Chapter 1
Divide and Conquer

Algorithm Theory
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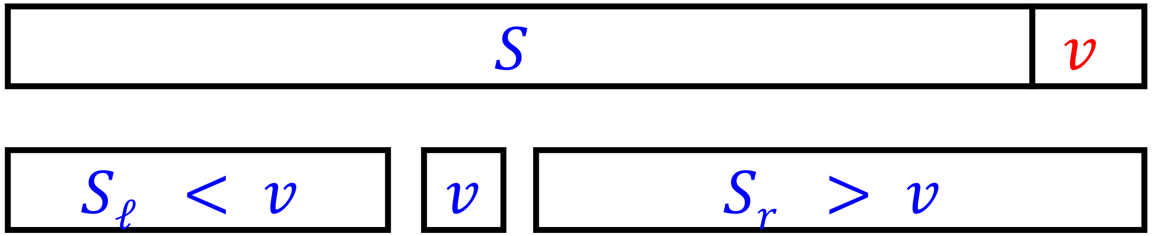
Divide-And-Conquer Principle

- Important algorithm design method

- Examples from Informatik 2:
  - Sorting: Mergesort, Quicksort
  - Binary search can be considered as a divide and conquer algorithm

- Further examples
  - Median
  - Compairing orders
  - Delaunay triangulation / Voronoi diagram
  - Closest pairs
  - Line intersections
  - Polynomial multiplication / FFT
  - ...

Example 1: Quicksort



function Quick (S: sequence): sequence;

{returns the sorted sequence S}

begin

    if $\#S \leq 1$ then **return** S

    else { choose pivot element v in S;

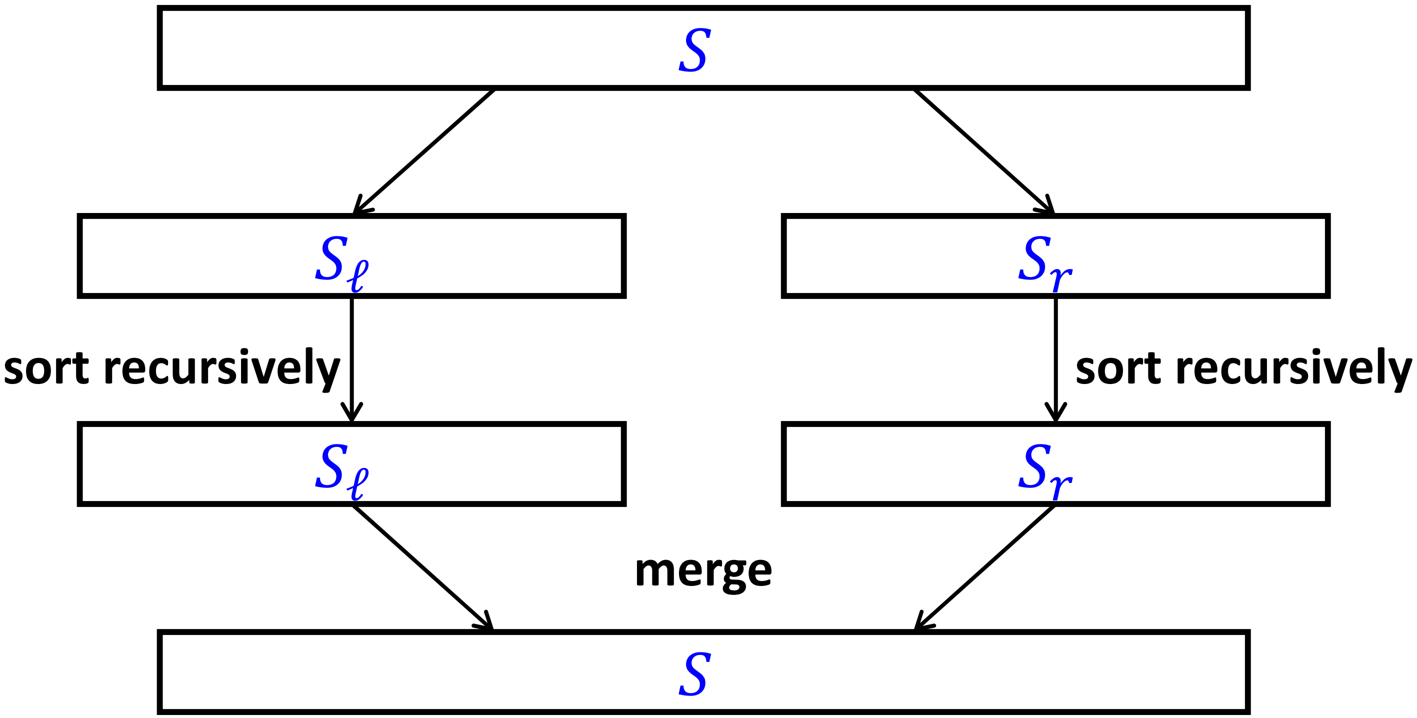
        partition S into S_ℓ with elements $< v$,

