Chapter 1
Divide and Conquer

Polynomial Multiplication

Algorithm Theory
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Formulation of the D&C principle

Divide-and-conquer method for solving a problem instance of size n:

1. Divide

$n \leq c$: Solve the problem directly.

$n > c$: Divide the problem into k subproblems of sizes $\underline{n_1}, \dots, \underline{n_k} < n$ $(k \geq 2)$.

2. Conquer

Solve the k subproblems in the same way (recursively).

3. Combine

Combine the partial solutions to generate a solution for the original instance.

Analysis

Recurrence relation:

- $\boldsymbol{T(n)}$: max. number of steps necessary for solving an instance of size n

- $$\boldsymbol{T(n) = \begin{cases} a & \textbf{if } n \le c \\ T(n_1) + \cdots + T(n_k) & \textbf{if } n > c \\ + \textbf{cost for divide and combine} & \end{cases}}$$

Special case: $\boldsymbol{k = 2, n_1 = n_2 = {}^n/_2}$

- cost for divide and combine: $\mathrm{DC}(n)$
- $T(1) = a$
- $T(n) = 2T(n/2) + \mathrm{DC}(n)$

Recurrence Relations: Master Theorem

Recurrence relation

$$\boldsymbol{T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)}, \qquad \boldsymbol{T(n) = O(1)} \text{ for } \boldsymbol{n \leq n_0}$$

$T(n) = 2T(n/2) + n^{\cancel{0.99}\ 1.01}$

$f\ \frac{n}{\log n}$

Cases

- $f(n) = O(n^c),\ c < \log_b a$

  $$\boldsymbol{T(n) = \Theta\left(n^{\log_b a}\right)}$$

- $f(n) = \Omega(n^c),\ c > \log_b a$

  $$\boldsymbol{T(n) = \Theta\left(f(n)\right)}$$

- $f(n) = \Theta\left(n^c \cdot \log^k n\right),\ c = \log_b a$, $k \geq 0$ $\quad T(n) = 2T(n/2) + O(n)$

  $$\boldsymbol{T(n) = \Theta\left(n^c \cdot \log^{k+1} n\right)}$$

Polynomials

Real polynomial p in one variable x:

$$p(x) = a_n x^{n} + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0$$

Coefficients of p: $a_0, a_1, \ldots, a_n \in \mathbb{R}$, $\mathbb{C}$

Degree of p: largest power of x in p (n in the above case)

Example:

degree

$$p(x) = 3x^{3} - 15x^2 + 18x$$

$a_0 = 0$, $a_1 = 18$, $a_2 = -15$, $a_3 = 3$

Set of all real-valued polynomials in x: $\mathbb{R}[x]$ (polynomial ring)

Operations: Addition

- Given: Polynomials $p, q \in \mathbb{R}[x]$ of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

- Compute sum $p(x) + q(x)$:

$$\begin{aligned} p(x) + q(x) &= (a_n x^n + \cdots + a_0) + (b_n x^n + \cdots + b_0) \\ &= (\underline{a_n + b_n}) x^n + \cdots + (\underline{a_1 + b_1}) x + (a_0 + b_0) \end{aligned}$$

Operations: Multiplication

- Given: Polynomials $p, q \in \mathbb{R}[x]$ of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

$c_{2n} = a_n \cdot b_n$

- Product $p(x) \cdot q(x)$:

$c_{2n-1} = a_n \cdot b_{n-1} + a_{n-1} b_n$

$c_5 = a_5 b_0 + a_4 b_1 + a_3 b_2 + a_2 b_3 + a_1 b_4 + a_0 b_5$

$$p(x) \cdot q(x) = (a_n x^n + \cdots + a_0) \cdot (b_n x^n + \cdots + b_0)$$

$$= c_{2n} x^{2n} + c_{2n-1} x^{2n-1} + \cdots + c_1 x + c_0$$

- Obtaining c_i: what products of monomials have degree i?

$$\text{For } 0 \leq i \leq 2n\text{: } c_i = \underline{\sum_{j=0}^{i} a_j b_{i-j}}$$

where $a_i = b_i = 0$ for $i > n$.

