Chapter 6
Graph Algorithms

Applications of Max Flow

Algorithm Theory
WS 2015/16

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Baseball Elimination

Team i	Wins w_i	Losses ℓ_i	To Play r_i	Against = r_{ij}				
				NY	Balt.	T. Bay	Tor.	Bost.
New York	81	70	11	-	2	4	2	3
Baltimore	79	77	6	2	-	2	1	1
Tampa Bay	79	75	8	4	2	-	1	1
Toronto	76	80	6	2	1	1	-	2
Boston	71	84	7	3	1	1	2	-

- Only wins/losses possible (no ties), winner: team with most wins
- Which teams can still win (as least as many wins as top team)?
- Boston is eliminated (cannot win):
  - Boston can get at most 78 wins, New York already has 81 wins
- If for some i, j: $w_i + r_i < w_j$ → team i is eliminated
- Sufficient condition, but not a necessary one!

Baseball Elimination

Team i	Wins w_i	Losses ℓ_i	To Play r_i	Against = r_{ij}				
				NY	Balt.	T. Bay	Tor.	Bost.
New York	81	70	11	-	2	4 5	2	3
Baltimore	79	77	6	2	-	2	1	1
Tampa Bay	79	75	8	4 5	2	-	1	1
Toronto	76	80	6	2	1	1	-	2
Boston	71	84	7	3	1	1	2	-

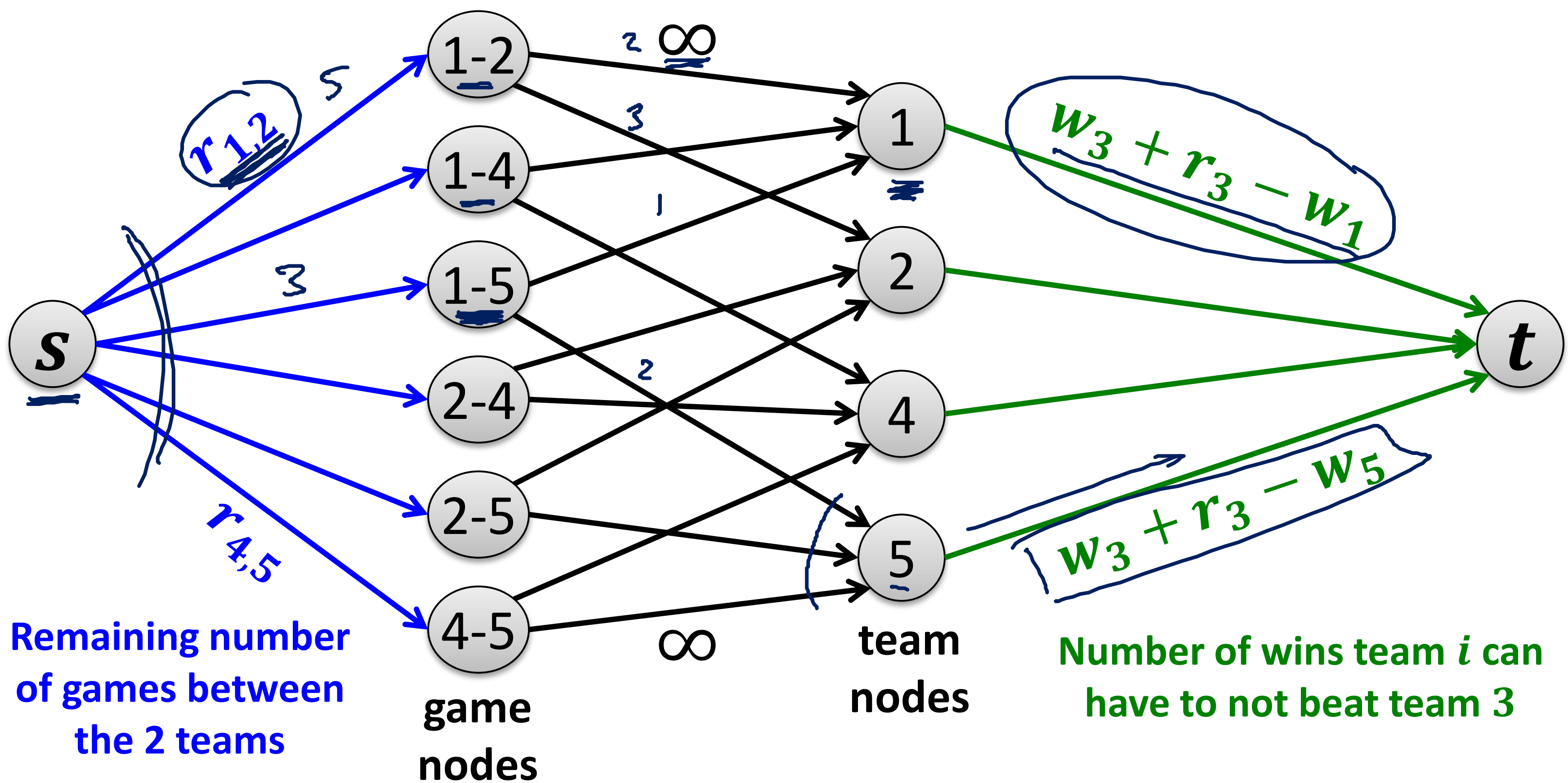
- Can Toronto still finish first?
- Toronto can get $82 > 81$ wins, but:
  NY and Tampa have to play 4 more times against each other
  → if NY wins two, it gets 83 wins, otherwise, Tampa has 83 wins
- Hence: Toronto cannot finish first
- How about the others? How can we solve this in general?