        and S_r with elements $> v$

        return | Quick(S_ℓ) | v | Quick(S_r) |

end;

Example 2: Mergesort



Formulation of the D&C principle

Divide-and-conquer method for solving a problem instance of size n:

1. Divide

$n \leq c$: Solve the problem directly.

$n > c$: Divide the problem into k subproblems of sizes $n_1, \ldots, n_k < n$ $(k \geq 2)$.

2. Conquer

Solve the k subproblems in the same way (recursively).

3. Combine

Combine the partial solutions to generate a solution for the original instance.

Analysis

Recurrence relation:

- $\boldsymbol{T(n)}$: max. number of steps necessary for solving an instance of size n
- $$\boldsymbol{T(n)} = \begin{cases} \boldsymbol{a} & \textbf{if } \boldsymbol{n \leq c} \\ \boldsymbol{T(n_1) + \cdots + T(n_k)} & \textbf{if } \boldsymbol{n > c} \\ \boldsymbol{+ \textbf{ cost for divide and combine}} & \end{cases}$$

Special case: $\boldsymbol{k = 2, n_1 = n_2 = {}^{n}/_{2}}$

- cost for divide and combine: $\mathrm{DC}(n)$
- $T(1) = a$
- $T(n) = 2T(n/2) + \mathrm{DC}(n)$

Analysis, Example

Recurrence relation:

$$T(n) \leq 2 \cdot T(n/2) + cn^2, \qquad T(1) \leq a$$

Guess the solution by repeated substitution:

Analysis, Example

Recurrence relation:

$$T(n) \leq 2 \cdot T(n/2) + cn^2, \qquad T(1) \leq a$$

Verify by induction:

Analysis, Example

Recurrence relation:

$$T(n) \leq 2 \cdot T(n/2) + cn^2, \qquad T(1) \leq a$$

Guess the solution by drawing the recursion tree:

Comparing Orders

- Many web systems maintain user preferences / rankings on things like books, movies, restaurants, …

- Collaborative filtering:
  - Predict user taste by comparing rankings of different users.
  - If the system finds users with similar tastes, it can make recommendations (e.g., Amazon)

- Core issue: Compare two rankings
  - Intuitively, two rankings (of movies) are more similar, the more pairs are ordered in the same way
  - Label the first user's movies from 1 to n according to ranking
  - Order labels according to second user's ranking
  - How far is this from the ascending order (of the first user)?

Number of Inversions

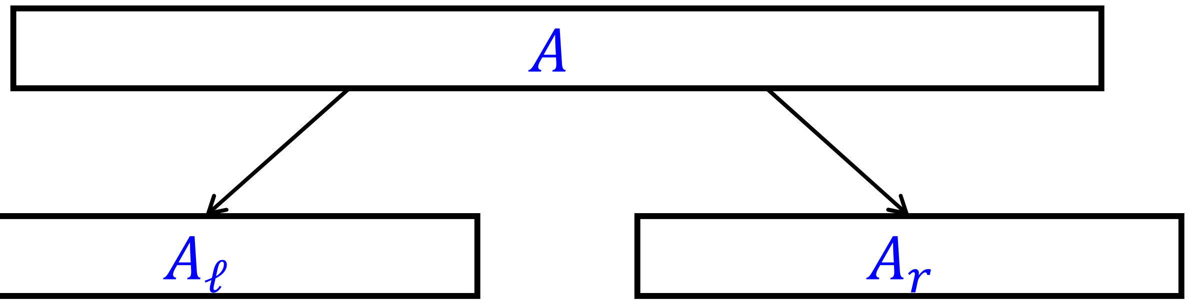
Formal problem:

- **Given**: array $A = [a_1, a_2, a_3, \dots, a_n]$ of distinct elements
- **Objective**: Compute number of inversions I

$$I := \left|\{0 \leq i < j \leq n \mid a_i > a_j)\}\right|$$

- **Example**: $A = [\ 4\ ,\ 1\ ,\ 5\ ,\ 2\ ,\ 7\ ,\ 10\ ,\ 6\ ]$
- **Naive solution**:

Divide and conquer



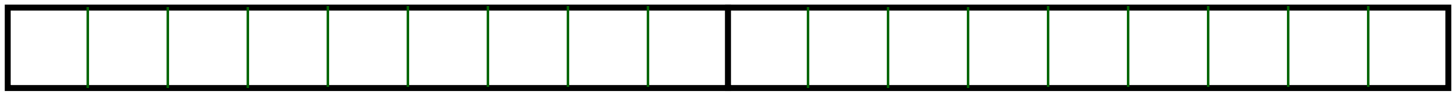
1. Divide array into 2 equal parts A_ℓ and A_r
2. Recursively compute #inversions in A_ℓ and A_r
3. Combine: add #pairs $a_i \in A_\ell$, $a_j \in A_r$ such that $a_i > a_j$

Combine Step: Example

- Assume A_ℓ and A_r are sorted

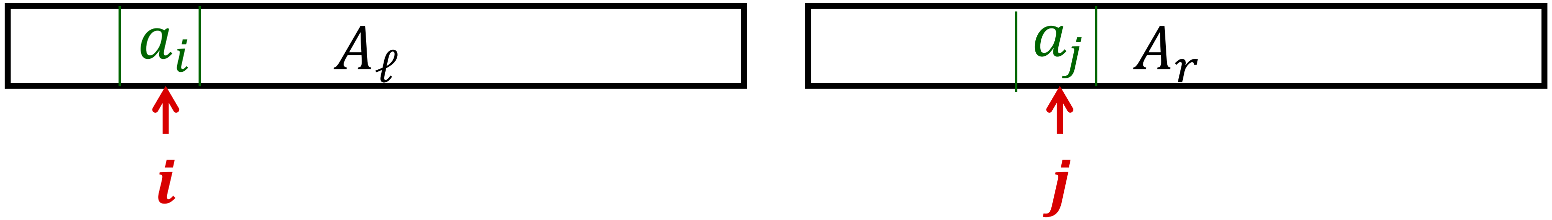
3	5	8	13	14	18	24	25	30

6	7	9	19	21	23	28	32	33



Combine Step

Assume A_ℓ and A_r are sorted



Idea:

- Maintain pointers i and j to go through the sorted parts
- While going through the sorted parts, we merge the two parts into one sorted part (like in MergeSort)

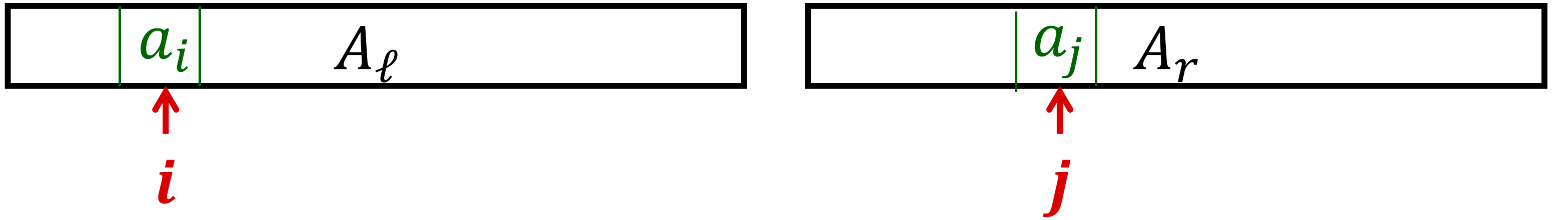
  and we count the number of inversions between the parts

Invariant:

- At each point in time, all inversions involving some element left of i (in A_ℓ) or left of j (in A_r) are counted
  - and all others still have to be counted...

