Chapter 8
Approximation Algorithms

Algorithm Theory
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Approximation Ratio

An approximation algorithm is an algorithm that computes a solution for an optimization with an objective value that is provably within a bounded factor of the optimal objective value.

Formally:

- $\mathrm{OPT} \geq 0$: optimal objective value
  $\mathrm{ALG} \geq 0$: objective value achieved by the algorithm

- **Approximation Ratio α:**

$$\textbf{Minimization: } \alpha := \max_{\text{input instances}} \frac{\mathrm{ALG}}{\mathrm{OPT}}$$

$$\textbf{Maximization: } \alpha := \max_{\text{input instances}} \frac{\mathrm{OPT}}{\mathrm{ALG}}$$

Example: Load Balancing

We are given:

- m machines $M_1, \dots, M_m$
- n jobs, processing time of job i is t_i

Goal:

- Assign each job to a machine such that the makespan is minimized

  makespan: largest total processing time of any machine

The above load balancing problem is NP-hard and we therefore want to get a good approximation for the problem.

Greedy Algorithm

There is a simple greedy algorithm:

- Go through the jobs in an arbitrary order
- When considering job i, assign the job to the machine that currently has the smallest load.

Example: 3 machines, 12 jobs

3	4	2	3	1	6	4	4	3	2	1	5

Greedy Assignment:

M_1: | 3 | 1 | 6 | 1 | 5 |

M_2: | 4 | 4 | 3 |

M_3: | 2 | 3 | 4 | 2 |

Optimal Assignment:

M_1: | 3 | 4 | 2 | 3 | 1 |

M_2: | 6 | 4 | 3 |

M_3: | 4 | 2 | 1 | 5 |

Greedy Analysis

- We will show that greedy gives a 2-approximation
- To show this, we need to compare the solution of greedy with an optimal solution (that we can't compute)
- Lower bound on the optimal makespan T^*:

$$T^* \geq \frac{1}{m} \cdot \sum_{i=1}^{n} t_i$$

- Second lower bound on optimal makespan T^*:

$$T^* \geq \max_{1 \leq i \leq n} t_i$$

Greedy Analysis

Theorem: The greedy algorithm has approximation ratio ≤ 2, i.e., for the makespan T of the greedy solution, we have $T \leq 2T^*$.

Proof:

- For machine k, let T_k be the time used by machine k
- Consider some machine M_i for which $T_i = T$
- Assume that job j is the last one schedule on M_i:

$\boldsymbol{M_i}$: [$T - t_j$ | t_j]

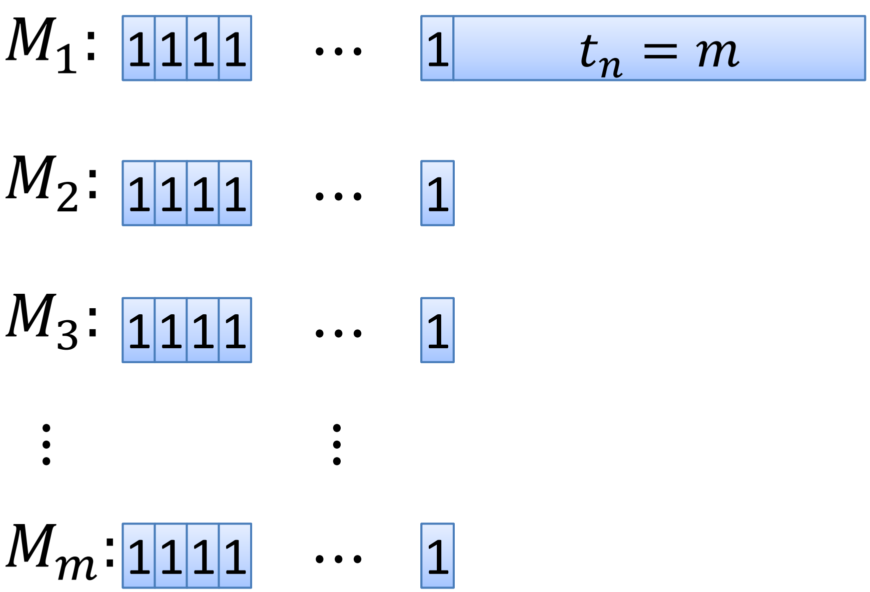
- When job j is scheduled, M_i has the minimum load

Can We Do Better?