Operations: Evaluation

$a_n x_0^3 + a_{n-1} x_0^2 + a_{n-2} x_0$

- Given: Polynomial $p \in \mathbb{R}[x]$ of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- **Horner's method** for evaluation at specific value x_0:

$$p(x_0) = \Big(\ldots\big((a_n x_0 + a_{n-1})x_0 + a_{n-2}\big)x_0 + \cdots + a_1\Big)x_0 + a_0$$

- Pseudo-code:

$p := a_n;\ i := n;$
while $(i > 0)$ **do**
  $i := i - 1;$
  $p := p \cdot x_0 + a_i$
end

- Running time: $O(n)$

Representation of Polynomials

Coefficient representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree n is given by its $n + 1$ coefficients $a_0, \dots, a_n$:

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

- Example:

$$p(x) = 3x^3 - 15x^2 + 18x$$

- The most typical (and probably most natural) representation of polynomials

Representation of Polynomials

Product of linear factors:

- Polynomial $p(x) \in \mathbb{C}[x]$of degree n is given by its n roots

$\in \mathbb{C}$

$$p(x) = a_n \cdot (x - x_1) \cdot (x - x_2) \cdot \ldots \cdot (x - x_n)$$

- Example:

$$p(x) = 3x(x-2)(x-3)$$

- Every polynomial has exactly n roots $x_i \in \mathbb{C}$ for which $p(x_i) = 0$
  - Polynomial is uniquely defined by the n roots and a_n

- We will not use this representation…

Representation of Polynomials

Point-value representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree n is given by $n + 1$ point-value pairs:

$$p = \{(x_0, p(x_0)), (x_1, p(x_1)), \dots, (x_n, p(x_n))\}$$

  where $x_i \neq x_j$ for $i \neq j$.

- Example: The polynomial

$$p(x) = 3x(x-2)(x-3)$$

  is uniquely defined by the four point-value pairs $(0,0), (1,6), (2,0), (3,0)$. $(4, 24)$

Operations: Coefficient Representation

Deg.-n polynomials $p(x) = a_n x^n + \cdots + a_0$, $q(x) = b_n x^n + \cdots + b_0$

$(a_i + b_i) x^i$

Addition:

$$p(x) + q(x) = (a_n + b_n)x^n + \cdots + (a_0 + b_0)$$

- Time: $O(n)$

Multiplication:

$$p(x) \cdot q(x) = c_{2n}x^{2n} + \cdots + c_0, \qquad \text{where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

- Naive solution: Need to compute product $a_i b_j$ for all $0 \leq i, j \leq n$
- Time: $O(n^2)$

Operations Point-Value Representation

Degree-n polynomials $\quad p+q \quad\quad p \cdot q$

y_{2n}

$$p = \{(x_0, p(x_0)), \dots, (x_n, p(x_n))\}, q = \{(x_0, q(x_0)), \dots, (x_n, q(x_n))\}$$

- Note: we use the same points $x_0, \dots, x_n$ for both polynomials

Addition:

$$p + q = \{(x_0, p(x_0) + q(x_0)), \dots, (x_n, p(x_n) + q(x_n))\}$$

- Time: $O(n)$

Multiplication:

$$p \cdot q = \{(x_0, p(x_0) \cdot q(x_0)), \dots, (x_{2n}, p(x_{2n}) \cdot q(x_{2n}))\}$$

- Time: $O(n)$

Faster Multiplication?

$a_0, \ldots, a_{n-1}$

- Multiplication is slow $\big(\Theta(n^2)\big)$ when using the standard coefficient representation

- Try divide-and-conquer to get a faster algorithm

- Assume: degree is $\underline{n-1}$, n is even
- Divide polynomial $p(x) = a_{n-1}x^{n-1} + \cdots + a_0$ into 2 polynomials of degree $n/_2 - 1$:

$$p_0(x) = a_{n/_2-1}x^{n/_2-1} + \cdots + a_0$$
$$p_1(x) = a_{n-1}x^{n/_2-1} + \cdots + a_{n/_2}$$
$$\underline{p(x)} = \underline{p_1(x)} \cdot \underline{x^{n/_2}} + \underline{p_0(x)}$$

example

$p(x) = \underbrace{3x^3 - 2x^2}_{} + \underbrace{4x + 7}_{}$

$p_0(x) = 4x + 7$

$p_1(x) = 3x - 2$

$\underline{p(x)} = \underline{p_0(x)} + x^2\underline{p_1(x)}$

- Similarly: $q(x) = q_1(x) \cdot x^{n/_2} + q_0(x)$

Use Divide-And-Conquer

- **Divide:**

$$p(x) = p_1(x) \cdot x^{n/2} + p_0(x), \qquad q(x) = q_1(x) \cdot x^{n/2} + q_0(x)$$

