tour
- If G has only even degree nodes, a tour can be found recursively:

1. Start at some node
2. As long as possible, follow an unvisited edge
   - Gives a partial tour, the remaining graph still has even degree

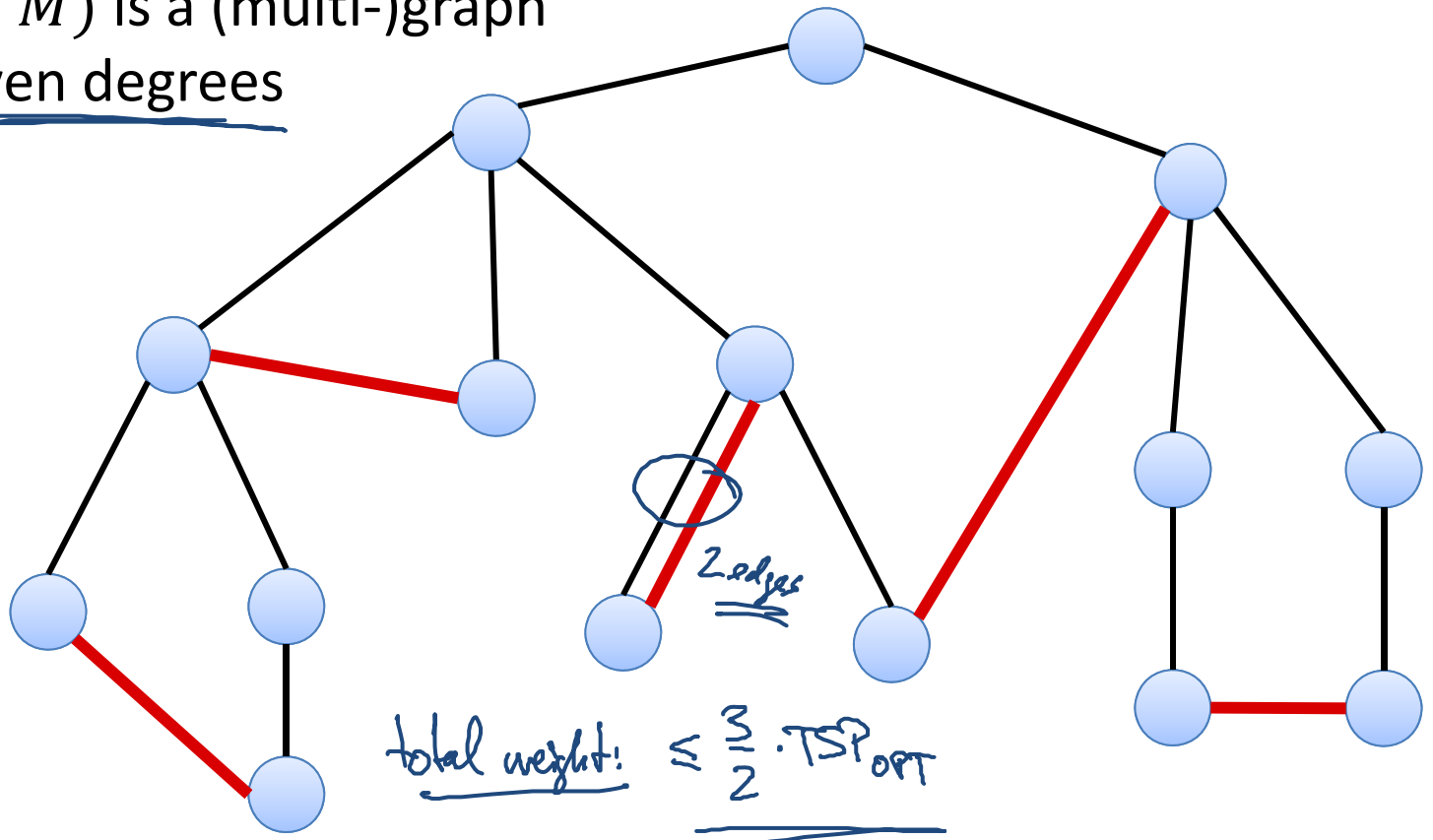


3. Solve problem on remaining components recursively
4. Merge the obtained tours into one tour that visits all edges

TSP Algorithm

$\sum_{v \in V} \deg(v) = 2 \cdot m$

1. Compute MST T
2. V_{odd}: nodes that have an odd degree in T ($|V_{\text{odd}}|$ is even)
3. Compute min weight perfect matching M of (V_{odd}, d)
4. $(V, T \cup M)$ is a (multi-)graph with even degrees



black edges:

$MST \leq TSP_{OPT}$

red edges:

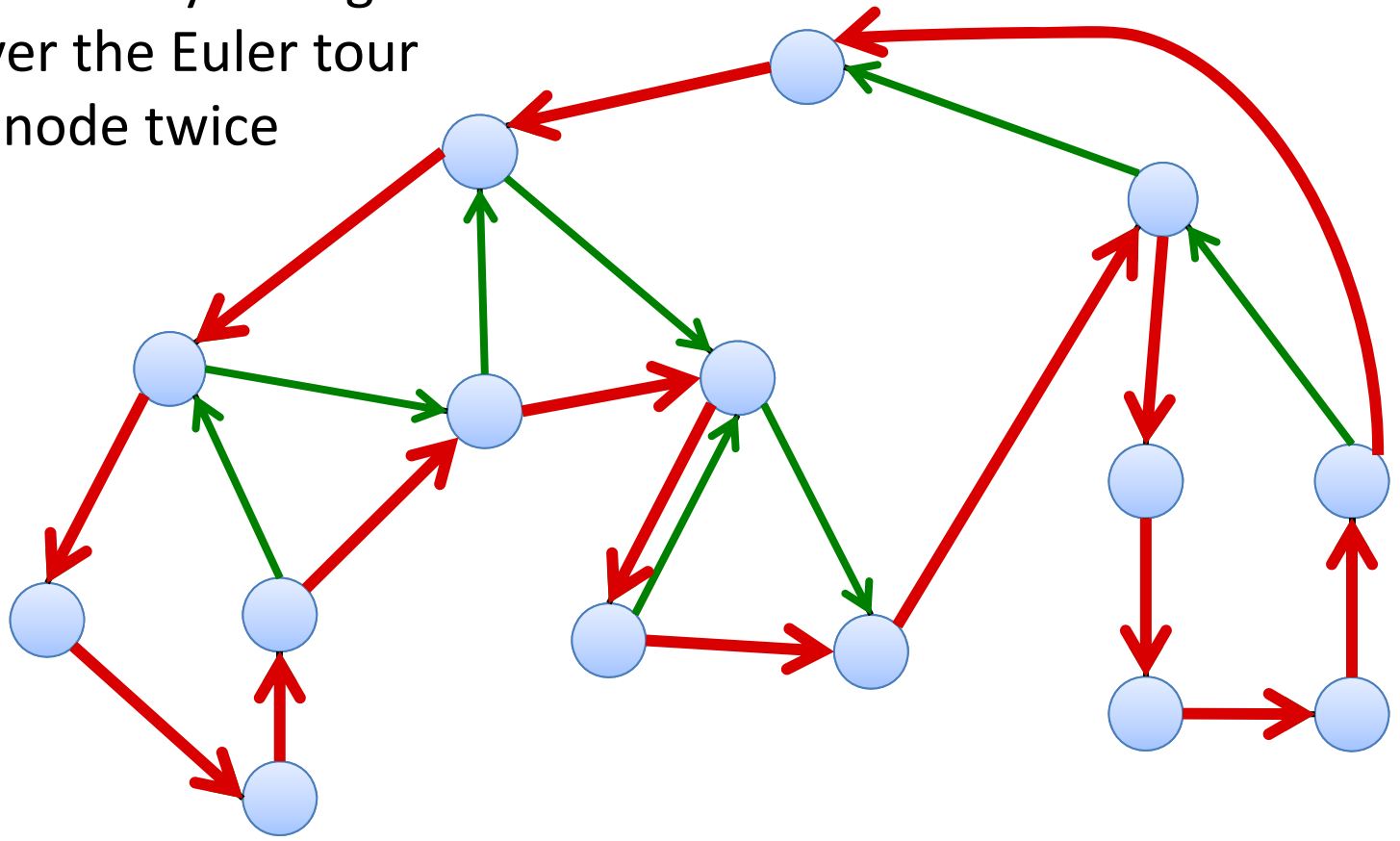
$PM \leq \frac{1}{2} TSP_{OPT}(V_{odd}, d)$

$\leq \frac{1}{2} TSP_{OPT}$

total weight: $\leq \frac{3}{2} \cdot TSP_{OPT}$

TSP Algorithm

5. Compute Euler tour on $(V, T \cup M)$
6. Total length of Euler tour $\leq \frac{3}{2} \cdot \mathbf{TSP_{OPT}}$
7. Get TSP tour by taking shortcuts wherever the Euler tour visits a node twice



TSP Algorithm

- The described algorithm is by Christofides

Theorem: The Christofides algorithm achieves an approximation ratio of at most $^3/_2$.

Proof:

- The length of the Euler tour is $\leq {}^3/_2 \cdot \mathrm{TSP}_{\mathrm{OPT}}$
- Because of the triangle inequality, taking shortcuts can only make the tour shorter

Knapsack

- n items $1, \dots, n$, each item has weight $w_i > 0$ and value $v_i > 0$
- Knapsack (bag) of capacity W

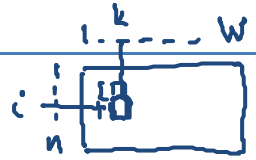
- Goal: pack items into knapsack such that total weight is at most W and total value is maximized:

$$\max \sum_{i \in S} v_i$$

$$\text{s.t. } S \subseteq \{1, \dots, n\} \text{ and } \sum_{i \in S} w_i \leq W$$

- E.g.: jobs of length w_i and value v_i, server available for W time units, try to execute a set of jobs that maximizes the total value

Knapsack: Dynamic Programming Alg.



We have shown:

- If all item weights w_i are integers, using dynamic programming, the knapsack problem can be solved in time $O(nW)$
- If all values v_i are integers, there is another dynamic progr. algorithm that runs in time $O(n^2V)$, where V is the max. value.

min. total weight to achieve total value exactly k with only elem. 1,..,i

Problems:

- If W and V are large, the algorithms are not polynomial in n
- If the values or weights are not integers, things are even worse (and in general, the algorithms cannot even be applied at all)

Idea:

- Can we adapt one of the algorithms to at least compute an approximate solution?