Chapter 1
Divide and Conquer

Polynomial Multiplication

Algorithm Theory
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Formulation of the D&C principle

Divide-and-conquer method for solving a problem instance of size n:

1. Divide

$n \leq c$: Solve the problem directly.

$n > c$: Divide the problem into k subproblems of sizes $n_1, \dots, n_k < n$ ($k \geq 2$).

2. Conquer

Solve the k subproblems in the same way (recursively).

3. Combine

Combine the partial solutions to generate a solution for the original instance.

Analysis

Recurrence relation:

- $\boldsymbol{T(n)}$: max. number of steps necessary for solving an instance of size n
- $$\boldsymbol{T(n)} = \begin{cases} \boldsymbol{a} & \textbf{if } \boldsymbol{n \leq c} \\ \boldsymbol{T(n_1) + \cdots + T(n_k)} & \textbf{if } \boldsymbol{n > c} \\ \boldsymbol{+ \text{ cost for divide and combine}} & \end{cases}$$

Special case: $\boldsymbol{k = 2, n_1 = n_2 = {}^{n}/_{2}}$

- cost for divide and combine: $\mathrm{DC}(n)$
- $T(1) = a$
- $T(n) = 2T(n/2) + \mathrm{DC}(n)$

Recurrence Relations: Master Theorem

Recurrence relation

$$\boldsymbol{T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \qquad T(n) = O(1) \text{ for } n \leq n_0}$$

Cases

- $f(n) = O(n^c),\ c < \log_b a$

$$\boldsymbol{T(n) = \Theta\left(n^{\log_b a}\right)}$$

- $f(n) = \Omega(n^c),\ c > \log_b a$

$$\boldsymbol{T(n) = \Theta\left(f(n)\right)}$$

- $f(n) = \Theta\left(n^c \cdot \log^k n\right),\ c = \log_b a$

$$\boldsymbol{T(n) = \Theta\left(n^c \cdot \log^{k+1} n\right)}$$

Polynomials

Real polynomial p in one variable x:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$$

Coefficients of p: $a_0, a_1, \dots, a_n \in \mathbb{R}$

Degree of p: largest power of x in p (n in the above case)

Example:

$$p(x) = 3x^3 - 15x^2 + 18x$$

Set of all real-valued polynomials in x: $\mathbb{R}[x]$ (polynomial ring)

Operations: Addition

- Given: Polynomials $p, q \in \mathbb{R}[x]$ of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

- Compute sum $p(x) + q(x)$:

$$\begin{aligned} p(x) + q(x) &= (a_n x^n + \cdots + a_0) + (b_n x^n + \cdots + b_0) \\ &= (a_n + b_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0) \end{aligned}$$

Operations: Multiplication

- Given: Polynomials $p, q \in \mathbb{R}[x]$ of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

- Product $p(x) \cdot q(x)$:

$$p(x) \cdot q(x) = (a_n x^n + \cdots + a_0) \cdot (b_n x^n + \cdots + b_0)$$

$$= c_{2n} x^{2n} + c_{2n-1} x^{2n-1} + \cdots + c_1 x + c_0$$

- Obtaining c_i: what products of monomials have degree i?

$$\text{For } 0 \leq i \leq 2n\text{: } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

where $a_i = b_i = 0$ for $i > n$.

Operations: Evaluation

- Given: Polynomial $p \in \mathbb{R}[x]$ of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- **Horner's method** for evaluation at specific value x_0:

$$p(x_0) = \big(\ldots\big((a_n x_0 + a_{n-1})x_0 + a_{n-2}\big)x_0 + \cdots + a_1\big)x_0 + a_0$$

- Pseudo-code:

$p \coloneqq a_n; i \coloneqq n;$
while $(i > 0)$ **do**
    $i \coloneqq i - 1;$
    $p \coloneqq p \cdot x_0 + a_i$
end

- Running time: $O(n)$

Representation of Polynomials

Coefficient representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree n is given by its $n+1$ coefficients $a_0, \dots, a_n$:

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

- Example:

$$p(x) = 3x^3 - 15x^2 + 18x$$

- The most typical (and probably most natural) representation of polynomials

Representation of Polynomials

Product of linear factors:

- Polynomial $p(x) \in \mathbb{C}[x]$ of degree n is given by its n roots

$$p(x) = a_n \cdot (x - x_1) \cdot (x - x_2) \cdot \ldots \cdot (x - x_n)$$

- Example:

$$p(x) = 3x(x-2)(x-3)$$

- Every polynomial has exactly n roots $x_i \in \mathbb{C}$ for which $p(x_i) = 0$
  - Polynomial is uniquely defined by the n roots and a_n

- We will not use this representation…

Representation of Polynomials

Point-value representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree n is given by $n+1$ point-value pairs:

$$p = \{(x_0, p(x_0)), (x_1, p(x_1)), \dots, (x_n, p(x_n))\}$$

where $x_i \neq x_j$ for $i \neq j$.

