Chapter 3
Dynamic Programming

Algorithm Theory
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Fabian Kuhn

Dynamic Programming

„Memoization“ for increasing the efficiency of a recursive solution:

- Only the *first time* a sub-problem is encountered, its solution is computed and then stored in a table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned
  (without repeated computation!).

- *Computing the solution*: For each sub-problem, store how the value is obtained (according to which recursive rule).

Dynamic Programming

Dynamic programming / memoization can be applied if

- Optimal solution contains optimal solutions to sub-problems (recursive structure)

- Number of sub-problems that need to be considered is small

Knapsack

- n items $1, \dots, n$, each item has weight w_i and value v_i
- Knapsack (bag) of capacity W

- Goal: pack items into knapsack such that total weight is at most W and total value is maximized:

$$\max \sum_{i \in S} v_i$$

$$\text{s.t. } S \subseteq \{1, \dots, n\} \text{ and } \sum_{i \in S} w_i \leq W$$

- E.g.: jobs of length w_i and value v_i, server available for W time units, try to execute a set of jobs that maximizes the total value

Recursive Structure?

OPT(k)

- Optimal solution: $\mathcal{O}$
- If $n \notin \mathcal{O}$: $\mathrm{OPT}(n) = \mathrm{OPT}(n-1)$
- What if $n \in \mathcal{O}$?
  - Taking n gives value v_n
  - But, n also occupies space w_n in the bag (knapsack)
  - There is space for $W - w_n$ total weight left!

$$\mathrm{OPT}(n) = v_n \text{ (value)} + \text{optimal solution with first } n-1 \text{ items and knapsack of capacity } W - w_n$$

A More Complicated Recursion

$OPT(n, W)$

$OPT(k, x)$: value of optimal solution with items $1, \dots, k$ and knapsack of capacity x

n possibilities

Recursion:

only makes sense if $x - w_k \geq 0$

$$OPT(k,x) = \max\{OPT(k-1,x),\ v_k + OPT(k-1, x-w_k)\}$$

items $1 \dots k$ | capacity

opt sol. if k is not cont.

remaining cap. after taking k

Initialization:

$OPT(0,x) = 0$

$OPT(k,0) = 0$

#possibilities for 2nd parameter: ∞, exp.

