Chapter 3
Dynamic Programming

Algorithm Theory
WS 2015/16

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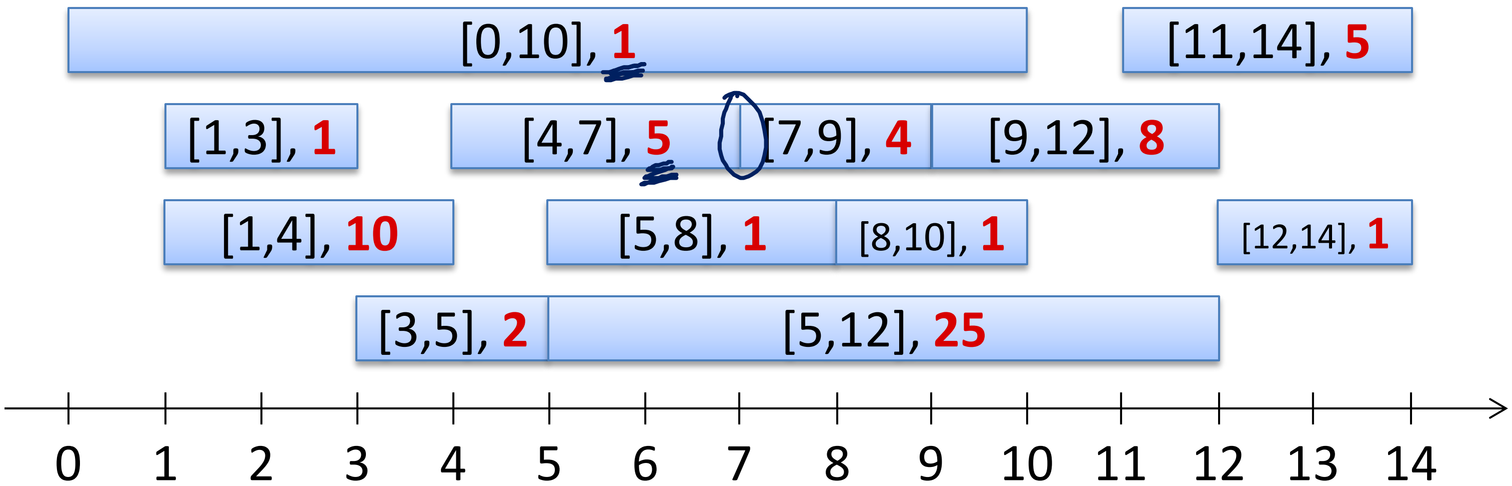
Weighted Interval Scheduling

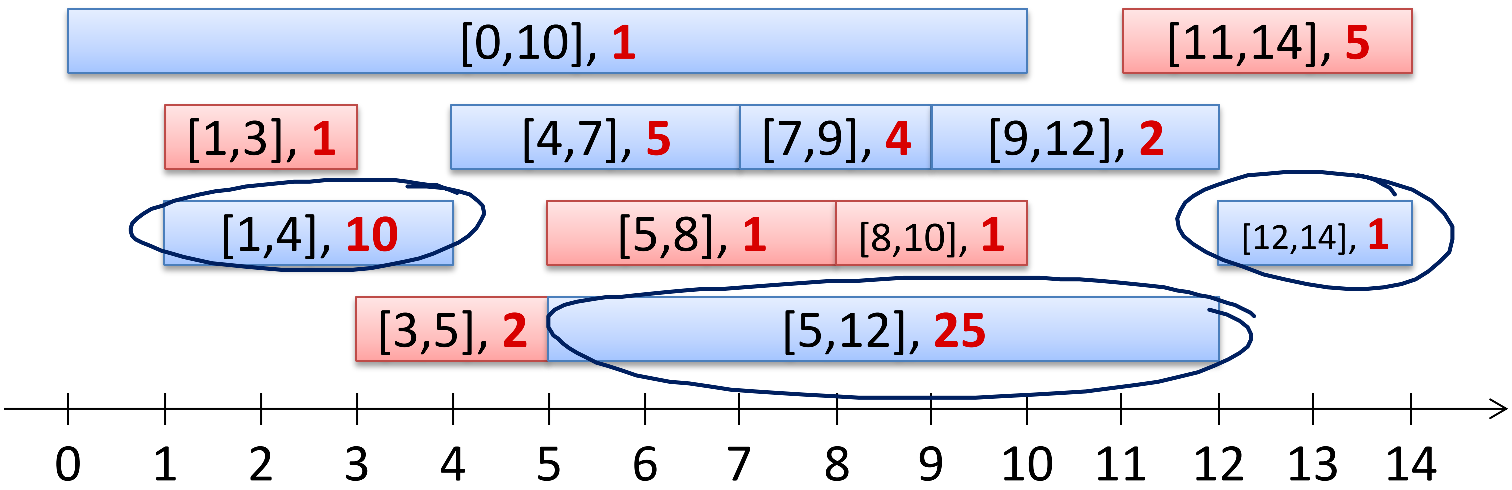
- **Given:** Set of intervals, e.g.
  [0,10],[1,3],[1,4],[3,5],[4,7],[5,8],[5,12],[7,9],[9,12],[8,10],[11,14],[12,14]
- Each interval has a **weight *w***