Max Flow Formulation

Teams 1, ..., 5

- Can team 3 finish with most wins?

$w_3 + r_3$



- Team 3 can finish first iff all source-game edges are saturated

Reason for Elimination

AL East: Aug 30, 1996

Team i	Wins w_i	Losses ℓ_i	To Play r_i	Against = r_{ij}				
				NY	Balt.	Bost.	Tor.	Detr.
New York	75	59	28	-	3	8	7	3
Baltimore	71	63	28	3	-	2	7	4
Boston	69	66	27	8	2	-	0	0
Toronto	63	72	27	7	7	0	-	0
Detroit	49	86	27	3	4	0	0	-

R

- Detroit could finish with $49 + 27 = 76$ wins
- Consider $R = \{\mathrm{NY}, \mathrm{Bal}, \mathrm{Bos}, \mathrm{Tor}\}$
  - Have together already won $w(R) = 278$ games
  - Must together win at least $r(R) = 27$ more games
- On average, teams in R win $\frac{278+27}{4} = 76.25$ games

Reason for Elimination

Certificate of elimination:

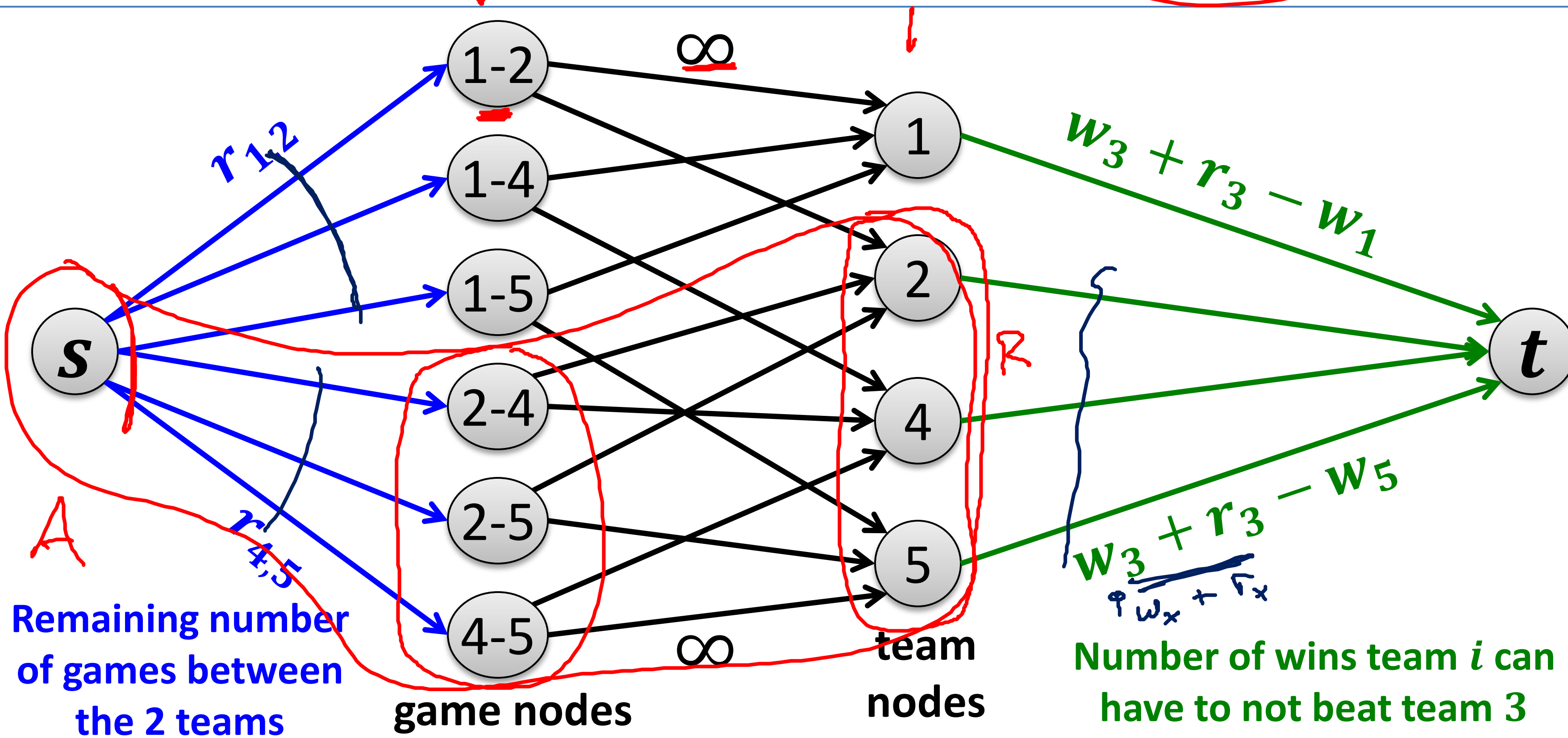
$$R \subseteq \underset{X \setminus \{x\}}{X}, \qquad w(R) := \underbrace{\sum_{i \in R} w_i}_{\substack{\text{\#wins of} \\ \text{nodes in } R}}, \qquad r(R) := \underbrace{\sum_{i,j \in R} r_{i,j}}_{\substack{\text{\#remaining games} \\ \text{among nodes in } R}}$$