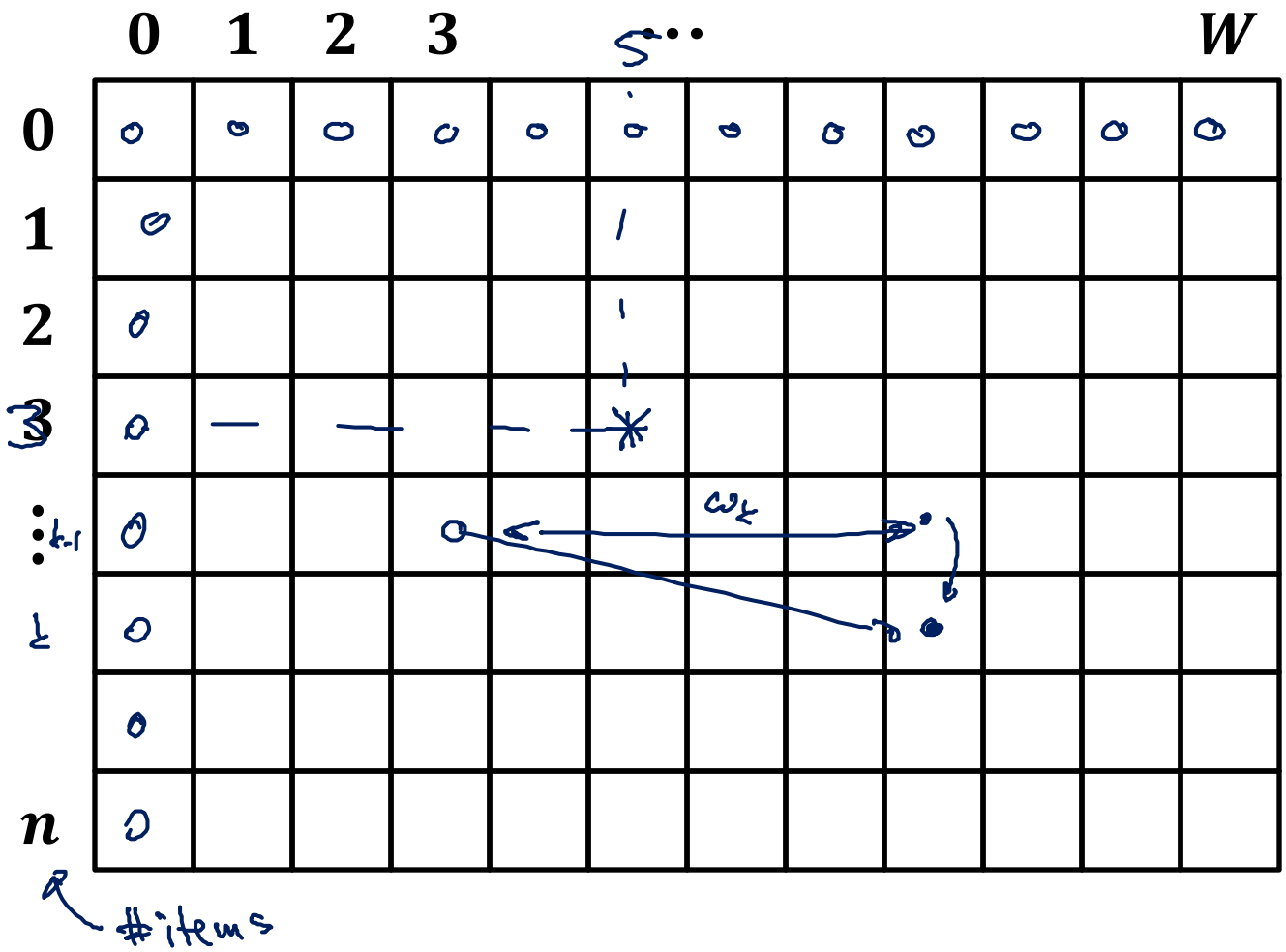
assumption: weights are integers

$\longrightarrow$ $W+1$ options for 2nd parameter

Dynamic Programming Algorithm

Set up table for all possible $\mathrm{OPT}(k, x)$-values

- Assume that all weights w_i are integers!



Example

- 8 items: $(3,2), (2,4), (4,1), (5,6), (3,3), (4,3), (5,4), (6,6)$
  Knapsack capacity: 12

  weight value

- $OPT(k,x) = \max\{OPT(k-1,x), OPT(k-1,x-w_k) + v_k\}$

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	2	2	2	2	2	2	2	2	2	2
2	0	4	4	4	6	6	6	6				
3												
4												
5												
6												
7												
8												

items

Running Time of Knapsack Algorithm

- **Size of table:** $O(n \cdot W)$
- Time per table entry: $O(1)$ → **overall time:** $\boldsymbol{O(nW)}$
- Computing solution (set of items to pick):
  Follow $\leq n$ arrows → $O(n)$ time (after filling table)
- Note: Time depends on W → can be exponential in n…
- And it is problematic if weights are not integers.

weights are rational would be possible

another special case: values are integers

- NP-hard problem
- but, one can compute an arbitrarily close to opt. solution in poly. time

String Matching Problems

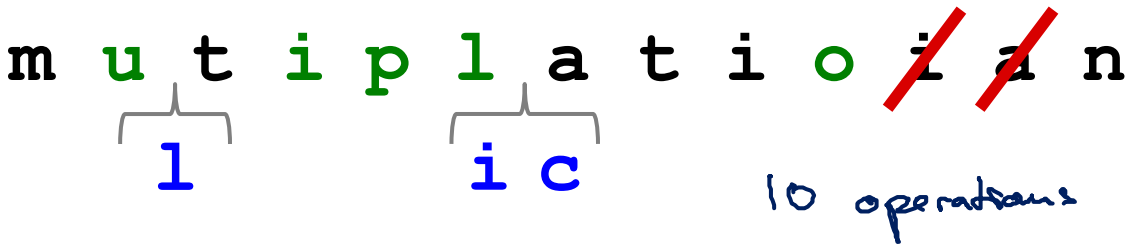
Edit distance:

- For two given strings A and B, efficiently compute the

  edit distance $D(A,B)$ (# edit operations to transform A into B)

  as well as a minimum sequence of edit operations that transform A into B.