- Example: The polynomial

$$p(x) = 3x(x-2)(x-3)$$

is uniquely defined by the four point-value pairs $(0,0), (1,6), (2,0), (3,0)$.

Operations: Coefficient Representation

Deg.-n polynomials $p(x) = a_n x^n + \cdots + a_0$, $q(x) = b_n x^n + \cdots + b_0$

Addition:

$$p(x) + q(x) = (a_n + b_n)x^n + \cdots + (a_0 + b_0)$$

- Time: $O(n)$

Multiplication:

$$p(x) \cdot q(x) = c_{2n}x^{2n} + \cdots + c_0, \qquad \text{where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

- Naive solution: Need to compute product $a_i b_j$ for all $0 \leq i, j \leq n$
- Time: $O(n^2)$

Operations Point-Value Representation

Degree-n polynomials

$$p = \{(x_0, p(x_0)), \dots, (x_n, p(x_n))\}, q = \{(x_0, q(x_0)), \dots, (x_n, q(x_n))\}$$

- Note: we use the same points $x_0, \dots, x_n$ for both polynomials

Addition:

$$p + q = \{(x_0, p(x_0) + q(x_0)), \dots, (x_n, p(x_n) + q(x_n))\}$$

- Time: $O(n)$

Multiplication:

$$p \cdot q = \{(x_0, p(x_0) \cdot q(x_0)), \dots, (x_{2n}, p(x_{2n}) \cdot q(x_{2n}))\}$$

- Time: $O(n)$

Faster Multiplication?

- Multiplication is slow $\left(\Theta(n^2)\right)$ when using the standard coefficient representation
- Try divide-and-conquer to get a faster algorithm
- Assume: degree is $n-1$, n is even
- Divide polynomial $p(x) = a_{n-1}x^{n-1} + \cdots + a_0$ into 2 polynomials of degree $n/2 - 1$:

$$p_0(x) = a_{n/2-1}x^{n/2-1} + \cdots + a_0$$
$$p_1(x) = a_{n-1}x^{n/2-1} + \cdots + a_{n/2}$$
$$p(x) = p_1(x) \cdot x^{n/2} + p_0(x)$$

- Similarly: $q(x) = q_1(x) \cdot x^{n/2} + q_0(x)$

Use Divide-And-Conquer

- **Divide:**
  $p(x) = p_1(x) \cdot x^{n/2} + p_0(x), \qquad q(x) = q_1(x) \cdot x^{n/2} + q_0(x)$

- **Multiplication:**
  $$p(x)q(x) = p_1(x)q_1(x) \cdot x^n + (p_0(x)q_1(x) + p_1(x)q_0(x)) \cdot x^{n/2} + p_0(x)q_0(x)$$

- 4 multiplications of degree $n/2 - 1$ polynomials:
  $$T(n) = 4T\left(n/2\right) + O(n)$$

- Leads to $T(n) = \Theta(n^2)$ like the naive algorithm…
  - follows immediately by using the master theorem

More Clever Recursive Solution

- **Recall that**

$$p(x)q(x) = p_1(x)q_1(x) \cdot x^n + (p_0(x)q_1(x) + p_1(x)q_0(x)) \cdot x^{n/2} + p_0(x)q_0(x)$$

- Compute $r(x) = (p_0(x) + p_1(x)) \cdot (q_0(x) + q_1(x))$:

Karatsuba Algorithm

- Recursive multiplication:

$$r(x) = (p_0(x) + p_1(x)) \cdot (q_0(x) + q_1(x))$$

$$p(x)q(x) = p_1(x)q_1(x) \cdot x^n + (r(x) - p_0(x)q_0(x) + p_1(x)q_1(x)) \cdot x^{n/2} + p_0(x)q_0(x)$$