- **Multiplication:**

$$p(x)q(x) = p_1(x)q_1(x) \cdot x^n + \left(p_0(x)q_1(x) + p_1(x)q_0(x)\right) \cdot x^{n/2} + p_0(x)q_0(x)$$

- 4 multiplications of degree $n/2 - 1$ polynomials:

$$T(n) = 4T(n/2) + O(n) \qquad \log_2 4 = 2$$

- Leads to $T(n) = \Theta(n^2)$ like the naive algorithm…
  - follows immediately by using the master theorem

More Clever Recursive Solution

- **Recall that**

$$\underline{p(x)q(x)} = \overbrace{p_1(x)q_1(x)}^{A} \cdot x^n + \underbrace{\big(p_0(x)q_1(x) + p_1(x)q_0(x)\big)}_{B} \cdot x^{n/2} + \underbrace{p_0(x)q_0(x)}_{C}$$

- Compute $r(x) = \underbrace{(p_0(x) + p_1(x))}_{\text{degr. } \frac{n}{2}-1} \cdot \underbrace{(q_0(x) + q_1(x))}_{\text{degr. } \frac{n}{2}-1}$:

$$r(x) = \underbrace{p_0(x)\cdot q_0(x)}_{C} + \underbrace{p_0(x)\cdot q_1(x) + p_1(x)q_0(x)}_{B} + \underbrace{p_1(x)\cdot q_1(x)}_{A} = A + B + C$$

$$B = \underset{\substack{\uparrow \\ 1 \text{ mult.}}}{r(x)} - \underset{\substack{\uparrow \\ 1 \text{ mult.}}}{A} - \underset{\substack{\uparrow \\ 1 \text{ mult.}}}{C}$$

Karatsuba Algorithm

problem size : # coefficients
size $n \Longleftrightarrow$ degree $n-1$

- Recursive multiplication:

$$r(x) = (p_0(x) + p_1(x)) \cdot (q_0(x) + q_1(x))$$

$$\begin{aligned} p(x)q(x) = {} & p_1(x)q_1(x) \cdot x^n \\ & + (r(x) - p_0(x)q_0(x) + p_1(x)q_1(x)) \cdot x^{n/2} \\ & + p_0(x)q_0(x) \end{aligned}$$

- Recursively do 3 multiplications of degr. $(n/2 - 1)$-polynomials

$$T(n) = 3T(n/2) + O(n)$$

- Gives: $T(n) = O(n^{1.59})$ (see Master theorem)

$\log_2 3$

Faster Polynomial Multiplication?

Multiplication is fast when using the point-value representation

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

free to choose $x_0, \dots, x_{2n-1}$

Evaluation at points $x_0, x_1, \dots, x_{2n-1}$

naively: $O(n^2)$

$2 \times 2n$ point-value pairs $(x_i, p(x_i))$ and $(x_i, q(x_i))$

Point-wise multiplication

$O(n)$

$2n$ point-value pairs $(x_i, p(x_i)q(x_i))$

Interpolation

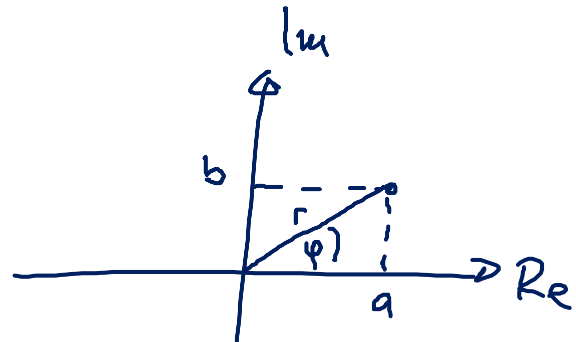
???