- **Example:** mathematician → multiplication:



Edit Distance

Given: Two strings $A = a_1 a_2 \dots a_m$ and $B = b_1 b_2 \dots b_n$

Goal: Determine the minimum number $D(A, B)$ of edit operations required to transform A into B

Edit operations:

a) **Replace** a character from string A by a character from B

b) **Delete** a character from string A

c) **Insert** a character from string B into A

```
m a - t h e m - - a t i c i a n
m u l t i p l i c a t i o - - n
```

Edit Distance – Cost Model

- Cost for **replacing** character a by b: $\boldsymbol{c(a,b) \geq 0}$

$c(a,a) = 0$

- Capture insert, delete by allowing $a = \varepsilon$ or $b = \varepsilon$:
  - Cost for **deleting** character a: $\boldsymbol{c(a,\varepsilon)}$
  - Cost for **inserting** character b: $\boldsymbol{c(\varepsilon,b)}$

- **Triangle inequality**:

$$c(a,c) \leq c(a,b) + c(b,c)$$

→ each character is changed at most once!

- **Unit cost model**: $c(a,b) = \begin{cases} 1, & \text{if } a \neq b \\ 0, & \text{if } a = b \end{cases}$

Recursive Structure

- Optimal "alignment" of strings (unit cost model)

  `bbcadfagikccm` and `abbagflrgikacc`:

```
- b b c a g f a | - g i k - c c m
a b b - a d f l | r g i k a c c -
```

- Consists of optimal "alignments" of sub-strings, e.g.:

```
-bbcagfa      -gik-ccm
abb-adfl  and rgikacc-
```

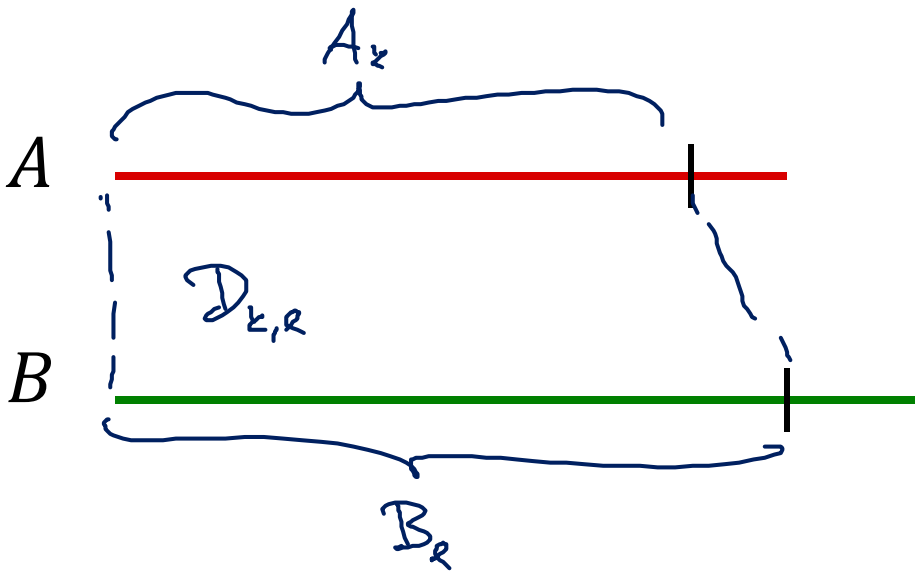
- Edit distance between $A_{1,m} = a_1 \dots a_m$ and $B_{1,n} = b_1 \dots b_n$:

$$D(A,B) = \min_{k,\ell}\{D(A_{1,k}, B_{1,\ell}) + D(A_{k+1,m}, B_{\ell+1,n})\}$$

Computation of the Edit Distance

Let $A_k := a_1 \dots a_k$, $B_\ell := b_1 \dots b_\ell$, and

$$D_{k,\ell} := D(A_k, B_\ell)$$

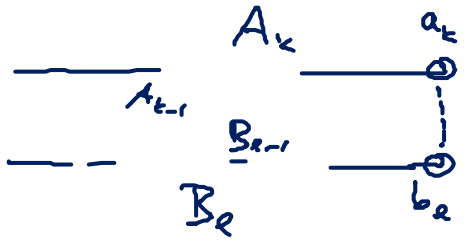


Computation of the Edit Distance

Three ways of ending an "alignment" between A_k and B_ℓ:

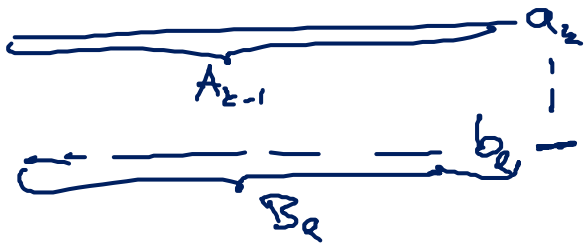
1. a_k is replaced by b_ℓ:

$$D_{k,\ell} = D_{k-1,\ell-1} + c(a_k, b_\ell)$$



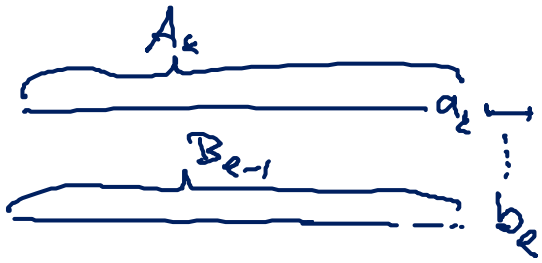
2. a_k is deleted:

$$D_{k,\ell} = D_{k-1,\ell} + c(a_k, \varepsilon)$$



3. b_ℓ is inserted:

$$D_{k,\ell} = D_{k,\ell-1} + c(\varepsilon, b_\ell)$$

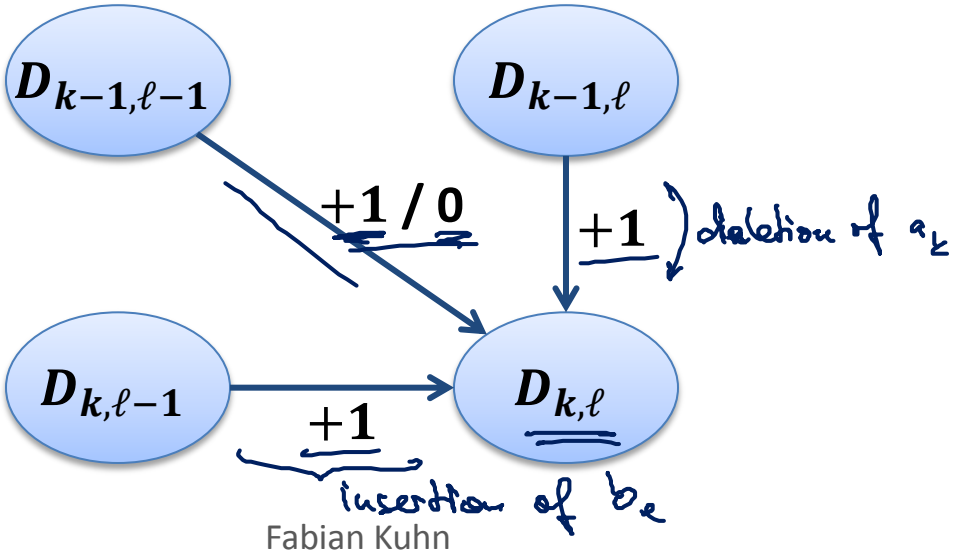


Computing the Edit Distance

- Recurrence relation (for $k, \ell \geq 1$)

$$D_{k,\ell} = \min \begin{Bmatrix} D_{k-1,\ell-1} + c(a_k, b_\ell) \\ D_{k-1,\ell} + c(a_k, \varepsilon) \\ D_{k,\ell-1} + c(\varepsilon, b_\ell) \end{Bmatrix} = \underbrace{\min \begin{Bmatrix} D_{k-1,\ell-1} + 1 \,/\, 0 \\ D_{k-1,\ell} + 1 \\ D_{k,\ell-1} + 1 \end{Bmatrix}}_{\textbf{unit cost model}}$$

- Need to compute $D_{i,j}$ for all $0 \leq i \leq k$, $0 \leq j \leq \ell$:



Recurrence Relation for the Edit Distance

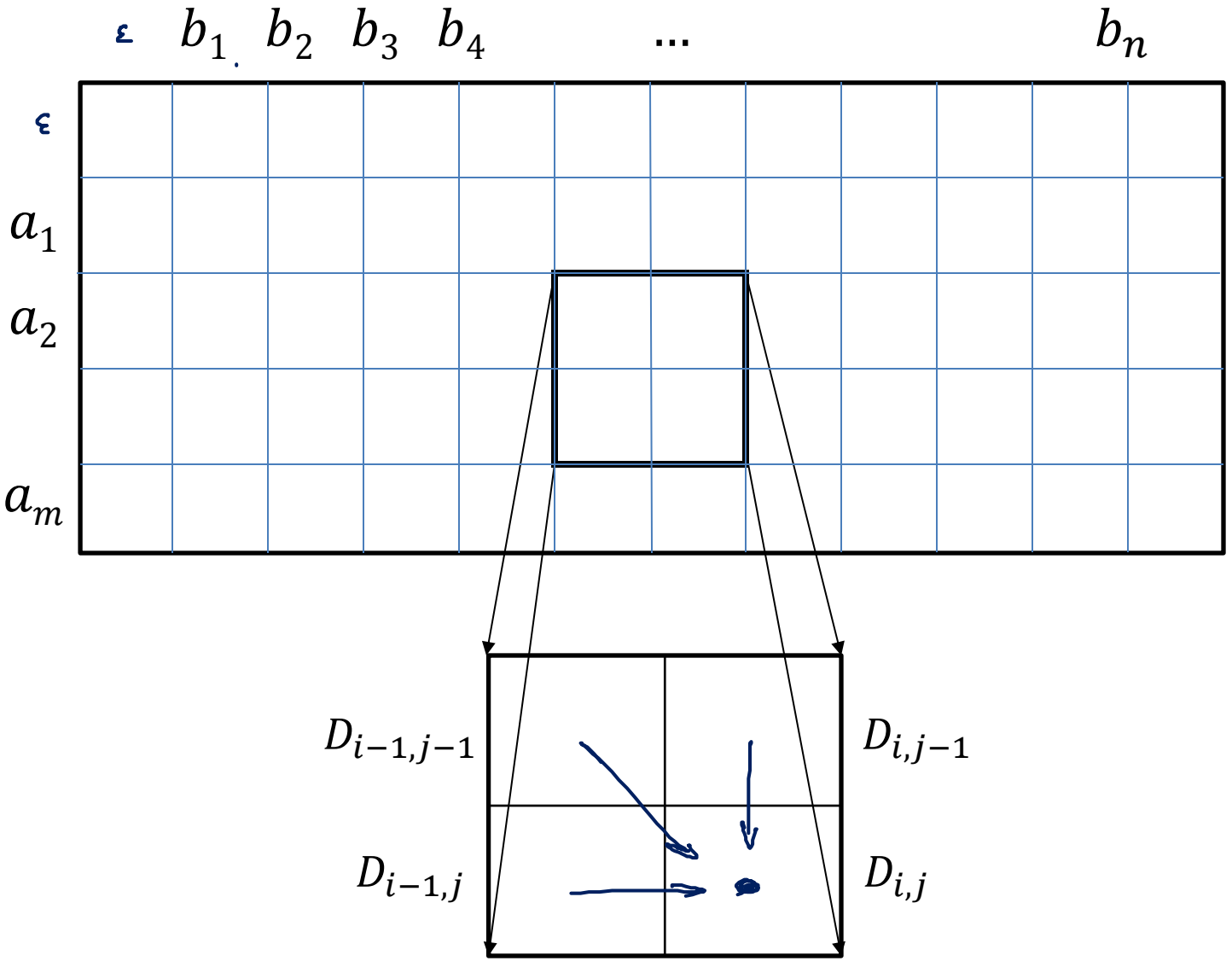
Base cases:

$$D_{0,0} = D(\varepsilon, \varepsilon) = 0$$
$$D_{0,j} = D(\varepsilon, B_j) = D_{0,j-1} + c(\varepsilon, b_j) \qquad (= j \text{ in unit cost})$$
$$D_{i,0} = D(A_i, \varepsilon) = D_{i-1,0} + c(a_i, \varepsilon) \qquad (= i \text{ " " " })$$

Recurrence relation:

$$D_{i,j} = \min \begin{cases} D_{k-1,\ell-1} + c(a_k, b_\ell) \\ D_{k-1,\ell} + c(a_k, \varepsilon) \\ D_{k,\ell-1} + c(\varepsilon, b_\ell) \end{cases}$$

Order of solving the subproblems



Algorithm for Computing the Edit Distance

Algorithm *Edit-Distance*

Input: 2 strings $A = a_1 \dots a_m$ and $B = b_1 \dots b_n$

Output: matrix $D = (D_{ij})$

$1\ D[0,0] \coloneqq 0;$

2 **for** $i \coloneqq 1$ **to** m **do** $D[i,0] \coloneqq i;$

3 **for** $j \coloneqq 1$ **to** n **do** $D[0,j] \coloneqq j;$

4 **for** $i \coloneqq 1$ **to** m **do**

5 **for** $j \coloneqq 1$ **to** n **do**

6 $$D[i,j] \coloneqq \min \begin{Bmatrix} D[i-1,j] & +1 \\ D[i,j-1] & +1 \\ D[i-1,j-1] & + c(a_i, b_j) \end{Bmatrix};$$