- Recursively do 3 multiplications of degr. $(n/2 - 1)$-polynomials

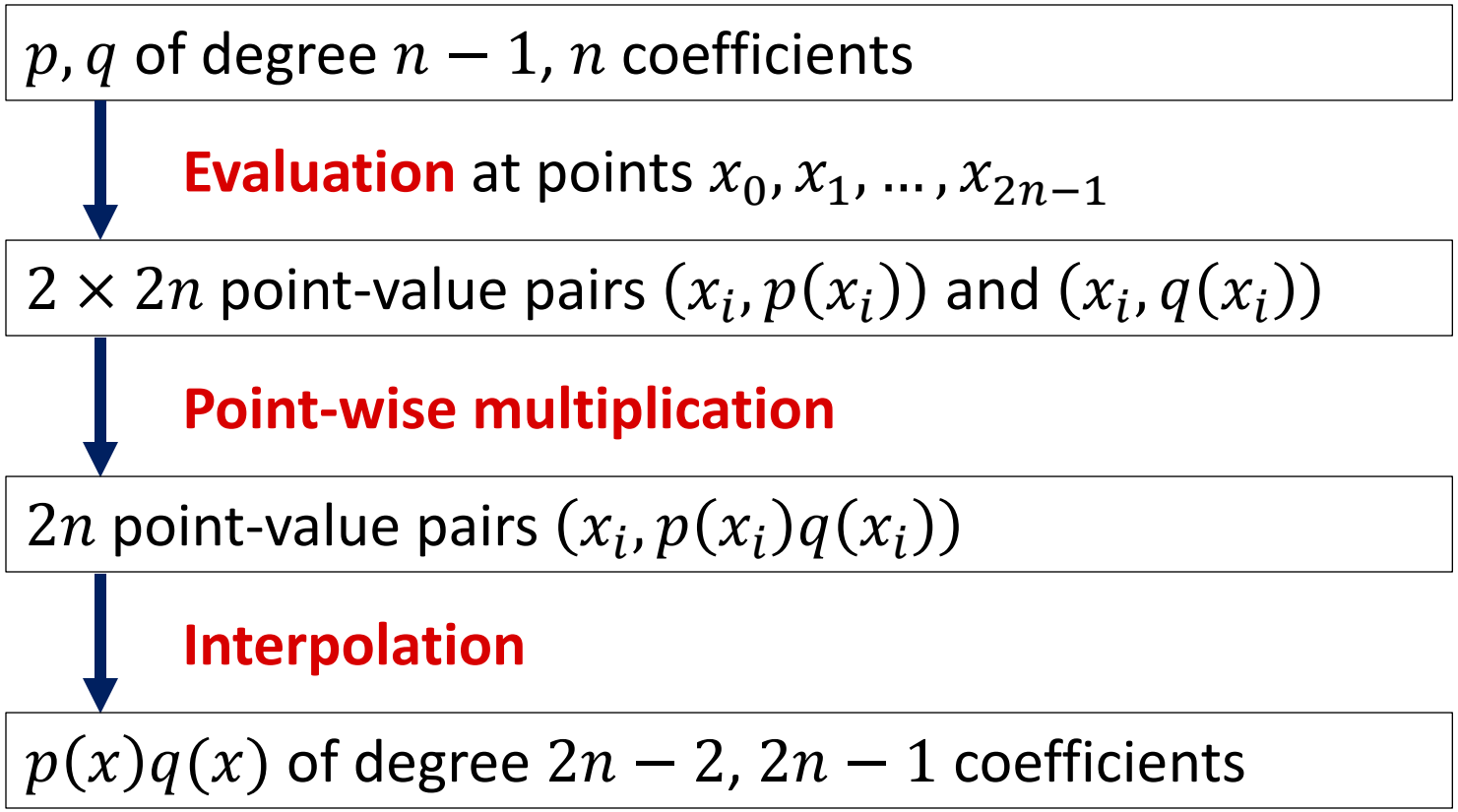
$$T(n) = 3T\left(n/2\right) + O(n)$$

- Gives: $T(n) = O(n^{1.59})$ (see Master theorem)

Faster Polynomial Multiplication?

