Exercise 1: Maximum Matching (6 points)

We are given bipartite graph \( B = (U \cup V, E) \) on two disjoint node sets \( U \) and \( V \); each edge connects a node in \( U \) and a node in \( V \). In the following, we define a 2-claw to be a set of three distinct nodes \( \{u, x, y\} \) such that \( u \in U \), \( x, y \in V \) and there is an edge from \( u \) to both nodes \( x \) and \( y \).

We consider the following maximization problem on the graph \( B \): Find a largest possible set of vertex-disjoint set of 2-claws. In other words, we want to find a largest possible subset of edges such that every node in \( U \) is incident to either 0 or 2 of the edges and each node in \( V \) is incident to either 0 or 1 of the edges (i.e., each node is either part of one 2-claw or it is not part of any 2-claw at all).

(a) (3 Points) Show that picking vertex-disjoint 2-claws in a greedy manner (as long as there is a 2-claw which is vertex-disjoint to all previously picked 2-claws, we pick it) results in a set of 2-claws which is at least one-third as large as an optimal set of vertex-disjoint 2-claws.

(b) (3 Points) In order to solve the problem optimally, let us now assume that in the given bipartite graph \( B \), each node in \( U \) has at most 3 neighbors in \( V \). Give a polynomial-time algorithm which computes a maximum set of vertex-disjoint 2-claws. You can use algorithms from the lecture as a subroutine.

Hint: Try to reduce the problem to the maximum matching problem in general graphs.

Exercise 2: Triangles in Random Graphs (10 points)

Given a fixed vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) with \( n \) being an even number. Then the following (random) process defines the (undirected) random graph \( G_p = (V, E_p) \):

For each vertex pair \( \{v_i, v_j\}, i \neq j \) we independently decide with probability \( p \) whether the edge defined by this pair is part of the graph, i.e., whether \( \{v_i, v_j\} \) is an element of the edge set \( E_p \).

Furthermore we say that a subset \( T = \{v_i, v_j, v_k\} \) of \( V \) of size 3 is a triangle of a graph, if all three edges \( \{v_i, v_j\}, \{v_i, v_k\}, \{v_j, v_k\} \) are in the edge set of the graph.

(a) (1 Point) Let \( Z \) be the random variable that counts the number of edges in \( G_p \). What kind of random variable is \( Z \)? What is the probability that \( Z \) has value \( k \), for some \( k \)?

(b) (1 Point) Calculate \( m_T \), the number of all triangles that could possibly occur in \( G_p \).

(c) (2 Points) Let \( X \) denote the number of triangles in \( G_p \). Calculate \( E[X] \).
The generation of the random graphs is now changed as follows. Before edges are determined each vertex is colored either red or green; we let $K$ be the random variable that counts the number of red vertices. Between two red vertices there is an edge with probability $p_{rr}$, between two green vertices with probability $p_{gg}$ and between vertices of different color with probability $p_{rg}$ (edges are still picked independently).

(d) (3 Points) Assume first that with probability $\frac{1}{7}$ all vertices are red, with probability $\frac{2}{7}$ all vertices are green and with probability $\frac{4}{7}$ each vertex independently gets color red or green with probability $1/2$ each. Also $p_{rr} = 1$, $p_{rg} = \frac{1}{\sqrt{3}}$ and $p_{gg} = 0$. Calculate $\mathbb{E}[X]$ under these conditions!

(e) (3 Points) Let us now assume that $K$ is not known, but it is known that $K \sim \text{Uniform}[1, n]$, i.e., in the painting process first the number of red vertices is determined and then $K$ vertices are being selected to be red. The edge probabilities are the same as in the question above. Consider a vertex $v \in V$. Conditioned on the event that $v$ is red, what is the probability that $v$ is not part of any triangle?