- **Goal:** Non-overlapping set of intervals of largest possible weight
  - Overlap at boundary ok, i.e., [4,7] and [7,9] are non-overlapping
- **Example:** Intervals are room requests of different importance

Greedy Algorithms

Choose available request with earliest finishing time:



- Algorithm is not optimal any more
  - It can even be arbitrarily bad…

- No greedy algorithm known that works

Solving Weighted Interval Scheduling

- Interval i: start time $s(i)$, finishing time: $f(i)$, weight: $w(i)$

- Assume intervals $1, \dots, n$ are sorted by increasing $f(i)$
  - $0 < f(1) \leq f(2) \leq \cdots \leq f(n)$, for convenience: $f(0) = 0$

$1, 2, 3, \dots, n-1, n$

- Simple observation:
  Opt. solution contains interval n or it doesn't contain interval n

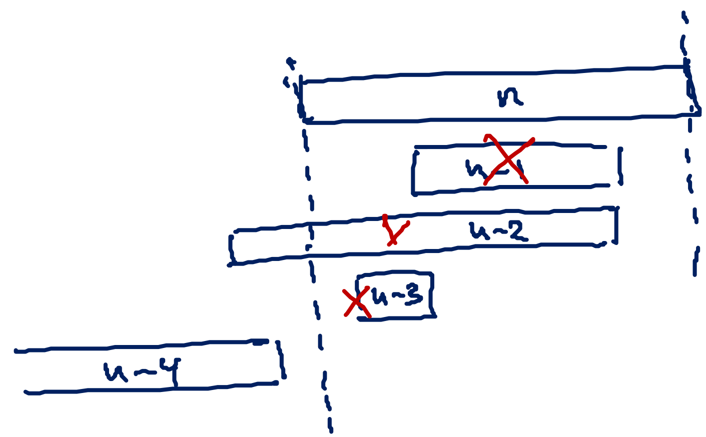
– opt. solution doesn't contain int. n

→ opt. sol. for intervals $1,\dots,n$ is the same as the opt. sol. for int. $1,\dots,n-1$

– opt. solution contains interval n

in example:

opt. sol. is composed of
opt. sol. for intervals $1,\dots,n-4$
and interval n



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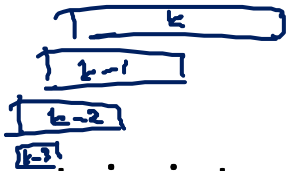


- Simple observation:
  Opt. solution contains interval n or it doesn't contain interval n

- Weight of optimal solution for only intervals $1, \dots, k$: $W(k)$
  Define $p(k) \coloneqq \max\{i \in \{0, \dots, k-1\} : f(i) \leq s(k)\}$