Team $x \in X$ is eliminated by R if

$$\boxed{\frac{w(R) + r(R)}{|R|} > w_x + r_x.}$$

Minimum Cut

A → B



$$cap(A,B) = |R|(w_x + r_x) - w(R) + r(X) - r(R) < r(X)$$

eliminated

$$\frac{w(R) + r(R)}{|R|} > w_x + r_x$$

Reason for Elimination

Theorem: Team x is eliminated if and only if there exists a subset $R \subseteq X$ of the teams X such that x is eliminated by R.

Proof Idea:

- Minimum cut gives a certificate…
- If x is eliminated, max flow solution does not saturate all outgoing edges of the source.
- Team nodes of unsaturated source-game edges are saturated
- Source side of min cut contains all teams of saturated team-dest. edges of unsaturated source-game edges
- Set of team nodes in source-side of min cut give a certificate R

Circulations with Demands

Given: Directed network with positive edge capacities

Sources & Sinks: Instead of one source and one destination, several sources that generate flow and several sinks that absorb flow.

Supply & Demand: sources have supply values, sinks demand values

Goal: Compute a flow such that source supplies and sink demands are exactly satisfied

- The circulation problem is a feasibility rather than a maximization problem

Circulations with Demands: Formally

Given: Directed network $G = (V, E)$ with

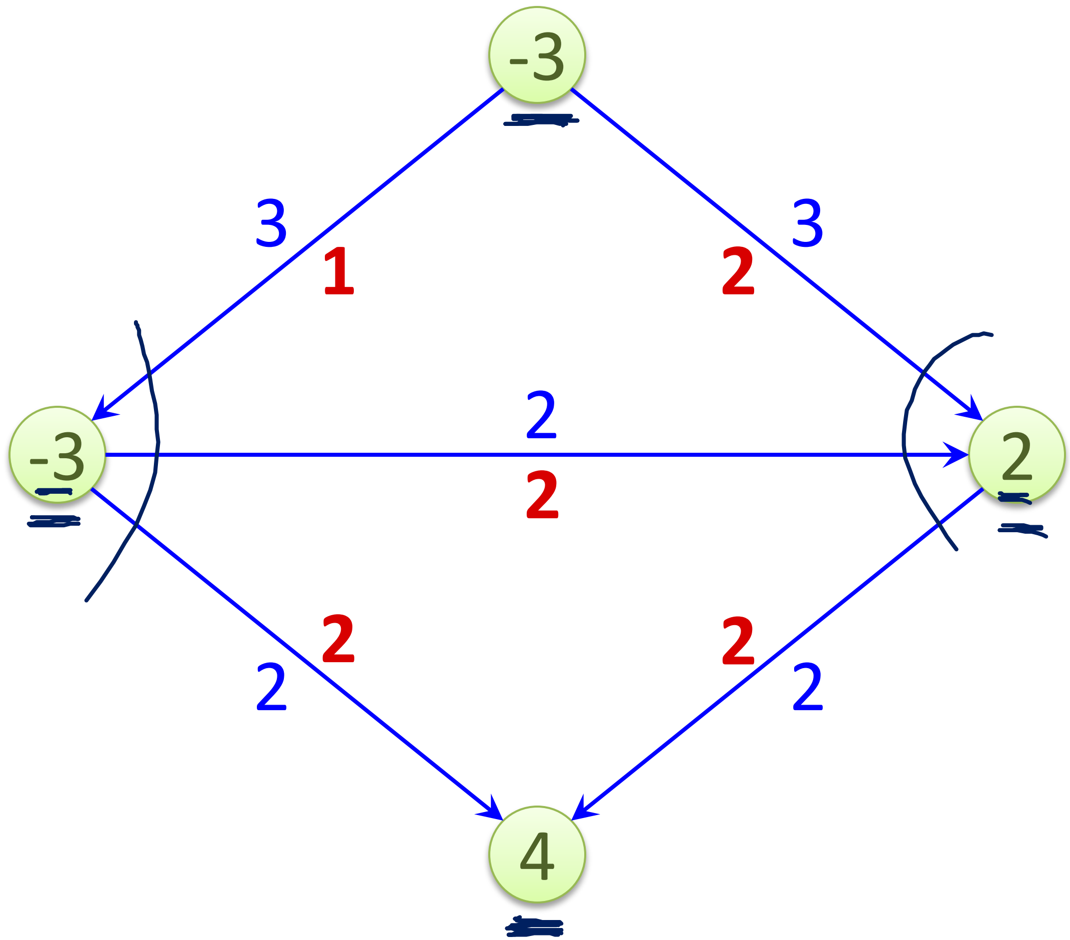
- Edge capacities $c_e > 0$ for all $e \in E$
- Node demands $d_v \in \mathbb{R}$ for all $v \in V$
  - $d_v > 0$: node needs flow d_v and therefore is a sink
  - $d_v < 0$: node has a supply of $-d_v$ and is therefore a source
  - $d_v = 0$: node is neither a source nor a sink