Multiplication is fast when using the point-value representation

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):



Reminder Complex Numbers

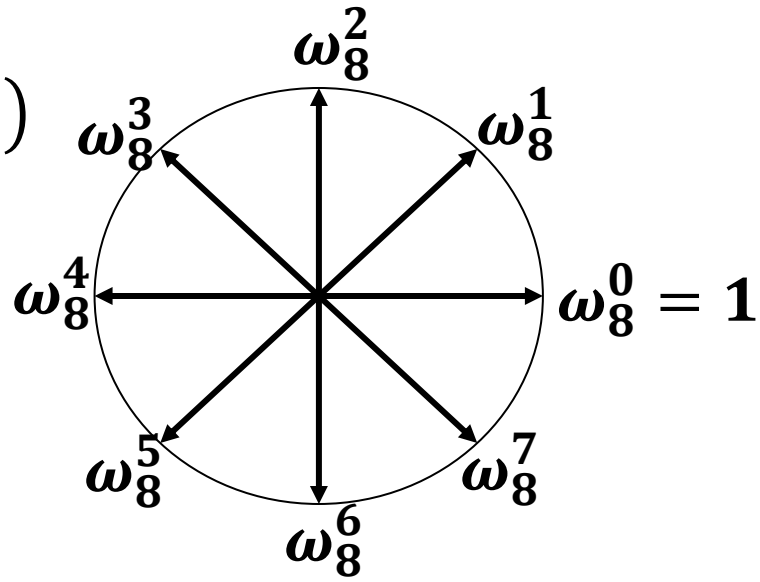
Point-Value Representation of p, q

- Select points $x_0, x_1, \dots, x_{N-1}$ to evaluate p and q in a clever way

Consider the N powers of the principle Nth root of unity:

Principle root of unity: $\omega_N = e^{2\pi i/N}$

$$\left(i = \sqrt{-1}, \qquad e^{2\pi i} = 1\right)$$



Powers of ω_N (roots of unity):

$$1 = \omega_N^0, \omega_N^1, \dots, \omega_N^{N-1}$$

Note: $\omega_N^k = e^{2\pi ik/N} = \cos\frac{2\pi k}{N} + i \cdot \sin\frac{2\pi k}{N}$

Discrete Fourier Transform

- The values $p\left(\omega_N^i\right)$ for $i = 0, \dots, N-1$ uniquely define a polynomial p of degree $< N$.

Discrete Fourier Transform (DFT):

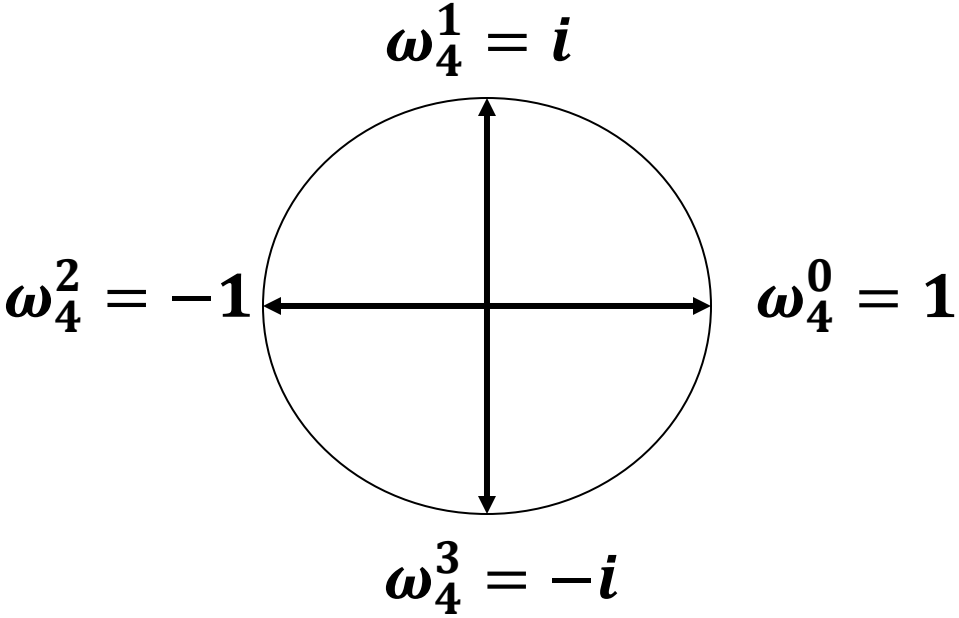
- Assume $a = (a_0, \dots, a_{N-1})$ is the coefficient vector of poly. p

$$(p(x) = a_{N-1}x^{N-1} + \cdots + a_1 x + a_0)$$

$$\mathrm{DFT}_N(a) := \left(p\left(\omega_N^0\right), p(\omega_N^1), \dots, p(\omega_N^{N-1})\right)$$

Example

- Consider polynomial $p(x) = 3x^3 - 15x^2 + 18x$
- Choose $N = 4$
- Roots of unity:



Example

- Consider polynomial $p(x) = 3x^3 - 15x^2 + 18x$
- $N = 4$, roots of unity: $\omega_4^0 = 1, \omega_4^1 = i, \omega_4^2 = -1, \omega_4^3 = -i$
- Evaluate $p(x)$ at ω_4^k:

$$\left(\omega_4^0, p(\omega_4^0)\right) = \left(1, p(1)\right) = (1,6)$$
$$\left(\omega_4^1, p(\omega_4^1)\right) = \left(i, p(i)\right) = (i, 15 + 15i)$$
$$\left(\omega_4^2, p(\omega_4^2)\right) = \left(-1, p(-1)\right) = (-1, -36)$$
$$\left(\omega_4^3, p(\omega_4^3)\right) = \left(-i, p(-i)\right) = (-i, 15 - 15i)$$

- For $a = (0,18,-15,3)$:

$$\mathbf{DFT_4}(\boldsymbol{a}) = (\mathbf{6}, \mathbf{15 + 15i}, \mathbf{-36}, \mathbf{15 - 15i})$$

Faster Polynomial Multiplication?

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

↓ **Evaluation** at points $\omega_{2n}^0, \omega_{2n}^1, \dots, \omega_{2n}^{2n-1}$

$2 \times 2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)\right)$ and $\left(\omega_{2n}^k, q\left(\omega_{2n}^k\right)\right)$

↓ **Point-wise multiplication**

$2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)q\left(\omega_{2n}^k\right)\right)$

↓ **Interpolation**

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

Properties of the Roots of Unity

- **Cancellation Lemma:**

  For all integers $n > 0$, $k \geq 0$, and $d > 0$, we have:

$$\omega_{dn}^{dk} = \omega_n^k \,, \qquad \omega_n^{k+n} = \omega_n^k$$

- **Proof:**

Divide-and-Conquer Approach

- Divide $p(x)$ of degree $N-1$ (N is even) into 2 polynomials of degree ${}^{N}/_{2}-1$ differently than in Karatsuba's algorithm

- $p_0(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{N-2} x^{{}^{N}/_{2}-1}$ (even coeff.)
  $p_1(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{N-1} x^{{}^{N}/_{2}-1}$ (odd coeff.)

Discrete Fourier Transform

Evaluation for $k = 0, \dots, N-1$:

$$p(\omega_N^k) = p_0((\omega_N^k)^2) + \omega_N^k \cdot p_1((\omega_N^k)^2)$$

$$= \begin{cases} p_0(\omega_{N/2}^k) + \omega_N^k \cdot p_1(\omega_{N/2}^k) & \text{if } k < N/2 \\ p_0\left(\omega_{N/2}^{k-N/2}\right) + \omega_N^k \cdot p_1\left(\omega_{N/2}^{k-N/2}\right) & \text{if } k \geq N/2 \end{cases}$$

For the coefficient vector a of $p(x)$:

$$\mathrm{DFT}_N(a) = \left(p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right), p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right)\right)$$
$$+ \left(\omega_N^0 p_1(\omega_{N/2}^0), \dots, \omega_N^{N/2-1} p_1\left(\omega_{N/2}^{N/2-1}\right), \omega_N^{N/2} p_1(\omega_{N/2}^0), \dots, \omega_N^{N-1} p_1\left(\omega_{N/2}^{N/2-1}\right)\right)$$

Example

For the coefficient vector a of $p(x)$:

$$\mathrm{DFT}_N(a) = \left(p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right), p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right)\right)$$
$$+ \left(\omega_N^0 p_1(\omega_{N/2}^0), \dots, \omega_N^{N/2-1} p_1\left(\omega_{N/2}^{N/2-1}\right), \omega_N^{N/2} p_1(\omega_{N/2}^0), \dots, \omega_N^{N-1} p_1\left(\omega_{N/2}^{N/2-1}\right)\right)$$

$N = 4$:

$$p(\omega_4^0) = p_0(\omega_2^0) + \omega_4^0 p_1(\omega_2^0)$$
$$p(\omega_4^1) = p_0(\omega_2^1) + \omega_4^1 p_1(\omega_2^1)$$
$$p(\omega_4^2) = p_0(\omega_2^0) + \omega_4^2 p_1(\omega_2^0)$$
$$p(\omega_4^3) = p_0(\omega_2^1) + \omega_4^3 p_1(\omega_2^1)$$

Need: $(p_0(\omega_2^0), p_0(\omega_2^1))$ and $(p_1(\omega_2^0), p_1(\omega_2^1))$

(DFTs of coefficient vectors of p_0 and p_1)