- Opt. solution does not contain interval n: $\boldsymbol{W(n) = W(n-1)}$

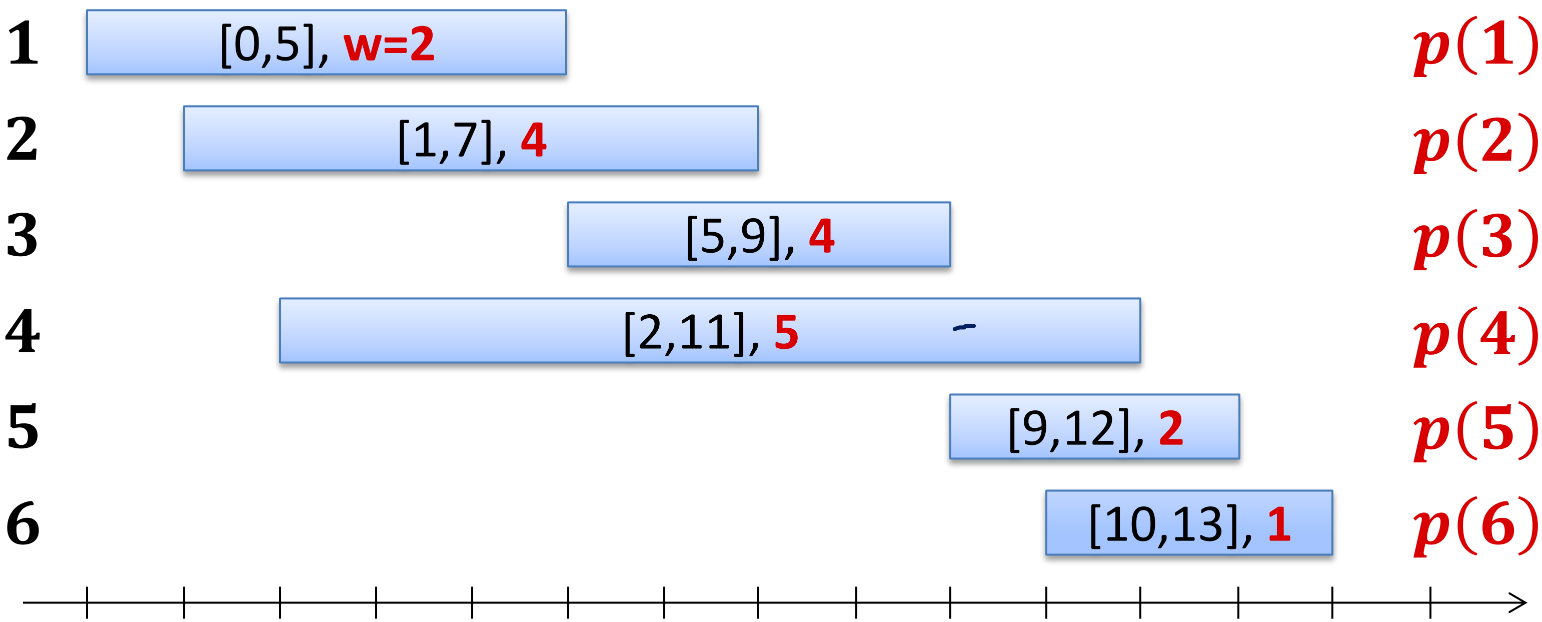
  Opt. solution contains interval n: $\boldsymbol{W(n) = w(n) + W(p(n))}$

$$W(n) = \max\{W(n-1),\ w(n) + W(p(n))\}$$

Example

$p(k)$

Interval:



$p(1) = 0$

$p(2) = 0$

$p(3) = 1$

$p(4) = 0$

$p(5) = 3$

$p(6) = 3$

compute $p(k)$:

binary search

all $p(k)$: $O(n \log n)$

Recursive Definition of Optimal Solution

- Recall:
  - $W(k)$: weight of optimal solution with intervals $1, \dots, k$
  - $p(k)$: last interval to finish before interval k starts

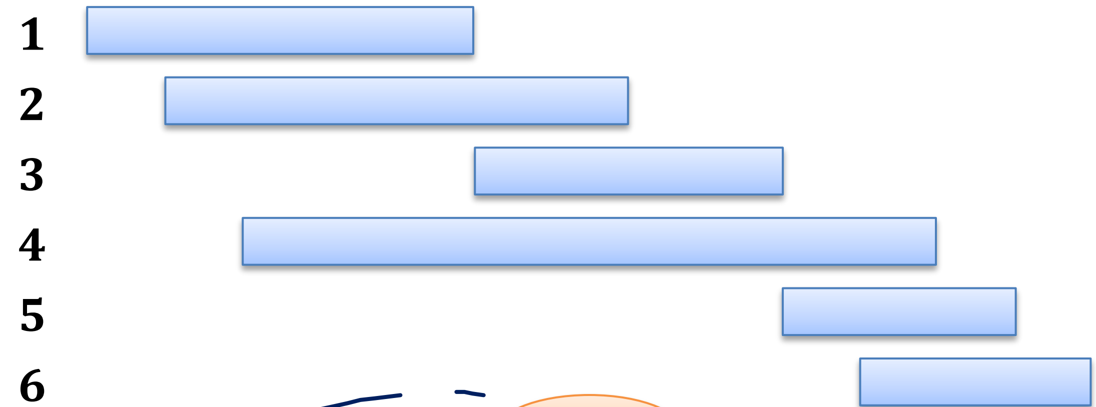
- Recursive definition of optimal weight:

$$\forall k > 1:\ \underline{W(k)} = \max\{\underline{W(k-1)}, \underline{w(k)} + \underline{W(p(k))}\}$$

$$\underline{W(1) = w(1)} \qquad W(0) = 0$$

- Immediately gives a simple, recursive algorithm

Running Time of Recursive Algorithm



$p(1) = 0$

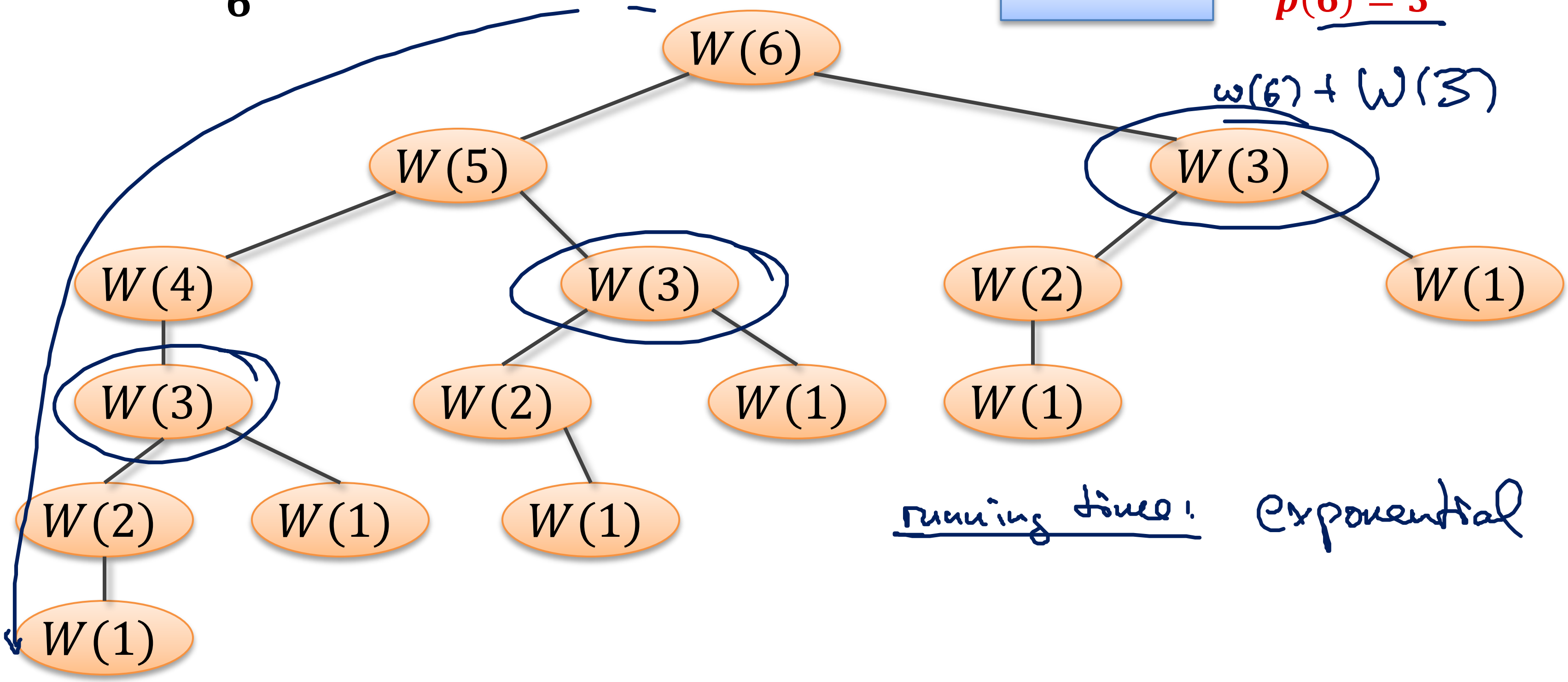
$p(2) = 0$

$p(3) = 1$

$p(4) = 0$

$p(5) = 3$

$p(6) = 3$



$w(6) + W(3)$

running time: exponential

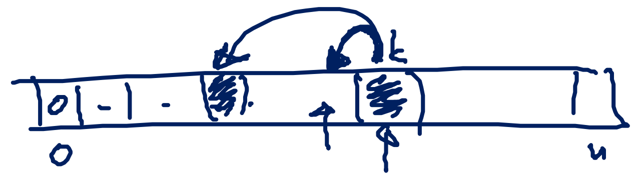
Memoizing the Recursion

- Running time of recursive algorithm: exponential!
- But, alg. only solves n different sub-problems: $W(1), \dots, W(n)$
- There is no need to compute them multiple times

Memoization:

- **Store already computed values** for future use (recursive calls)

Efficient algorithm:



1. $W[0] \coloneqq 0$; compute values $p(i)$
2. **for** $i \coloneqq 1$ **to** n **do**
3. $\quad W[i] \coloneqq \max\{W[i-1], w(i) + W[p(i)]\}$
4. **end**

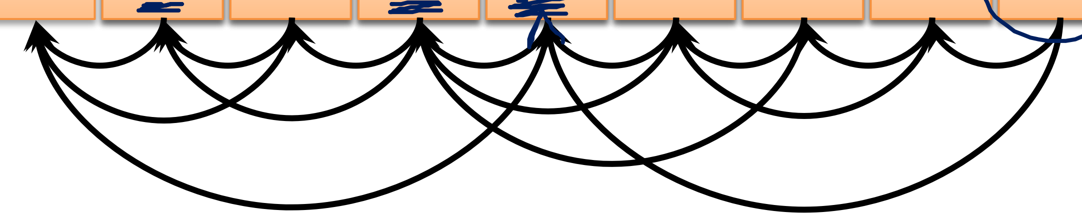
O(n) time

Example

1	$w = 2$	$p(1) = 0$
2	$w = 4$	$p(2) = 0$
3	$w = 4$	$p(3) = 1$
4	$w = 5$	$p(4) = 0$
5	$w = 2$	$p(5) = 3$
6	$w = 1$	$p(6) = 3$
7	$w = 3$	$p(7) = 5$