Flow: Function $f: E \to \mathbb{R}_{\geq 0}$ satisfying

- *Capacity Conditions*: $\forall e \in E$: $\quad 0 \leq f(e) \leq c_e$
- *Demand Conditions*: $\forall v \in V$: $\quad f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

Objective: Does a flow f satisfying all conditions exist?
If yes, find such a flow f.

Example



Condition on Demands

Claim: If there exists a feasible circulation with demands d_v for $v \in V$, then

$$\sum_{v \in V} d_v = 0.$$

$d_v = f^{in}(v) - f^{out}(v)$

Proof:

- $\sum_v d_v = \sum_v \left(f^{\text{in}}(v) - f^{\text{out}}(v)\right)$

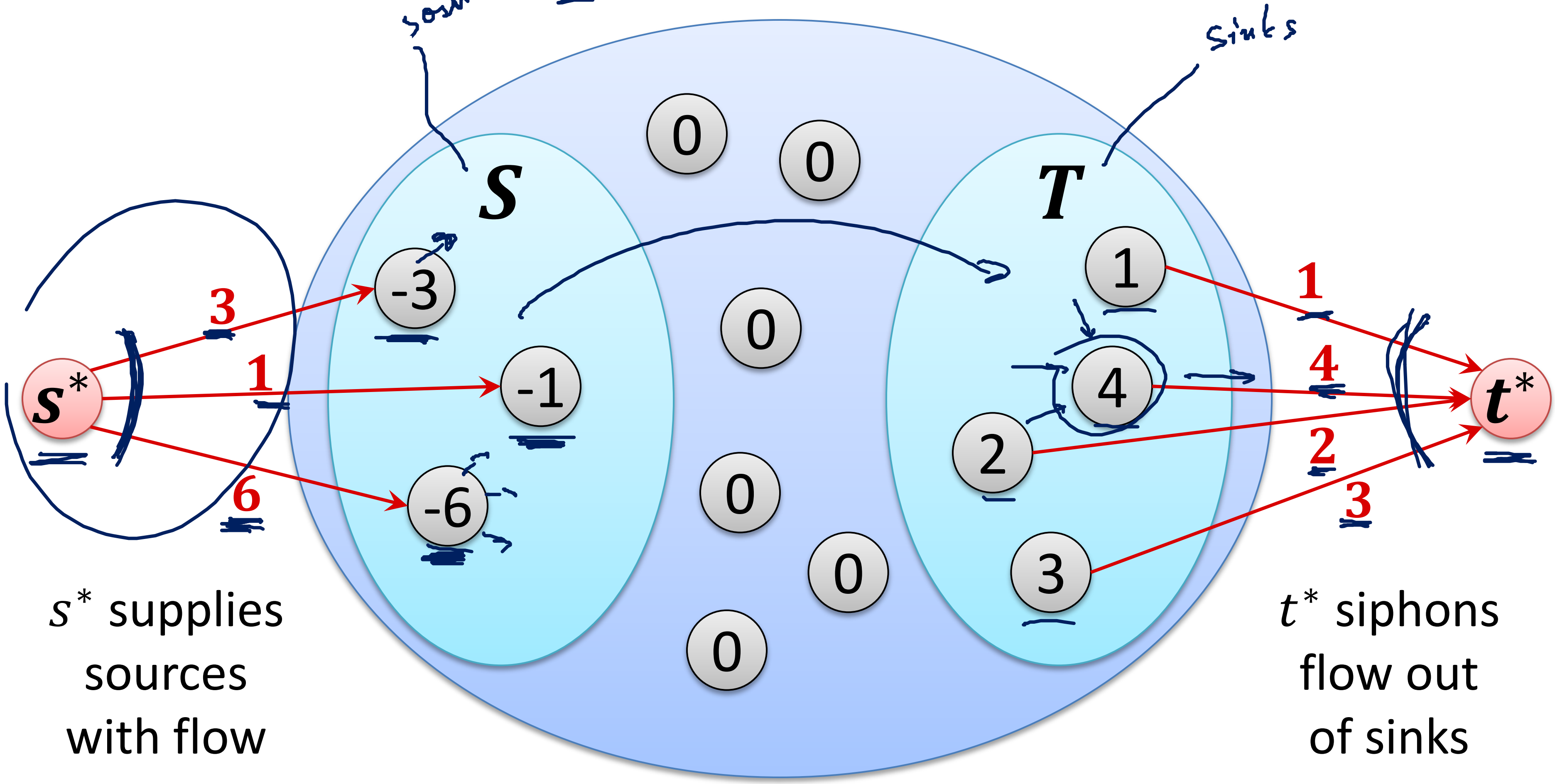
$u \xrightarrow{f(e)} v$

- $f(e)$ of each edge e appears twice in the above sum with different signs → overall sum is 0

