Chapter 7
Randomization

Algorithm Theory
WS 2015/16

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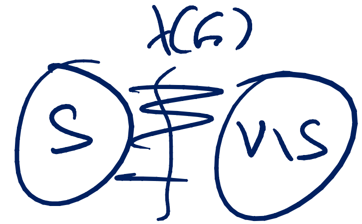
Randomized Contraction Algorithm

Algorithm:

to compute a min. cut in a graph



while there are >2 nodes **do**

$O(n^2)$ contract a uniformly random edge

return cut induced by the last two remaining nodes

(cut defined by the original node sets represented by the last 2 nodes)

Theorem: The random contraction algorithm returns a **specific** minimum cut with probability at least $\frac{2}{n(n-1)}$.

Theorem: The random contraction algorithm can be implemented in time $O(n^2)$.

- There are $n-2$ contractions, each can be done in time $O(n)$.

Randomized Min Cut Algorithm

Theorem: If the contraction algorithm is repeated $O(n^2 \log n)$ times, one of the $O(n^2 \log n)$ instances returns a min. cut w.h.p.

Corollary: The contraction algorithm allows to compute a minimum cut in $O(n^4 \log n)$ time w.h.p.

- One instance consists of $n - 2$ edge contractions
- Each edge contraction can be carried out in time $O(n)$
  - Actually: $O(\text{current \#nodes})$
- Time per instance of the contraction algorithm: $O(n^2)$

Can We Do Better?

- Time $O(n^4 \log n)$ is not very spectacular, a simple max flow based implementation has time $O(n^4)$.

However, we will see that the contraction algorithm is nevertheless very interesting because:

1. The algorithm can be improved to beat every known deterministic algorithm.

1. It allows to obtain strong statements about the distribution of cuts in graphs.

Improving the Contraction Algorithm

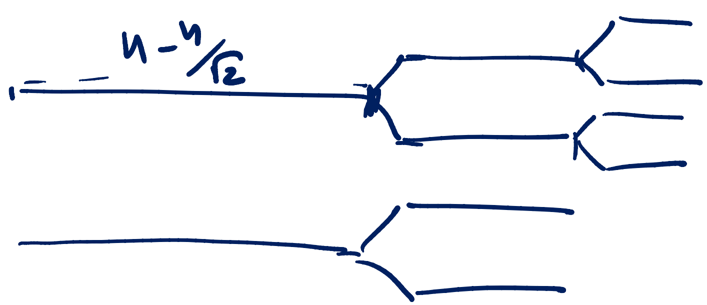
- For a specific min cut (A, B), if (A, B) survives the first i contractions,

$$\mathbb{P}(\text{edge crossing } (A, B) \text{ in contraction } i + 1) \leq \frac{2}{n - i}.$$

- **Observation:** The probability only gets large for large i

- **Idea:** The early steps are much safer than the late steps.

  Maybe we can repeat the late steps more often than the early ones.



Safe Contraction Phase

Lemma: A given min cut (A, B) of an n-node graph G survives the first $n - \left\lceil {}^{n}/_{\sqrt{2}} + 1 \right\rceil$ contractions, with probability $\geq {}^{1}/_{2}$.

Proof:

- Event $\mathcal{E}_i$: cut (A, B) survives contraction i
- Probability that (A, B) survives the first $n - t$ contractions:

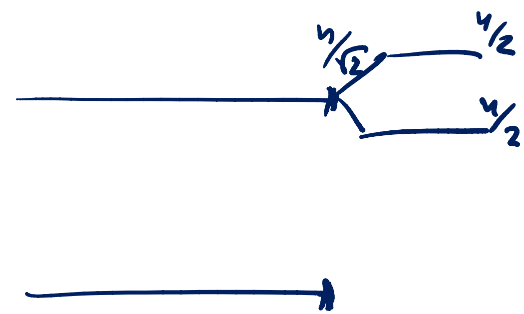
Better Randomized Algorithm

Let's simplify a bit:

- Pretend that $n/\sqrt{2}$ is an integer (for all n we will need it).
- Assume that a given min cut survives the first $n - {}^{n}/_{\sqrt{2}}$ contractions with probability $\geq {}^{1}/_{2}$.