8	$w = 6$	$p(8) = 4$

	$W[0]$	$W[1]$	$W[2]$	$W[3]$	$W[4]$	$W[5]$	$W[6]$	$W[7]$	$W[8]$
W:	0	2	4	6	6	8	8	11	12

8, 3, 1



Computing the schedule: **store where you come from!**

Matrix-chain multiplication

Given: sequence (chain) $\langle A_1, A_2, \dots, A_n \rangle$ of matrices

Goal: compute the product $A_1 \cdot A_2 \cdot \dots \cdot A_n$

$(\_\_\_) \cdot (\_\_\_)$

Problem: Parenthesize the product in a way that minimizes the number of scalar multiplications.

Definition: A product of matrices is *fully parenthesized* if it is

- a single matrix
- or the product of two fully parenthesized matrix products, surrounded by parentheses.

Example

All possible fully parenthesized matrix products of the chain $\langle A_1, A_2, A_3, A_4 \rangle$:

$$(A_1(A_2(A_3A_4)))$$

$$(A_1((A_2A_3)A_4))$$

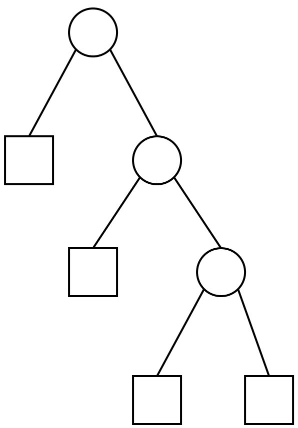
$$((A_1A_2)(A_3A_4))$$

$$((A_1(A_2A_3))A_4)$$

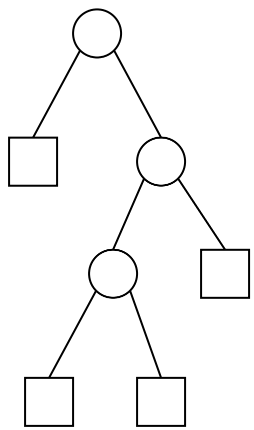
$$(((A_1A_2)A_3)A_4)$$

Different parenthesizations

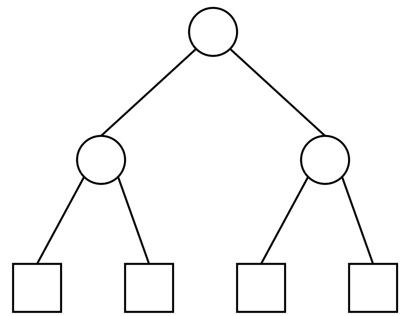
Different parenthesizations correspond to different trees:



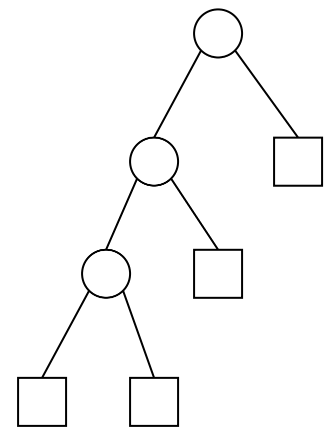
$(A_1(A_2(A_3A_4)))$



$(A_1((A_2A_3)A_4))$



$((A_1A_2)(A_3A_4))$



$(((A_1A_2)A_3)A_4)$

Number of different parenthesizations

- Let $P(n)$ be the number of alternative parenthesizations of the product $A_1 \cdot \ldots \cdot A_n$:

$$P(1) = 1$$

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k), \qquad \text{for } n \geq 2$$

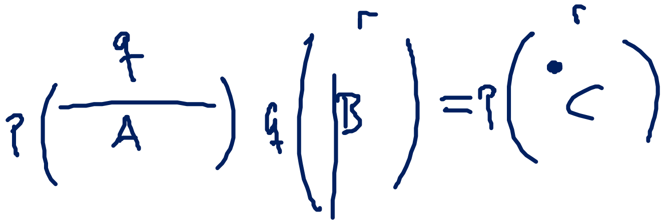
$$P(n+1) = \frac{1}{n+1}\binom{2n}{n} \approx \frac{4^n}{n\sqrt{\pi n}} + O\left(\frac{4^n}{\sqrt{n^5}}\right)$$

$$P(n+1) = C_n \qquad (n^{th} \text{ Catalan number})$$

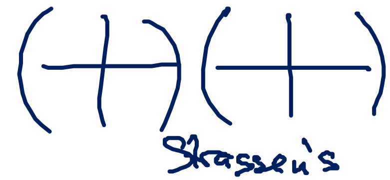
- Thus: Exhaustive search needs exponential time!

Multiplying Two Matrices

$$A = (a_{ij})_{p\times q}, \qquad B = (b_{ij})_{q\times r}, \qquad A\cdot B = C = (c_{ij})_{p\times r}$$



$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}$$



Algorithm *Matrix-Mult*

Input: $(p\times q)$ matrix A, $(q \times r)$ matrix B

Output: $(p \times r)$ matrix $C = A \cdot B$

```
for i := 1 to p do
    for j := 1 to r do
        C[i,j] := 0;
        for k := 1 to q do
            C[i,j] := C[i,j] + A[i,k] · B[k,j]
```

Remark:

Using this algorithm, multiplying two $(n \times n)$ matrices requires n^3 multiplications. This can also be done using $O(n^{2.376})$ multiplications.

Number of multiplications and additions: $\cancel{p \cdot q \cdot r}$

Matrix-chain multiplication: Example

Computation of the product $A_1 A_2 A_3$, where

$A_1 : (\underline{50} \times \underline{5})$ matrix

$A_2 : (5 \times \underline{100})$ matrix

$A_3 : (\underline{100 \times} 10)$ matrix

a) Parenthesization $((A_1 A_2)A_3)$ and $(A_1(A_2 A_3))$ require:

$A' = (A_1 A_2)$: $50 \cdot 5 \cdot 100 = 25000$

50×100

$A'' = (A_2 A_3)$: $5 \cdot 100 \cdot 10 = 5000$

$5 \cdot 10$

$A' A_3$: $50 \cdot 100 \cdot 10 = 50000$

$A_1 A''$: $50 \cdot 5 \cdot 10 = 2500$

Sum: 75'000 7'500

Structure of an Optimal Parenthesization

$1, \ldots, n$

- $(A_{\ell \ldots r})$: optimal parenthesization of $A_\ell \cdot \ldots \cdot A_r$