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

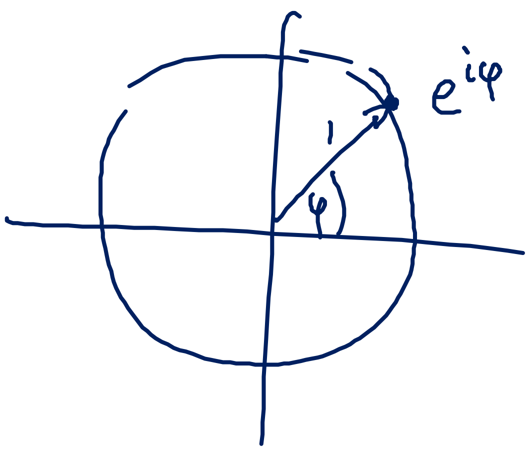
Reminder Complex Numbers

$x^2 = -1 \qquad i = \sqrt{-1}$

$\underbrace{a}_{\text{real}} + \underbrace{b}_{\text{im.}} \cdot i \qquad a, b \in \mathbb{R}$



$r\cos\varphi + i \cdot r\sin\varphi = r \cdot e^{i\varphi}$



$e^{i \cdot 2\pi} = 1$

Point-Value Representation of p, q

- Select points $x_0, x_1, \dots, x_{N-1}$ to evaluate p and q in a clever way

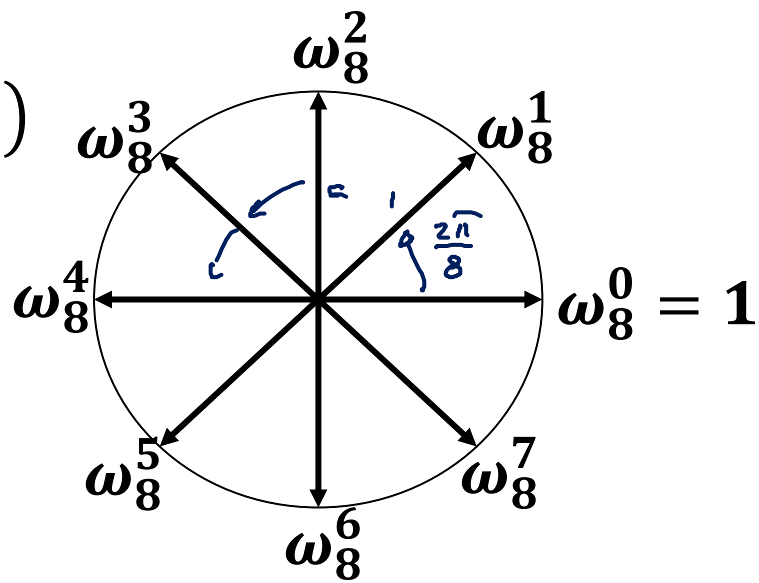
Consider the N powers of the principle Nth root of unity:

$x^N = 1 \leftarrow$ N solutions

$e^{\frac{2\pi i \cdot k}{N}}$

Principle root of unity: $\omega_N = e^{2\pi i/N}$

$$\left(i = \sqrt{-1}, \qquad e^{2\pi i} = 1\right)$$



Powers of ω_N (roots of unity):

$$1 = \omega_N^0, \omega_N^1, \dots, \omega_N^{N-1}$$

Note: $\omega_N^k = e^{2\pi i k/N} = \cos\frac{2\pi k}{N} + i \cdot \sin\frac{2\pi k}{N}$

Discrete Fourier Transform

- The values $p(\omega_N^i)$ for $i = 0, \ldots, N-1$ uniquely define a polynomial p of degree $< N$.

Discrete Fourier Transform (DFT):

- Assume $a = (a_0, \ldots, a_{N-1})$ is the coefficient vector of poly. p

$$(p(x) = a_{N-1}x^{N-1} + \cdots + a_1 x + a_0)$$

$$\mathrm{DFT}_N(a) \coloneqq \left(p(\omega_N^0), p(\omega_N^1), \ldots, p(\omega_N^{N-1})\right)$$