Total supply = total demand:

$$\text{Define } \boldsymbol{D} := \sum_{v:d_v>0} \boldsymbol{d_v} = \sum_{v:d_v<0} -\boldsymbol{d_v}$$

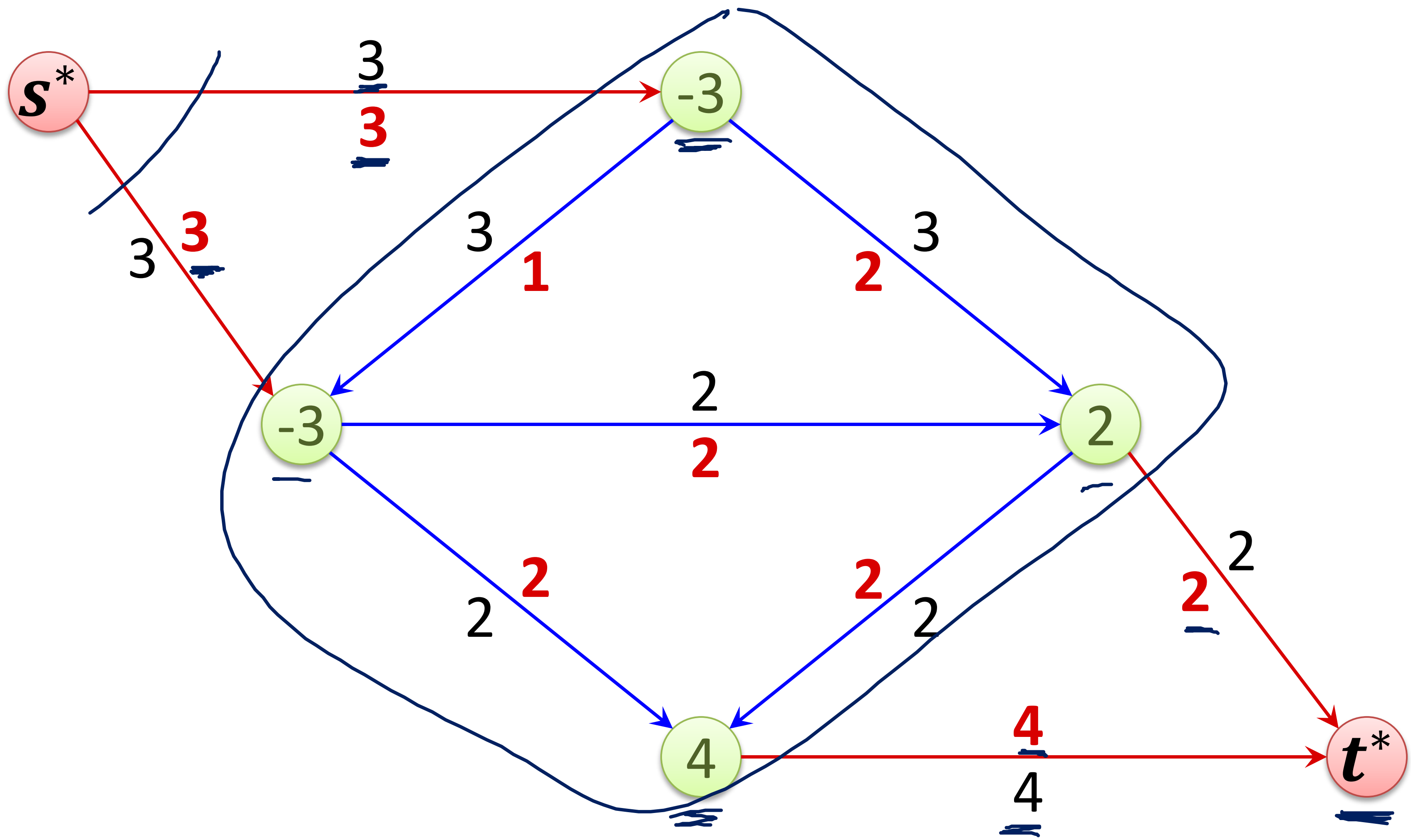
- Add "super-source" s^* and "super-sink" t^* to network



s^* supplies sources with flow

t^* siphons flow out of sinks

Example



Formally…

Reduction: Get graph G' from graph as follows

- Node set of G' is $V \cup \{s^*, t^*\}$
- Edge set is E and edges
  - (s^*, v) for all v with $d_v < 0$, capacity of edge is d_v
  - (v, t^*) for all v with $d_v > 0$, capacity of edge is d_v

Observations:

- Capacity of min s^*-t^* cut is at most D (e.g., the cut $(s^*, V \cup \{t^*\})$)
- A feasible circulation on G can be turned into a feasible flow of value D of G' by saturating all (s^*, v) and (v, t^*) edges.
- Any flow of G' of value D induces a feasible circulation on G
  - (s^*, v) and (v, t^*) edges are saturated
  - By removing these edges, we get exactly the demand constraints

Circulation with Demands

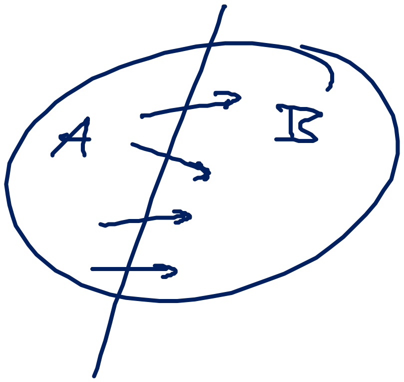
Theorem: There is a feasible circulation with demands d_v, $v \in V$ on graph G if and only if there is a flow of value D on G'.