  For some $1 \leq k < n$: $(\boldsymbol{A_{1 \ldots n}}) = ((\boldsymbol{A_{1 \ldots k}}) \cdot (\boldsymbol{A_{k+1 \ldots n}}))$

- Any optimal solution contains optimal solutions for sub-problems

- Assume matrix A_i is a $(d_{i-1} \times d_i)$-matrix

- Cost to solve sub-problem $A_\ell \cdot \ldots \cdot A_r, \ell \leq r$ optimally: $C(\ell, r)$

- Then:

$a \ldots k \quad k+1 \ldots b \qquad d_{a-1} \times d_k \quad d_k \times d_b$

$$\boldsymbol{C(a,b) = \min_{a \leq k < b} C(a,k) + C(k+1,b) + d_{a-1} d_k d_b}$$

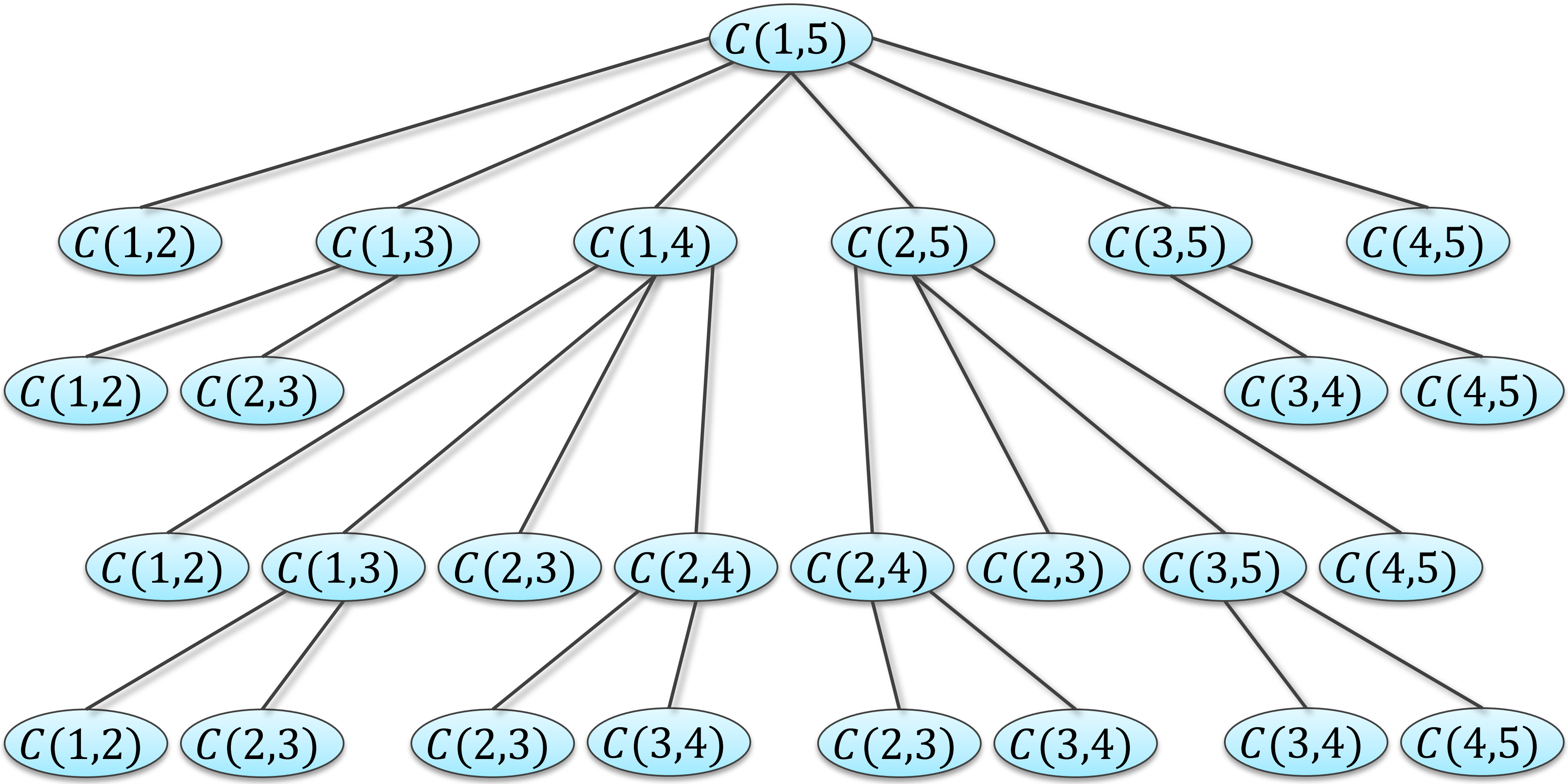
$$\boldsymbol{C(a,a) = 0}$$

k

$C(1,n) \qquad \underbrace{C(1,k)}_{k'} \qquad \underbrace{C(k+1,n)}_{k''}$

Recursive Computation of Opt. Solution

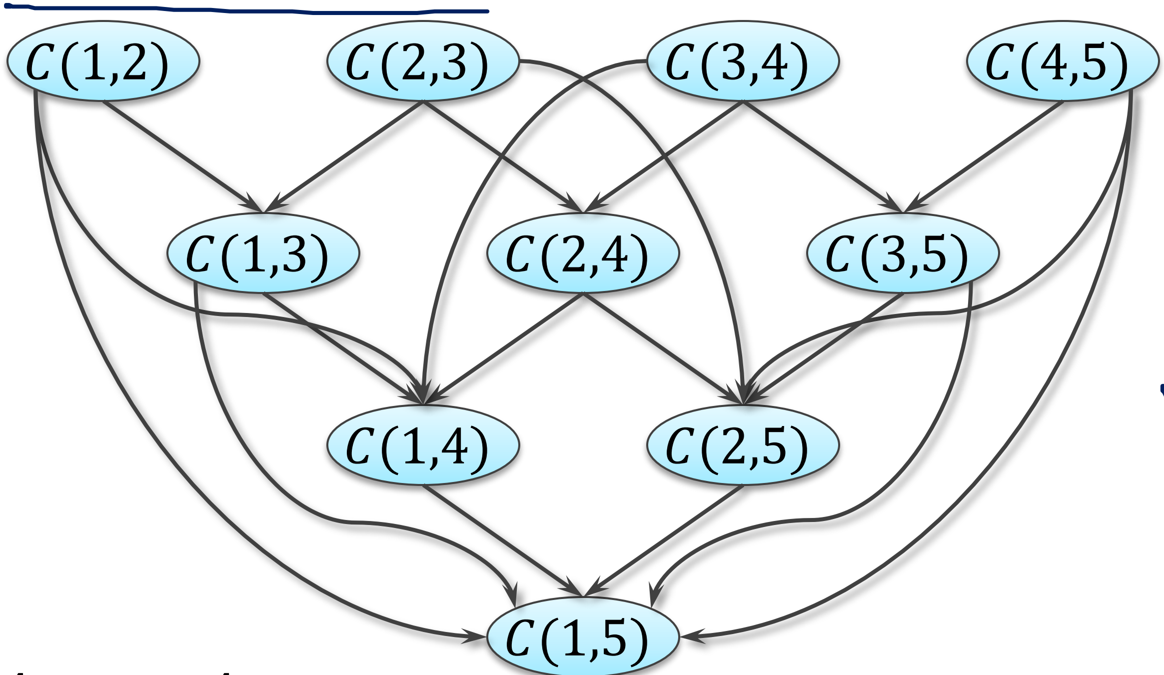
Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:



Using Meomization

C(1,5)

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:



Compute $A_1 \cdot \ldots \cdot A_n$:

- Each $C(i,j)$, $i < j$ is computed exactly once → $O(n^2)$ values
- Each $C(i,j)$ dir. depends on $C(i,k)$, $C(k,j)$ for $i < k < j$

  Cost for each $C(i,j)$: $O(n)$ → overall time: $\boldsymbol{O(n^3)}$

$j-1$

Dynamic Programming

„Memoization“ for increasing the efficiency of a recursive solution:

- Only the *first time* a sub-problem is encountered, its solution is computed and then stored in a table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned

  (without repeated computation!).

- *Computing the solution*: For each sub-problem, store how the value is obtained (according to which recursive rule).

Dynamic Programming

Dynamic programming / memoization can be applied if

- Optimal solution contains optimal solutions to sub-problems (recursive structure)

- Number of sub-problems that need to be considered is small

Remarks about matrix-chain multiplication

1. There is an algorithm that determines an optimal parenthesization in time
$$O(n \cdot \log n).$$

2. There is a linear time algorithm that determines a parenthesization using at most
$$1.155 \cdot C(1,n)$$
multiplications.