contract$(\boldsymbol{G}, \boldsymbol{t})$**:**

- Starting with n-node graph G, perform $n - t$ edge contractions such that the new graph has t nodes.



mincut$(\boldsymbol{G})$**:**

1. $X_1 := \text{mincut}\left(\text{contract}\left(G, n/\sqrt{2}\right)\right);$
2. $X_2 := \text{mincut}\left(\text{contract}\left(G, n/\sqrt{2}\right)\right);$
3. **return** $\min\{X_1, X_2\};$

Success Probability

$\mathbf{mincut}(\boldsymbol{G})$:

1. $X_1 \coloneqq \mathrm{mincut}\left(\mathrm{contract}\left(G, n/\sqrt{2}\right)\right);$
2. $X_2 \coloneqq \mathrm{mincut}\left(\mathrm{contract}\left(G, n/\sqrt{2}\right)\right);$
3. **return** $\min\{X_1, X_2\};$

$\boldsymbol{P(n)}$: probability that the above algorithm returns a min cut when applied to a graph with n nodes.

Theorem: The recursive randomized min cut algorithm returns a minimum cut with probability at least $P(n) \geq 1/\log_2 n$.

Running Time

1. $X_1 := \text{mincut}\left(\text{contract}\left(G, n/\sqrt{2}\right)\right);$
2. $X_2 := \text{mincut}\left(\text{contract}\left(G, n/\sqrt{2}\right)\right);$
3. **return** $\min\{X_1, X_2\};$

$n \longrightarrow n/\sqrt{2}$

$\longrightarrow k$

Master thm!

$c = \log_b a$

$T(n) = a \cdot T(n/b) + O(n^c)$ $\hookrightarrow O(n^c \cdot \log n)$

Recursion:

- $T(n)$: time to apply algorithm to n-node graphs
- Recursive calls: $2T\left({}^{n}/_{\sqrt{2}}\right)$
- Number of contractions to get to ${}^{n}/_{\sqrt{2}}$ nodes: $O(n)$

$$T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2), \qquad T(2) = O(1)$$

$\rightarrow T(n) = O(n^2 \log n)$

Running Time

$1-\varepsilon < e^{-\varepsilon}$

Theorem: The running time of the recursive, randomized min cut algorithm is $O(\underline{n^2 \log n})$.

Proof:

(Master Thm)

$t = \Theta(\log^2 n)$

- Can be shown in the usual way, by induction on n

$$\left(1-\frac{1}{\log n}\right)^t < e^{-t/\log n} \stackrel{!}{=} \frac{1}{n^c} = e^{-c\ln n}$$

Remark:

- The running time is only by an $O(\underline{\log n})$-factor slower than the basic contraction algorithm.

succ. prob. $\frac{1}{\log n}$

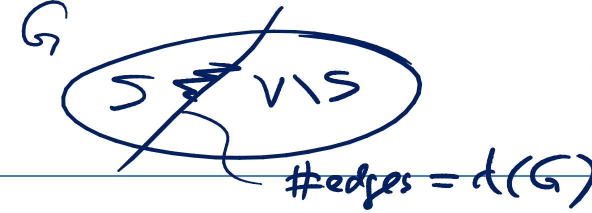
- The success probability is exponentially better!

If we want a min. cut w.h.p. $\left(1-\frac{1}{n^c}\right)$: we need $O(\log^2 n)$ rep.

$\Rightarrow$ running time: $O(n^2 \cdot \log^3 n)$ $\leftarrow$

best det. alg.: $\underline{O(m \cdot n + n^2 \log n)}$

Number of Minimum Cuts



- Given a graph G, how many minimum cuts can there be?