- If all capacities and demands are integers, there is an integer circulation

The max flow min cut theorem also implies the following:

Theorem: The graph G has a feasible circulation with demands d_v, $v \in V$ if and only if for all cuts (A, B),

$$\sum_{v \in B} d_v \leq c(A, B).$$



Circulation: Demands and Lower Bounds

Given: Directed network $G = (V, E)$ with

- Edge capacities $c_e > 0$ and **lower bounds $\mathbf{0 \leq \ell_e \leq c_e}$ for $\boldsymbol{e \in E}$**
- Node demands $d_v \in \mathbb{R}$ for all $v \in V$
  - $d_v > 0$: node needs flow and therefore is a sink
  - $d_v < 0$: node has a supply of $-d_v$ and is therefore a source
  - $d_v = 0$: node is neither a source nor a sink

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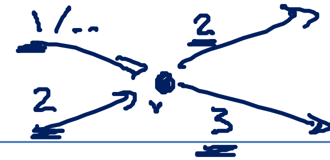
Flow: Function $f : E \to \mathbb{R}_{\geq 0}$ satisfying

- *Capacity Conditions*: $\forall e \in E$: $\boxed{\boldsymbol{\ell_e \leq f(e) \leq c_e}}$
- *Demand Conditions*: $\forall v \in V$: $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

Objective: Does a flow f satisfying all conditions exist?
If yes, find such a flow f.

Solution Idea



$L_v = -2$
$d_v = -2$

- Define initial circulation $f_0(e) = \ell_e$
  Satisfies capacity constraints: $\forall e \in E: \ell_e \leq f_0(e) \leq c_e$

$f(e) := f_0(e) + f_1(e)$

$f^{in}(v) - f^{out}(v) = \underbrace{\left(f_0^{in}(v) - f_0^{out}(v)\right)}_{L_v} + \underbrace{\left(f_1^{in}(v) - f_1^{out}(v)\right)}_{d_v - L_v} \stackrel{!}{=} d(v)$

- Define

$$L_v := f_0^{\text{in}}(v) - f_0^{\text{out}}(v) = \sum_{e \text{ into } v} \ell_e - \sum_{e \text{ out of } v} \ell_e$$

- If $L_v = d_v$, demand condition is satisfied at v by f_0, otherwise, we need to superimpose another circulation f_1 such that

