Algorithm Theory, Winter Term 2015/16
Problem Set 12

hand in (hard copy or electronically) by 10:15, Thursday January 28, 2016,
tutorial session will be on February 1, 2016

Exercise 1: Covering as many Elements as Possible (8 points)

We consider the following variant of the set cover problem discussed in the lecture. We are given a set of elements \( X \) and a collection \( S \subseteq 2^X \) of subsets of \( X \) such that \( \bigcup_{S \in S} S = X \). In addition, we are given an integer parameter \( k \geq 2 \).

Instead of finding a collection \( C \subseteq S \) of the sets which covers all elements, the goal is to find a set of (at most) \( k \) sets \( S_1, \ldots, S_k \in S \) such that the number of covered elements \( |S_1 \cup \cdots \cup S_k| \) is maximized.

We consider the greedy set cover algorithm from the lecture, but we stop the algorithm after adding \( k \) sets.

(a) (2 points) Show that for \( k = 2 \), the described greedy algorithm has approximation ratio at most \( \frac{4}{3} \).

(b) (4 points) Let us now consider a general parameter \( k \geq 2 \). Show that if an optimal choice of \( k \) sets \( S_1, \ldots, S_k \) covers \( \ell \) elements, after adding \( t \) sets, the greedy algorithm covers at least \( \frac{\ell}{k} \cdot \sum_{i=1}^{t} (1 - \frac{1}{k})^{i-1} \) elements.

(c) (2 points) Prove that the approximation ratio of the greedy algorithm is at most \( \frac{e}{e-1} \). You can use that \( (1 - 1/k)^k < e^{-1} \).

Exercise 2: TSP in Graphs with Edge Weights 1 and 2 (4 points)

Consider the family of complete undirected graph \( G \) in which all edges have length either 1 or 2. Give a 4/3-approximation for the TSP problem for this family of graphs. Note that \( G \) satisfies the triangle inequality.

Hint: Start with a minimum 2-matching in \( G \). A 2-matching is a subset \( M \) of edges so that every vertex in \( G \) is incident to exactly two edges in \( M \). You can assume that a minimum 2-matching can be computed in polynomial time.