Example

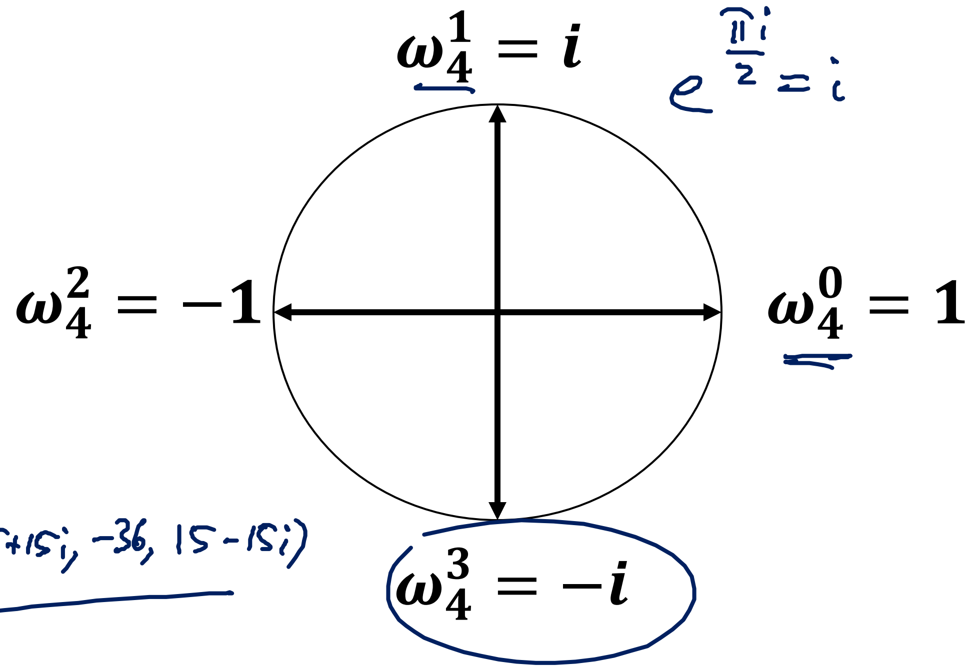
- Consider polynomial $p(x) = 3x^3 - 15x^2 + 18x$

$p(\omega_4^0) = 6$
$p(\omega_4^1) = p(i) = 15 + 15i$
$p(\omega_4^2) = p(-1) = -36$
$p(\omega_4^3) = p(-i) = 15 - 15i$

- Choose $N = 4$

- Roots of unity:

$e^{\frac{\pi i}{2}} = i$



$DFT_4((0, 18, -15, 3)) = (6, 15+15i, -36, 15-15i)$

Example

- Consider polynomial $p(x) = 3x^3 - 15x^2 + 18x$

- $N = 4$, roots of unity: $\omega_4^0 = 1, \omega_4^1 = i, \omega_4^2 = -1, \omega_4^3 = -i$

- Evaluate $p(x)$ at ω_4^k:

$$\left(\omega_4^0, p(\omega_4^0)\right) = \left(1, p(1)\right) = (1,6)$$
$$\left(\omega_4^1, p(\omega_4^1)\right) = \left(i, p(i)\right) = (i, 15 + 15i)$$
$$\left(\omega_4^2, p(\omega_4^2)\right) = \left(-1, p(-1)\right) = (-1, -36)$$
$$\left(\omega_4^3, p(\omega_4^3)\right) = \left(-i, p(-i)\right) = (-i, 15 - 15i)$$

- For $a = (0,18,-15,3)$:

$$\mathbf{DFT_4}(\boldsymbol{a}) = (\mathbf{6}, \mathbf{15} + \mathbf{15}\boldsymbol{i}, -\mathbf{36}, \mathbf{15} - \mathbf{15}\boldsymbol{i})$$

Faster Polynomial Multiplication?

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

Evaluation at points $\omega_{2n}^0, \omega_{2n}^1, \dots, \omega_{2n}^{2n-1}$

$2 \times 2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)\right)$ and $\left(\omega_{2n}^k, q\left(\omega_{2n}^k\right)\right)$

Point-wise multiplication

$2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)q\left(\omega_{2n}^k\right)\right)$

Interpolation

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

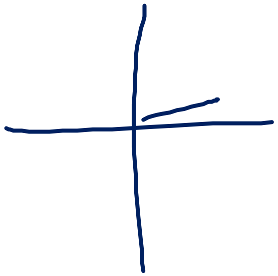
Properties of the Roots of Unity

$e^{a+b} = e^a \cdot e^b$

- **Cancellation Lemma:**

  For all integers $n > 0$, $k \geq 0$, and $d > 0$, we have:

$$(*) \quad \omega_{dn}^{dk} = \omega_n^k, \qquad (**) \quad \omega_n^{k+n} = \omega_n^k$$



- **Proof:**

$$\omega_N = e^{\frac{2\pi i}{N}}$$

$$(*) \quad \omega_{dn}^{dk} = \left(e^{\frac{2\pi i}{dn}}\right)^{dk} = e^{\frac{2\pi i\, \cancel{d}k}{\cancel{d}n}} = \left(e^{\frac{2\pi i}{n}}\right)^{k} = \omega_n^k$$

$$(**) \quad \omega_n^{k+n} = \left(e^{\frac{2\pi i}{n}}\right)^{k+n} = e^{\frac{2\pi i k}{n}} \cdot \underbrace{e^{\frac{2\pi i \cancel{n}}{\cancel{n}}}}_{1} = \left(e^{\frac{2\pi i}{n}}\right)^{k} = \omega_n^k$$