Recursive Structure

For simplicity, we abuse notation in the following:

- Poly. $p(x) = a_{N-1}x^{N-1} + \cdots + a_0$ with coefficient vector a

  Let $\mathrm{DFT}_N(p) := \mathrm{DFT}_N(a)$

Recursive structure:

- For $N = 4$:

$$\big(\mathrm{DFT}_4(p)\big)_k = p\big(\omega_4^k\big)$$
$$= \big(\mathrm{DFT}_2(p_0)\big)_{k \bmod 2} + \omega_4^k \cdot \big(\mathrm{DFT}_2(p_1)\big)_{k \bmod 2}$$

- General N (assume N is even):

$$\big(\mathrm{DFT}_N(p)\big)_k = p\big(\omega_N^k\big)$$
$$= \Big(\mathrm{DFT}_{N/2}(p_0)\Big)_{k \bmod N/2} + \omega_N^k \cdot \Big(\mathrm{DFT}_{N/2}(p_1)\Big)_{k \bmod N/2}$$

Computation of DFT_N

- Divide-and-conquer algorithm for $\mathrm{DFT}_N(p)$:

1. Divide

$N \leq 1$: $\mathrm{DFT}_1(p) = a_0$

$N > 1$: Divide p into p_0 (even coeff.) and p_1 (odd coeff).

2. Conquer

Solve $\mathrm{DFT}_{N/2}(p_0)$ and $\mathrm{DFT}_{N/2}(p_1)$ recursively

3. Combine

Compute $\mathrm{DFT}_N(p)$ based on $\mathrm{DFT}_{N/2}(p_0)$ and $\mathrm{DFT}_{N/2}(p_1)$

Analysis

- $T(N)$: time to compute $\mathrm{DFT}_N(p)$:

$$T(N) = 2T\left({}^{N}/_{2}\right) + O(N), \qquad T(1) = O(1)$$

- As for mergesort, comparing orders, closest pair of points:

$$T(N) = O(N \cdot \log N)$$

Small Improvement

Polynomial p of degree $N-1$:

$$p(\omega_N^k) = \begin{cases} p_0(\omega_{N/2}^k) + \omega_N^k \cdot p_1(\omega_{N/2}^k) & \text{if } k < N/2 \\ p_0\left(\omega_{N/2}^{k-N/2}\right) + \omega_N^k \cdot p_1\left(\omega_{N/2}^{k-N/2}\right) & \text{if } k \geq N/2 \end{cases}$$

$$= \begin{cases} p_0(\omega_{N/2}^k) + \omega_N^k \cdot p_1(\omega_{N/2}^k) & \text{if } k < N/2 \\ p_0\left(\omega_{N/2}^{k-N/2}\right) - \omega_N^{k-N/2} \cdot p_1\left(\omega_{N/2}^{k-N/2}\right) & \text{if } k \geq N/2 \end{cases}$$

Need to compute $p_0(\omega_{N/2}^k)$ and $\omega_N^k \cdot p_1(\omega_{N/2}^k)$ for $0 \leq k < N/2$.

Example

$$p(\omega_4^0) = p_0(\omega_2^0) + \omega_4^0 \cdot p_1(\omega_2^0)$$
$$p(\omega_4^1) = p_0(\omega_2^1) + \omega_4^1 \cdot p_1(\omega_2^1)$$
$$p(\omega_4^2) = p_0(\omega_2^0) - \omega_4^0 \cdot p_1(\omega_2^0)$$
$$p(\omega_4^3) = p_0(\omega_2^1) - \omega_4^1 \cdot p_1(\omega_2^1)$$

Fast Fourier Transform (FFT) Algorithm

Algorithm FFT(a)

- Input: Array a of length N, where N is a power of 2
- Output: $\mathrm{DFT}_N(a)$

if $n = 1$ **then return** a_0; // $a = [a_0]$

$d^{[0]} := \mathrm{FFT}([a_0, a_2, \dots, a_{N-2}])$;

$d^{[1]} := \mathrm{FFT}([a_1, a_3, \dots, a_{N-1}])$;

$\omega_N := e^{2\pi i/N}$; $\omega := 1$;

for $k = 0$ **to** $N/2 - 1$ **do** // $\omega = \omega_N^k$

$x := \omega \cdot d_k^{[1]}$;

$d_k := d_k^{[0]} + x$; $d_{k+N/2} := d_k^{[0]} - x$;

$\omega := \omega \cdot \omega_N$

end;

return $d = [d_0, d_1, \dots, d_{N-1}]$;