Chapter 2
Greedy Algorithms

Algorithm Theory
WS 2015/16

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Matroids

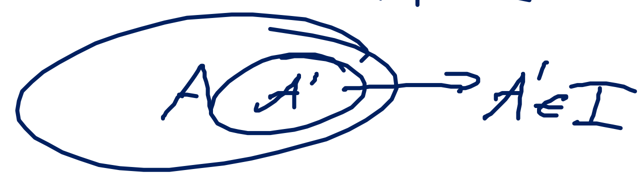
- Same, but more abstract…

Matroid: **pair (E, I)** (finite)

- $\boldsymbol{E}$: set, called the **ground set** set of elements
- $\boldsymbol{I}$: finite family of finite subsets of E (i.e., $I \subseteq 2^E$),
  called **independent sets**

$A \in I$

(E, I) needs to satisfy 3 properties:



1. Empty set is independent, i.e., $\boldsymbol{\emptyset \in I}$ (implies that $I \neq \emptyset$)
2. **Hereditary property**: For all $A \subseteq E$ and all $A' \subseteq A$,

   if $\boldsymbol{A \in I}$, then also $\boldsymbol{A' \in I}$

3. **Augmentation / Independent set exchange property:**
   If $A, B \in I$ and $|A| > |B|$, there exists $x \in A \setminus B$ such that

$$\mathbf{B'} := \boldsymbol{B \cup \{x\} \in I}$$

Matroids and Greedy Algorithms

Weighted matroid: each $e \in E$ has a weight $w(e) > 0$

Goal: find maximum weight independent set

Greedy algorithm:

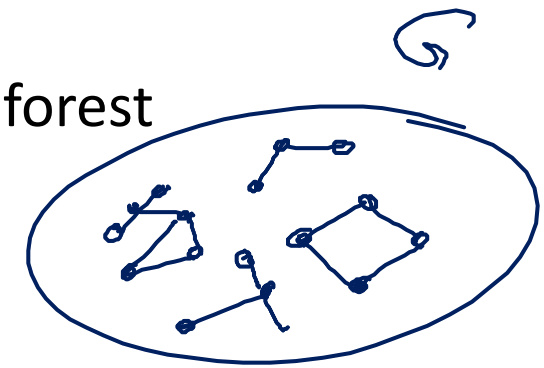
1. Start with $S = \emptyset$
2. Add max. weight $e \in E \setminus S$ to S such that $S \cup \{e\} \in I$

Claim: greedy algorithm computes optimal solution

Matroids: Examples

Forests of a graph $G = (V, E)$:

- forest F: subgraph with no cycles (i.e., $F \subseteq E$)
- $\mathcal{F}$: set of all forests → $(E, \mathcal{F})$ is a matroid
- Greedy algorithm gives maximum weight forest (equivalent to MST problem)



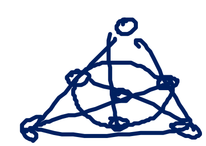
Bicircular matroid of a graph $G = (V, E)$:

sets of

- $\mathcal{B}$: set of edges such that every connected subset has ≤ 1 cycle
- $(E, \mathcal{B})$ is a matroid → greedy gives max. weight such subgraph

Linearly independent vectors:

- Vector space V, E: finite set of vectors, I: sets of lin. indep. vect.
- Fano matroid can be defined like that



Forest Matroid $G=(V,E)$

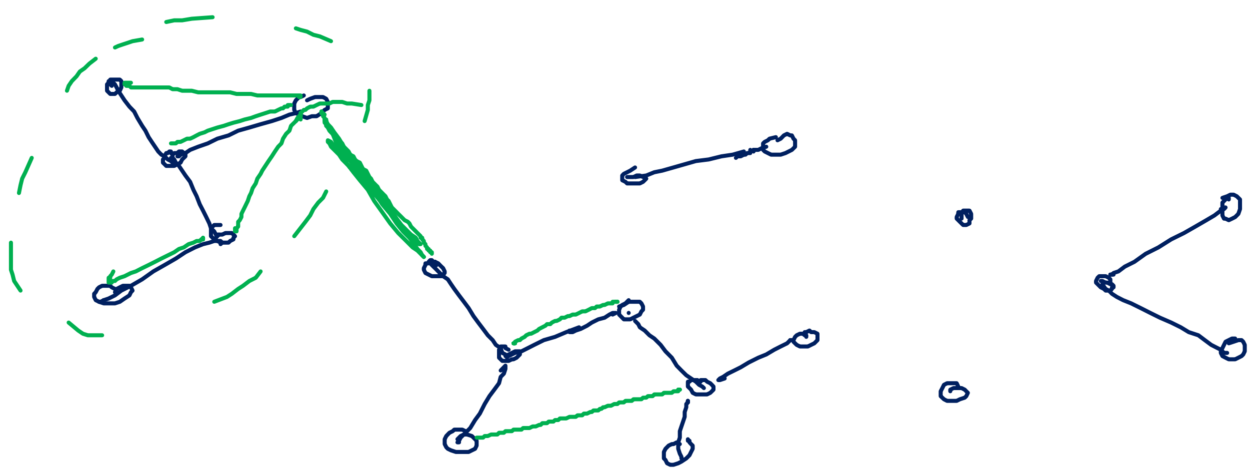
E: edge set of a graph I: all subsets of E with no cycles

(E,I) is a matroid:

1. $\emptyset \in I$: ✓
2. $A \in I,\ A' \subseteq A \Rightarrow A' \in I$: ✓
3. $F, F' \in I \quad |F| < |F'| \qquad \exists e \in F' \setminus F : F \cup \{e\}$ is a forest

F

F'



Greedoid

- Matroids can be generalized even more

- Relax hereditary property:

Replace $A' \subseteq A \in I \implies A' \in I$

by $\emptyset \neq \underline{A} \in I \implies \exists a \in A,\ \text{s.t. } A \setminus \{a\} \in I$

- Exchange property holds as before

- Under certain conditions on the weights, greedy is optimal for computing the max. weight $A \in I$ of a greedoid.
  - Additional conditions automatically satisfied by hereditary property

- More general than matroids