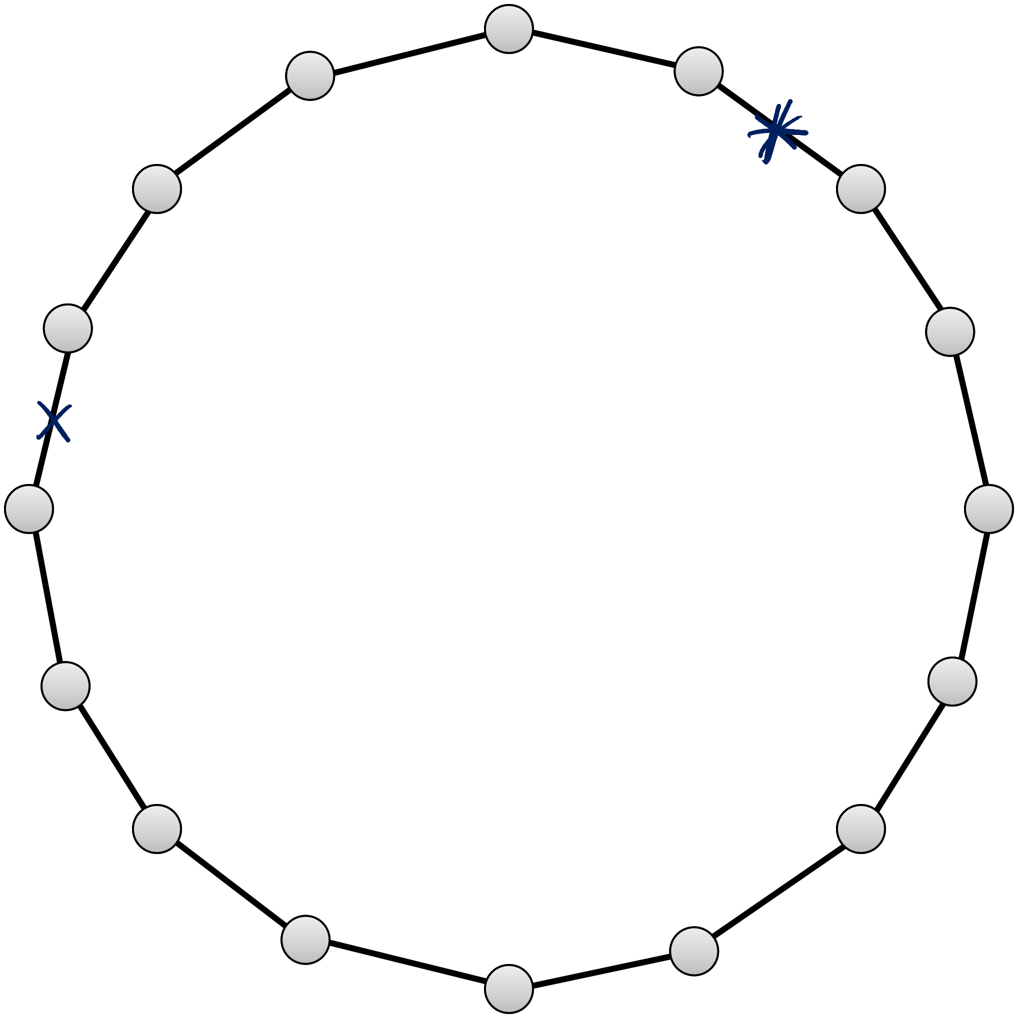
- Or alternatively: If G has edge connectivity k, how many ways are there to remove k edges to disconnect G?

- Note that the total number of cuts is large.

$2^{n-1} - 2$

Number of Minimum Cuts

Example: Ring with n nodes



- Minimum cut size: 2
- Every two edges induce a min cut
- Number of edge pairs:

$$\binom{n}{2}$$

- Are there graphs with more min cuts?

Number of Min Cuts

Theorem: The number of minimum cuts of a graph is at most $\binom{n}{2}$.

Proof:

$C_i \cap C_j = \emptyset$

- Assume there are s min cuts $1, \dots, s$
- For $i \in \{1, \dots, s\}$, define event $\mathcal{C}_i$:

  $\mathcal{C}_i :=$ {basic contraction algorithm returns min cut i}

- We know that for $i \in \{1, \dots, s\}$: $\mathbb{P}(\mathcal{C}_i) \geq 1/\binom{n}{2} = \frac{2}{n(n-1)}$
- Events $\mathcal{C}_1, \dots, \mathcal{C}_s$ are disjoint:

$$1 \geq \mathbb{P}\left(\bigcup_{i=1}^{s} \mathcal{C}_i\right) = \sum_{i=1}^{s} \mathbb{P}(\mathcal{C}_i) \geq \frac{s}{\binom{n}{2}}$$

$s \leq \binom{n}{2}$

can be generalized

#cuts of size $\leq \alpha \cdot \lambda(G)$

is at most $n^{2\alpha}$