Combine Step

Assume A_ℓ and A_r are sorted



- Pointers i and j, initially pointing to first elements of A_ℓ and A_r
- If $a_i \leq a_j$:
  - a_i is smallest among the remaining elements
  - No inversion of a_i and one of the remaining elements
  - Do not change count
- If $a_i > a_j$:
  - a_j is smallest among the remaining elements
  - a_j is smaller than all remaining elements in A_ℓ
  - Add number of remaining elements in A_ℓ to count
- Increment point, pointing to smaller element

Combine Step

- Need sub-sequences in sorted order
- Then, combine step is like merging in merge sort

- **Idea**: Solve sorting and #inversions at the same time!
  1. Partition A into two equal parts A_ℓ and A_r
  2. Recursively compute #inversions and sort A_ℓ and A_r

  3. Merge A_ℓ and A_r to sorted sequence, at the same time, compute number of inversions between elements a_i in A_ℓ and a_j in A_r

Analysis, Guessing

Recurrence relation:

$$T(n) \le 2 \cdot T(n/2) + c \cdot n, \qquad T(1) \le c$$

Repeated substitution:

Analysis, Induction

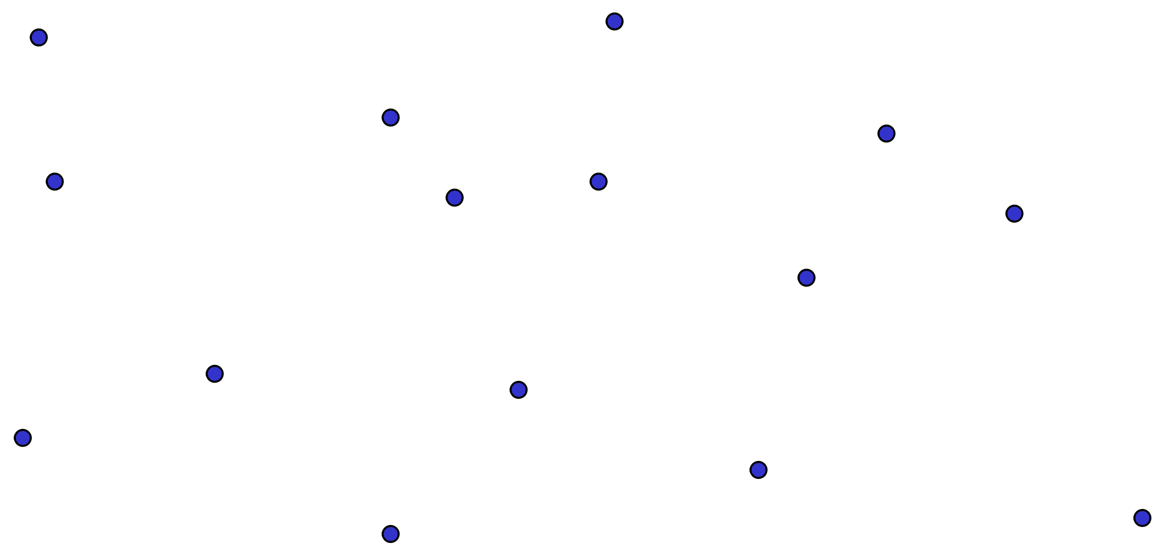
Recurrence relation:

$$T(n) \leq 2 \cdot T(n/2) + c \cdot n, \qquad T(1) \leq c$$

Verify by induction:

Geometric divide-and-conquer

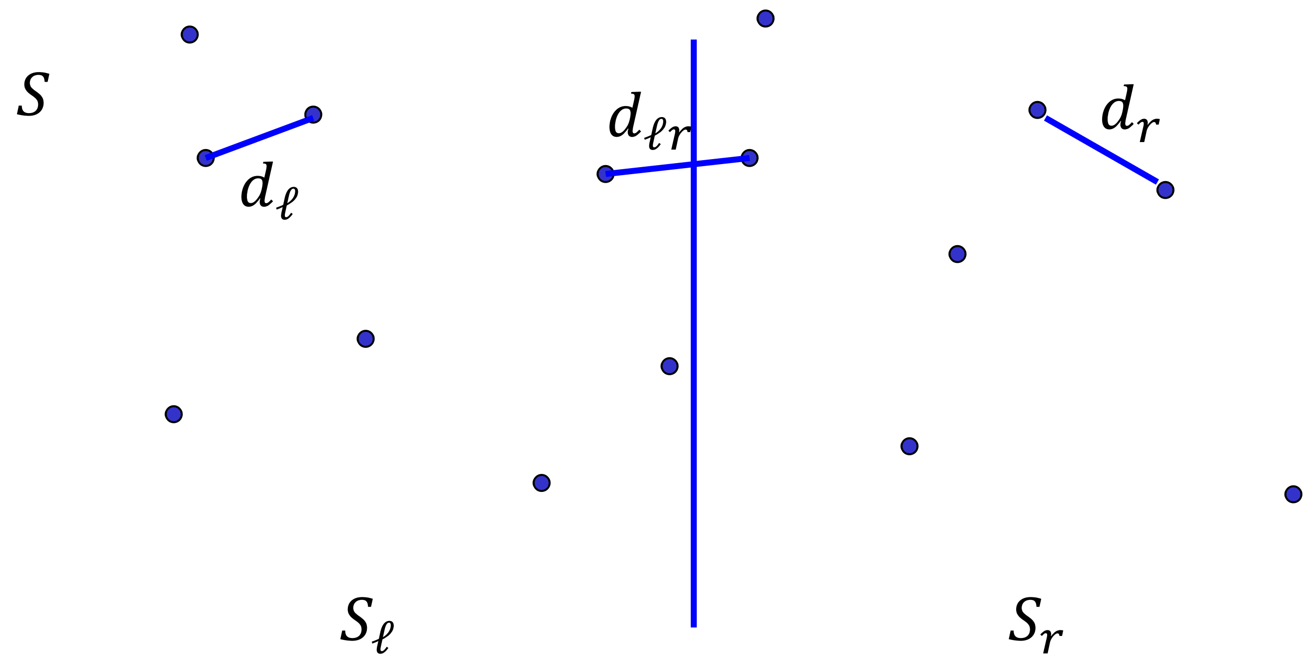
Closest Pair Problem: Given a set S of n points, find a pair of points with the smallest distance.



Naive solution:

Divide-and-conquer solution

1. **Divide:** Divide S into two equal sized sets S_ℓ und S_r.
2. **Conquer:** $d_\ell = \text{mindist}(S_\ell)$ $d_r = \text{mindist}(Sr)$
3. **Combine:** $d_{\ell r} = \min\{d(p_\ell, p_r) \mid p_\ell \in S_\ell, p_r \in S_r\}$
   return $\min\{d_\ell, d_r, d_{\ell r}\}$