Divide-and-Conquer Approach

- Divide $p(x)$ of degree $N-1$ (N is even) into 2 polynomials of degree $N/2 - 1$ differently than in Karatsuba's algorithm

- $p_0(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{N-2} x^{N/2-1}$ (even coeff.)
  $p_1(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{N-1} x^{N/2-1}$ (odd coeff.)

$$p(x) = a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{N-2} x^{N-2}$$
$$+ a_1 x + a_3 x^3 + a_5 x^5 + \cdots + a_{N-1} x^{N-1}$$

$$p(x) = p_0(x^2) + x \cdot p_1(x^2)$$

Discrete Fourier Transform

$p(\omega_N^k)$

Evaluation for $k = 0, \dots, N-1$:

$p(x) = p_0(x^2) + x p_1(x^2)$

$(\omega_N^k)^2 = \omega_N^{2k} = \omega_{N/2}^k$

$= \omega_{N/2}^{k-N/2}$

$$p(\omega_N^k) = p_0\big((\omega_N^k)^2\big) + \omega_N^k \cdot p_1\big((\omega_N^k)^2\big)$$

$$= \begin{cases} p_0(\omega_{N/2}^k) + \omega_N^k \cdot p_1(\omega_{N/2}^k) & \text{if } k < N/2 \\ \underline{p_0\left(\omega_{N/2}^{k-N/2}\right)} + \omega_N^k \cdot \underline{p_1\left(\omega_{N/2}^{k-N/2}\right)} & \text{if } k \geq N/2 \end{cases}$$

For the coefficient vector a of $\cancel{p(x)}$:

$$\mathrm{DFT}_N(a) = \left(p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right), p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right)\right)$$
$$+ \left(\omega_N^0 p_1(\omega_{N/2}^0), \dots, \omega_N^{N/2-1} p_1\left(\omega_{N/2}^{N/2-1}\right), \omega_N^{N/2} p_1(\omega_{N/2}^0), \dots, \omega_N^{N-1} p_1\left(\omega_{N/2}^{N/2-1}\right)\right)$$

$p_0(\omega_{N/2}^\ell) \leftarrow$ values of $\mathrm{DFT}_{N/2}$ (coeff. of p_0)

Example

For the coefficient vector a of $p(x)$:

$$\mathrm{DFT}_N(a) = \left(p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right), p_0(\omega_{N/2}^0), \dots, p_0\left(\omega_{N/2}^{N/2-1}\right)\right)$$
$$+\left(\omega_N^0 p_1(\omega_{N/2}^0), \dots, \omega_N^{N/2-1} p_1\left(\omega_{N/2}^{N/2-1}\right), \omega_N^{N/2} p_1(\omega_{N/2}^0), \dots, \omega_N^{N-1} p_1\left(\omega_{N/2}^{N/2-1}\right)\right)$$

$N = 4$:

$$p(\omega_4^0) = p_0(\omega_2^0) + \omega_4^0 p_1(\omega_2^0)$$
$$p(\omega_4^1) = p_0(\omega_2^1) + \omega_4^1 p_1(\omega_2^1)$$
$$p(\omega_4^2) = p_0(\omega_2^0) + \omega_4^2 p_1(\omega_2^0)$$
$$p(\omega_4^3) = p_0(\omega_2^1) + \omega_4^3 p_1(\omega_2^1)$$

Need: $(p_0(\omega_2^0), p_0(\omega_2^1))$ and $(p_1(\omega_2^0), p_1(\omega_2^1))$

(DFTs of coefficient vectors of p_0 and p_1)

Recursive Structure

For simplicity, we abuse notation in the following:

- Poly. $p(x) = a_{N-1}x^{N-1} + \cdots + a_0$ with coefficient vector a

  Let $\mathrm{DFT}_N(p) \coloneqq \mathrm{DFT}_N(a)$

Recursive structure:

- For $N = 4$:

$$\left(\mathrm{DFT}_4(p)\right)_k = p\left(\omega_4^k\right)$$
$$= \left(\mathrm{DFT}_2(p_0)\right)_{k \bmod 2} + \omega_4^k \cdot \left(\mathrm{DFT}_2(p_1)\right)_{k \bmod 2}$$

- General N (assume N is even):

$$\left(\mathrm{DFT_N}(p)\right)_k = p\left(\omega_\mathrm{N}^k\right)$$

$$= \left(\mathrm{DFT}_{\mathrm{N}/2}(p_0)\right)_{k \bmod N/2} + \omega_N^k \cdot \left(\mathrm{DFT}_{\mathrm{N}/2}(p_1)\right)_{k \bmod N/2}$$

Computation of DFT_N

- Divide-and-conquer algorithm for $\mathrm{DFT}_N(p)$:

1. Divide

$N \leq 1$: $\mathrm{DFT}_1(p) = a_0$

$N > 1$: Divide p into p_0 (even coeff.) and p_1 (odd coeff).

$O(n)$

2. Conquer

Solve $\mathrm{DFT}_{N/2}(p_0)$ and $\mathrm{DFT}_{N/2}(p_1)$ recursively

3. Combine

Compute $\mathrm{DFT}_N(p)$ based on $\mathrm{DFT}_{N/2}(p_0)$ and $\mathrm{DFT}_{N/2}(p_1)$ $O(n)$

Analysis

- $T(N)$: time to compute $\mathrm{DFT}_N(p)$:

$$T(N) = 2T\left({}^{N}/_{2}\right) + O(N), \qquad T(1) = O(1)$$

- As for mergesort, comparing orders, closest pair of points:

$$T(N) = O(N \cdot \log N)$$

Small Improvement

$\omega_N^k = -\omega_N^{k-N/2}$

Polynomial p of degree $N-1$:

$$p(\omega_N^k) = \boxed{\begin{cases} p_0(\omega_{N/2}^k) + \underline{\omega_N^k \cdot p_1(\omega_{N/2}^k)} & \text{if } k < N/2 \\ p_0\left(\omega_{N/2}^{k-N/2}\right) + \underline{\omega_N^k} \cdot p_1\left(\omega_{N/2}^{k-N/2}\right) & \text{if } \underline{k \geq N/2} \end{cases}}$$

$$= \begin{cases} p_0(\omega_{N/2}^k) + \omega_N^k \cdot p_1(\omega_{N/2}^k) & \text{if } k < N/2 \\ p_0\left(\omega_{N/2}^{k-N/2}\right) - \omega_N^{k-N/2} \cdot p_1\left(\omega_{N/2}^{k-N/2}\right) & \text{if } k \geq N/2 \end{cases}$$

$\omega_N^{k+N/2} \cdot p_1(\omega_{N/2}^k)$

Need to compute $\underline{p_0(\omega_{N/2}^k)}$ and $\underline{\omega_N^k \cdot p_1(\omega_{N/2}^k)}$ for $0 \leq k < N/2$.

Example

$$p(\omega_4^0) = p_0(\omega_2^0) + \omega_4^0 \cdot p_1(\omega_2^0)$$

$$p(\omega_4^1) = p_0(\omega_2^1) + \omega_4^1 \cdot p_1(\omega_2^1)$$

$$p(\omega_4^2) = p_0(\omega_2^0) - \omega_4^0 \cdot p_1(\omega_2^0)$$

$$p(\omega_4^3) = p_0(\omega_2^1) - \omega_4^1 \cdot p_1(\omega_2^1)$$

Fast Fourier Transform (FFT) Algorithm

Algorithm FFT(a)

- Input: Array a of length N, where N is a power of 2
- Output: $\mathrm{DFT}_N(a)$

if $n = 1$ **then return** a_0; // $a = [a_0]$
$d^{[0]} := \mathrm{FFT}([a_0, a_2, \dots, a_{N-2}]);$
$d^{[1]} := \mathrm{FFT}([a_1, a_3, \dots, a_{N-1}]);$
$\omega_N := e^{2\pi i / N};\ \omega := 1;$
for $k = 0$ **to** $N/2 - 1$ **do** // $\omega = \omega_N^k$

$\quad x := \omega \cdot d_k^{[1]};$
$\quad d_k := d_k^{[0]} + x;\ d_{k+N/2} := d_k^{[0]} - x;$
$\quad \omega := \omega \cdot \omega_N$

end;
return $d = [d_0, d_1, \dots, d_{N-1}]$;