$$d_v' := f_1^{\text{in}}(v) - f_1^{\text{out}}(v) = d_v - L_v$$

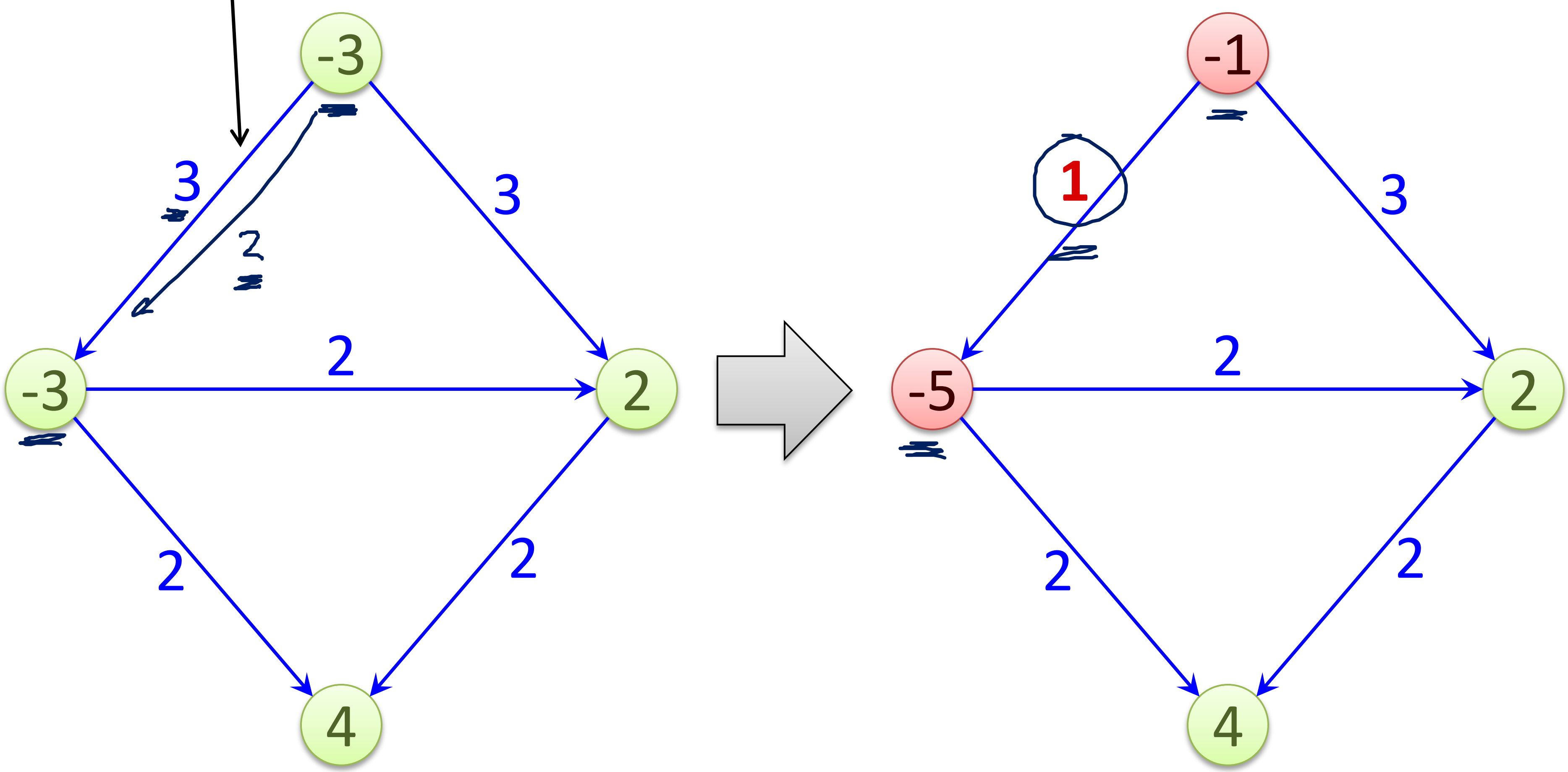
$0 \leq f_1(e) \leq \overbrace{c_e - \ell_e}^{\geq 0}$

- Remaining capacity of edge e: $c_e' := c_e - \ell_e$

- We get a circulation problem with new demands d_v', new capacities c_e', and no lower bounds

Eliminating a Lower Bound: Example

Lower bound of 2



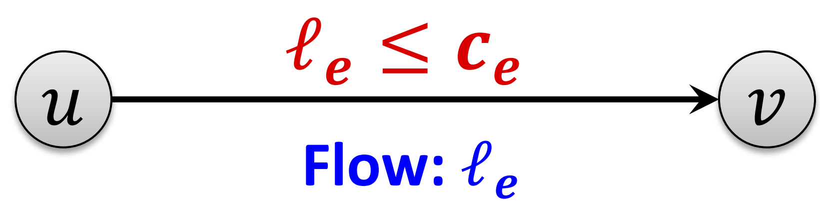
Reduce to Problem Without Lower Bounds

Graph $G = (V, E)$:

- Capacity: For each edge $e \in E$: $\ell_e \leq f(e) \leq c_e$
- Demand: For each node $v \in V$: $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

Model lower bounds with supplies & demands:

$L_v = f_0^{\text{in}}(v) - f_0^{\text{out}}(v)$



Create Network G' (without lower bounds):

- For each edge $e \in E$: $c'_e = c_e - \ell_e$
- For each node $v \in V$: $d'_v = d_v - L_v$

Circulation: Demands and Lower Bounds

Theorem: There is a feasible circulation in G (with lower bounds) if and only if there is feasible circulation in G' (without lower bounds).

- Given circulation f' in G', $f(e) = f'(e) + \ell_e$ is circulation in G
  - The capacity constraints are satisfied because $f'(e) \leq c_e - \ell_e$
  - Demand conditions:

$$f^{\text{in}}(v) - f^{\text{out}}(v) = \sum_{e \text{ into } v} (\ell_e + f'(e)) - \sum_{e \text{ out of } v} (\ell_e + f'(e))$$
$$= L_v + (d_v - L_v) = d_v$$

- Given circulation f in G, $f'(e) = f(e) - \ell_e$ is circulation in G'
  - The capacity constraints are satisfied because $\ell_e \leq f(e) \leq c_e$
  - Demand conditions:

$$f'^{\text{in}}(v) - f'^{\text{out}}(v) = \sum_{e \text{ into } v} (f(e) - \ell_e) - \sum_{e \text{ out of } v} (f(e) - \ell_e)$$
$$= d_v - L_v$$

Integrality

Theorem: Consider a circulation problem with integral capacities, flow lower bounds, and node demands. If the problem is feasible, then it also has an integral solution.

Proof:

- Graph G' has only integral capacities and demands
- Thus, the flow network used in the reduction to solve circulation with demands and no lower bounds has only integral capacities
- The theorem now follows because a max flow problem with integral capacities also has an optimal integral solution

(Ford Fulkerson)

- It also follows that with the max flow algorithms we studied, we get an integral feasible circulation solution.