Divide-and-conquer solution

1. **Divide:** Divide S into two equal sized sets S_ℓ und S_r.
2. **Conquer:** $d_\ell = \text{mindist}(S_\ell)$ $d_r = \text{mindist}(Sr)$
3. **Combine:** $d_{\ell r} = \min\{d(p_\ell, p_r) \mid p_\ell \in S_\ell, p_r \in S_r\}$
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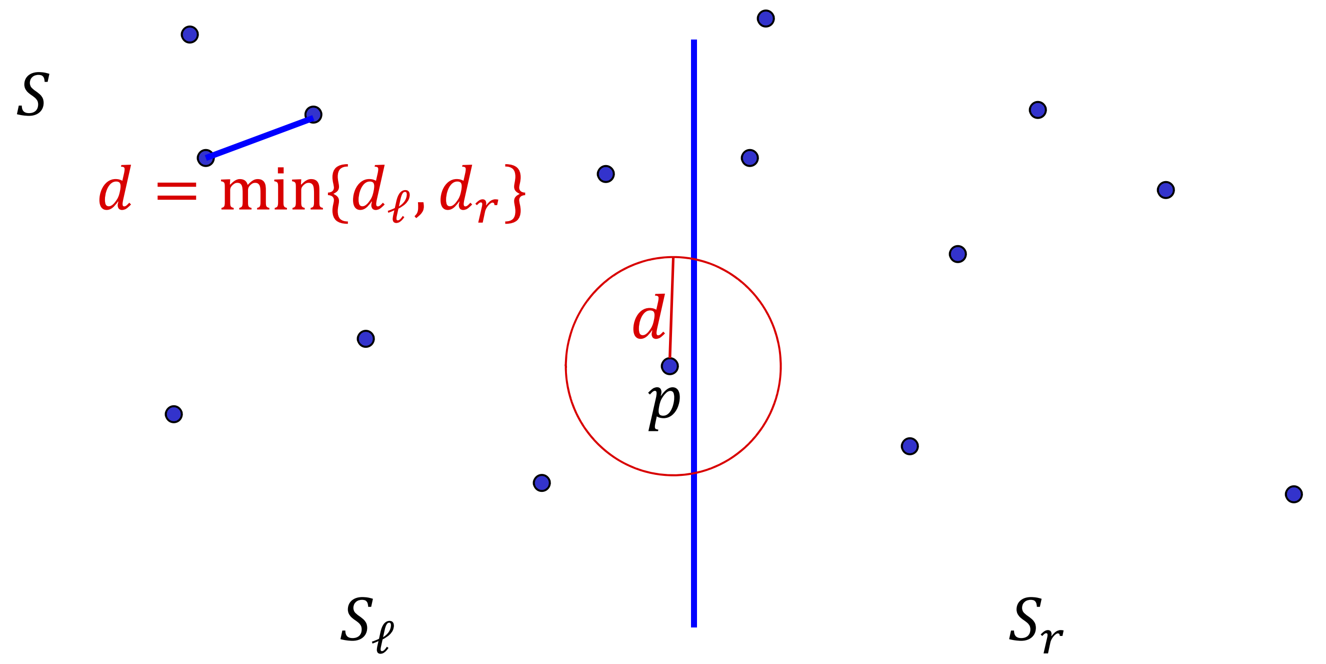
Computation of $d_{\ell r}$:



Merge step

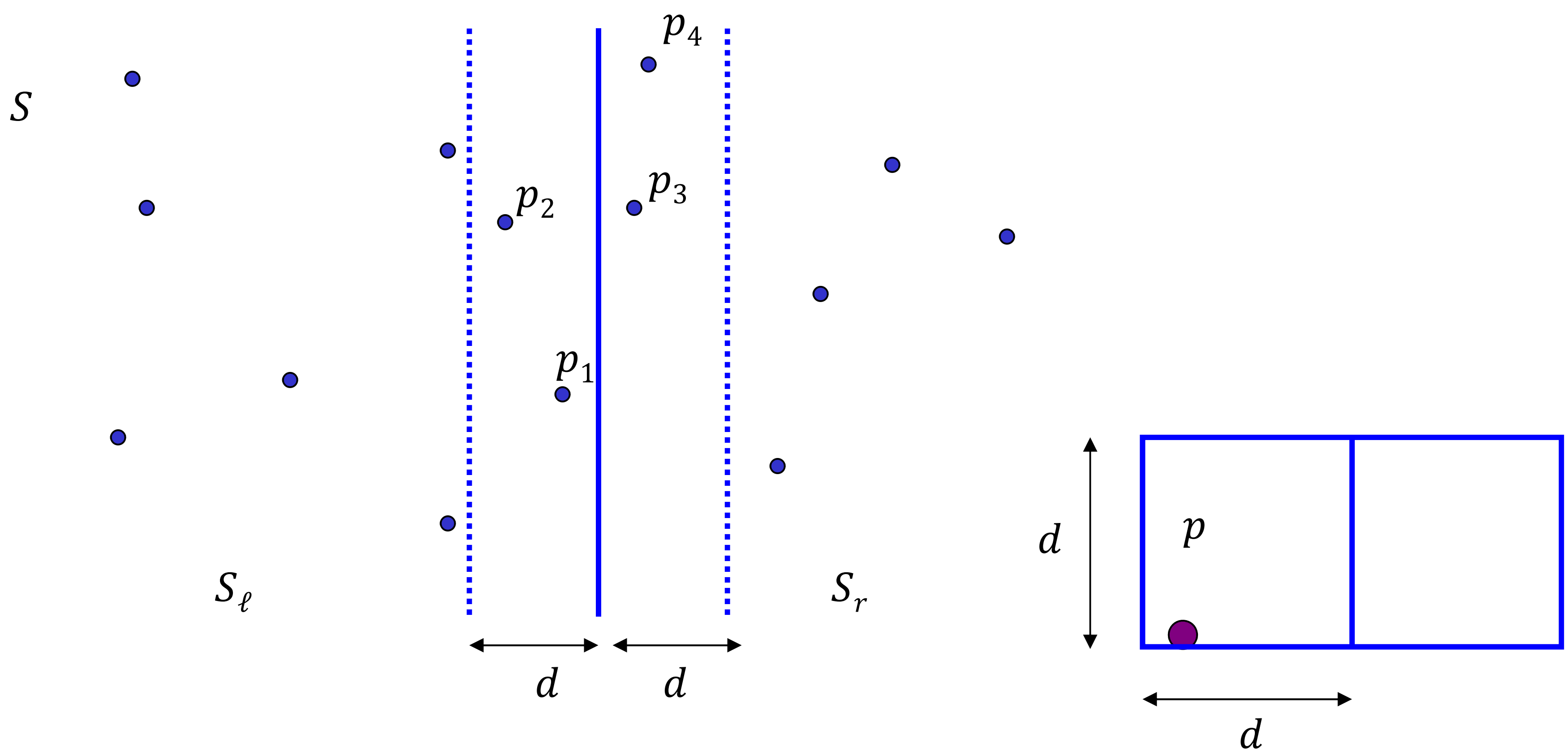
1. Consider only points within distance $< d$ of the bisection line, in the order of increasing y-coordinates.

2. For each point p consider all points q with

$$\boldsymbol{y_p \leq y_q \leq y_p + d}$$

3. There are at most 7 such points.

Combine step



$d = \min\{d_\ell, dr\}$

Implementation

- Initially sort the points in S in order of increasing x-coordinates

- While computing closest pair, also sort S according to y-coord.
  - Partition S into S_ℓ and S_r, solve and sort sub-problems recursively
  - Merge to get sorted S according to y-coordinates
  - Center points: points within x-distance $d = \min\{d_\ell, d_r\}$ of center
  - Go through center points in S in order of incr. y-coordinates

Running Time

Recurrence relation:

$$T(n) = 2 \cdot T(n/2) + c \cdot n, \qquad T(1) = a$$

Solution:

- Same as for computing number of number of inversions, merge sort (and many others…)

$$T(n) = O(n \cdot \log n)$$

Recurrence Relations: Master Theorem

Recurrence relation

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \qquad T(n) = O(1) \text{ for } n \leq n_0$$

Cases

- $f(n) = O(n^c),\ c < \log_b a$

$$T(n) = \Theta\left(n^{\log_b a}\right)$$

- $f(n) = \Omega(n^c),\ c > \log_b a$

$$T(n) = \Theta\big(f(n)\big)$$

- $f(n) = \Theta\left(n^c \cdot \log^k n\right),\ c = \log_b a$

$$T(n) = \Theta\left(n^c \cdot \log^{k+1} n\right)$$