Matrix Rounding

- **Given:** $p \times q$ matrix $D = \{d_{i,j}\}$ of real numbers
- **row i sum:** $a_i = \sum_j d_{i,j}$, **column j sum:** $b_j = \sum_i d_{i,j}$
- **Goal:** Round each $d_{i,j}$, as well as a_i and b_j up or down to the next integer so that the sum of rounded elements in each row (column) equals the rounded row (column) sum
- **Original application:** publishing census data

Example:

3.14	6.80	7.30	17.24
9.60	2.40	0.70	12.70
3.60	1.20	6.50	11.30
16.34	10.40	14.50	

original data

3	7	7	17
10	2	1	13
3	1	7	11
16	10	15	

possible rounding

Matrix Rounding

Theorem: For any matrix, there exists a feasible rounding.

Remark: Just rounding to the nearest integer doesn't work

0.35	0.35	0.35	1.05
0.55	0.55	0.55	1.65
0.90	0.90	0.90	

original data

0	0	0	0
1	1	1	3
1	1	1	

rounding to nearest integer

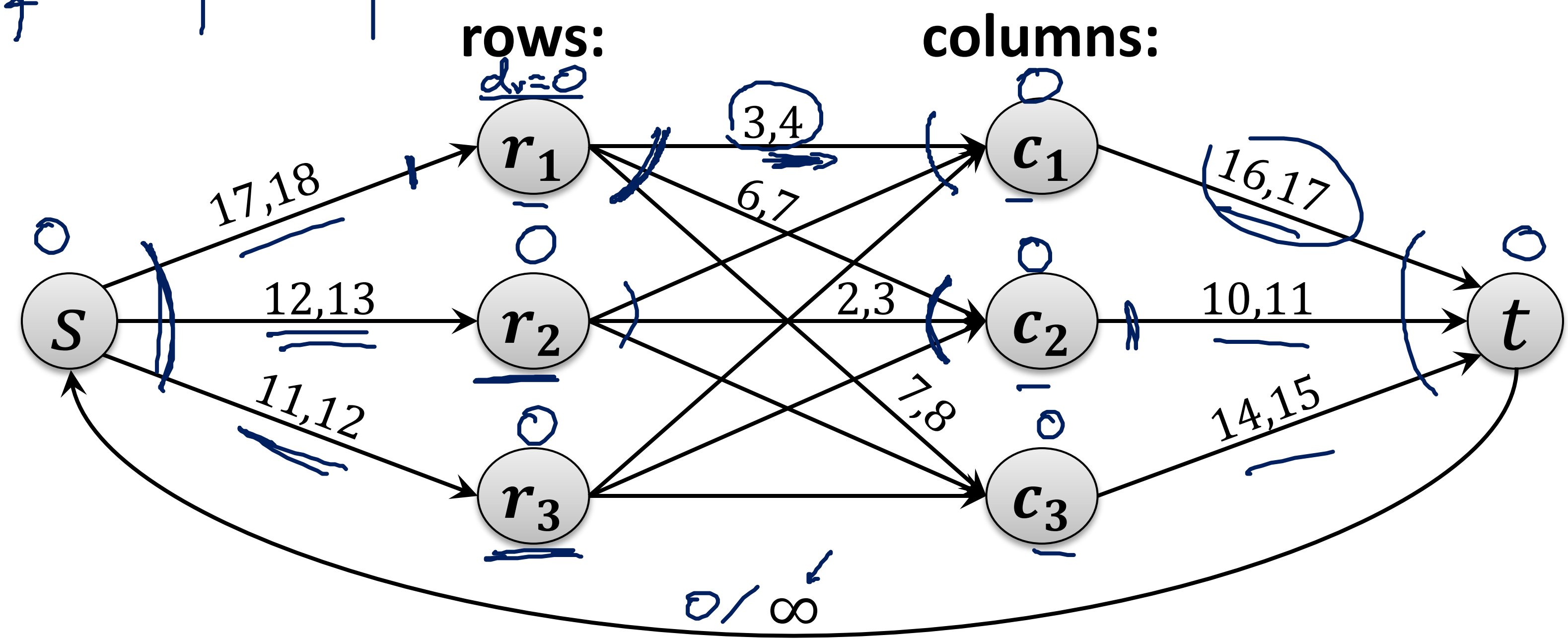
0	0	1	1
1	1	0	2
1	1	1	

feasible rounding

Reduction to Circulation

3.14	6.80	7.30	17.24
9.60	2.40	0.70	12.70
3.60	1.20	6.50	11.30
16.34	10.40	14.50	

Matrix elements and row/column sums give a feasible circulation that satisfies all lower bound, capacity, and demand constraints



all demands $d_v = 0$

Matrix Rounding

matching

Theorem: For any matrix, there exists a feasible rounding.

Proof:

- The matrix entries $d_{i,j}$ and the row and column sums a_i and b_j give a feasible circulation for the constructed network
- Every feasible circulation gives matrix entries with corresponding row and column sums (follows from demand constraints)
- Because all demands, capacities, and flow lower bounds are integral, there is an integral solution to the circulation problem

→ gives a feasible rounding!