The analysis of the greedy algorithm is almost tight:

- Example with $n = m(m-1) + 1$ jobs
- Jobs $1, \ldots, n-1 = m(m-1)$ have $t_i = 1$, job n has $t_n = m$

Greedy Schedule:



Improving Greedy

Bad case for the greedy algorithm:
One large job in the end can destroy everything

Idea: assign large jobs first

Modified Greedy Algorithm:

1. Sort jobs by decreasing length s.t. $t_1 \geq t_2 \geq \cdots \geq t_n$
2. Apply the greedy algorithm as before (in the sorted order)

Lemma: If $n > m$: $T^* \geq t_m + t_{m+1} \geq 2t_{m+1}$

Proof:

- Two of the first $m + 1$ jobs need to be scheduled on the same machine
- Jobs m and $m + 1$ are the shortest of these jobs

Analysis of the Modified Greedy Alg.

Theorem: The modified algorithm has approximation ratio $\leq {}^{3}/_{2}$.

Proof:

- We show that $T \leq {}^{3}/_{2} \cdot T^*$
- As before, we consider the machine M_i with $T_i = T$
- Job j (of length t_j) is the last one scheduled on machine M_i
- If j is the only job on M_i, we have $T = T^*$
- Otherwise, we have $j \geq m + 1$
  - The first m jobs are assigned to m distinct machines

Metric TSP

Input:

- Set V of n nodes (points, cities, locations, sites)
- Distance function $d: V \times V \rightarrow \mathbb{R}$, i.e., $d(u, v)$ is dist from u to v
- Distances define a metric on V:

$$d(u,v) = d(v,u) \geq 0, \qquad d(u,v) = 0 \Leftrightarrow u = v$$
$$\forall u,v,w \in V : d(u,v) \leq d(u,w) + d(w,v)$$

Solution:

- Ordering/permutation $v_1, v_2, \dots, v_n$ of the vertices
- Length of TSP path: $\sum_{i=1}^{n-1} d(v_i, v_{i+1})$
- Length of TSP tour: $d(v_1, v_n) + \sum_{i=1}^{n-1} d(v_i, v_{i+1})$

Goal:

- Minimize length of TSP path or TSP tour

Metric TSP

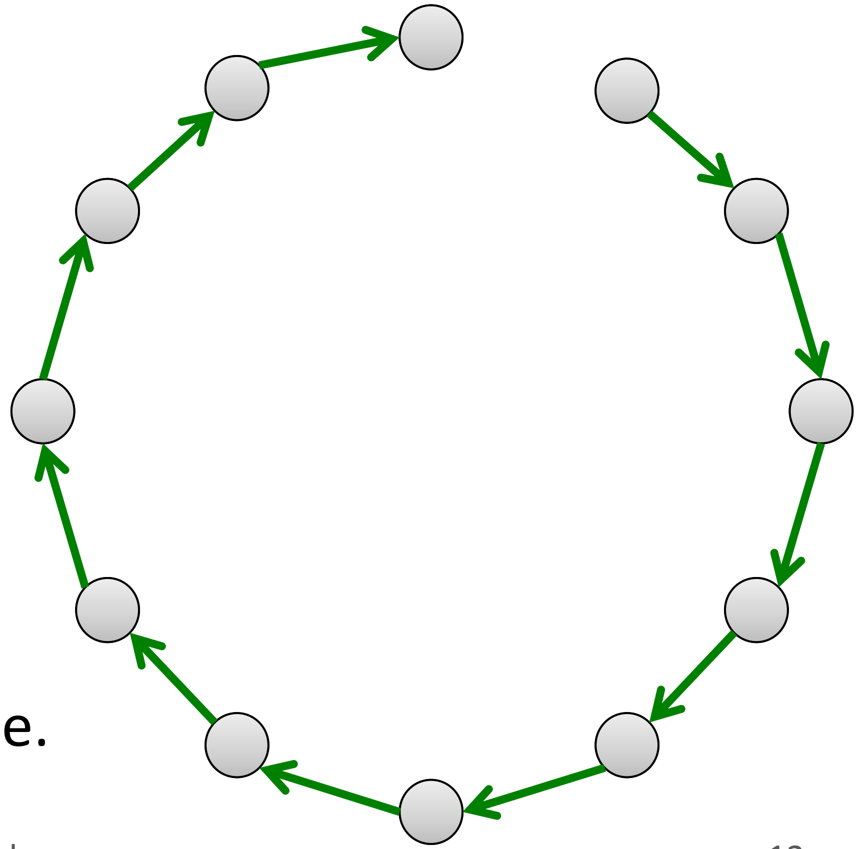
- The problem is NP-hard
- We have seen that the greedy algorithm (always going to the nearest unvisited node) gives an $O(\log n)$-approximation
- Can we get a constant approximation ratio?
- We will see that we can…

TSP and MST

Claim: The length of an optimal TSP path is lower bounded by the weight of a minimum spanning tree

Proof:

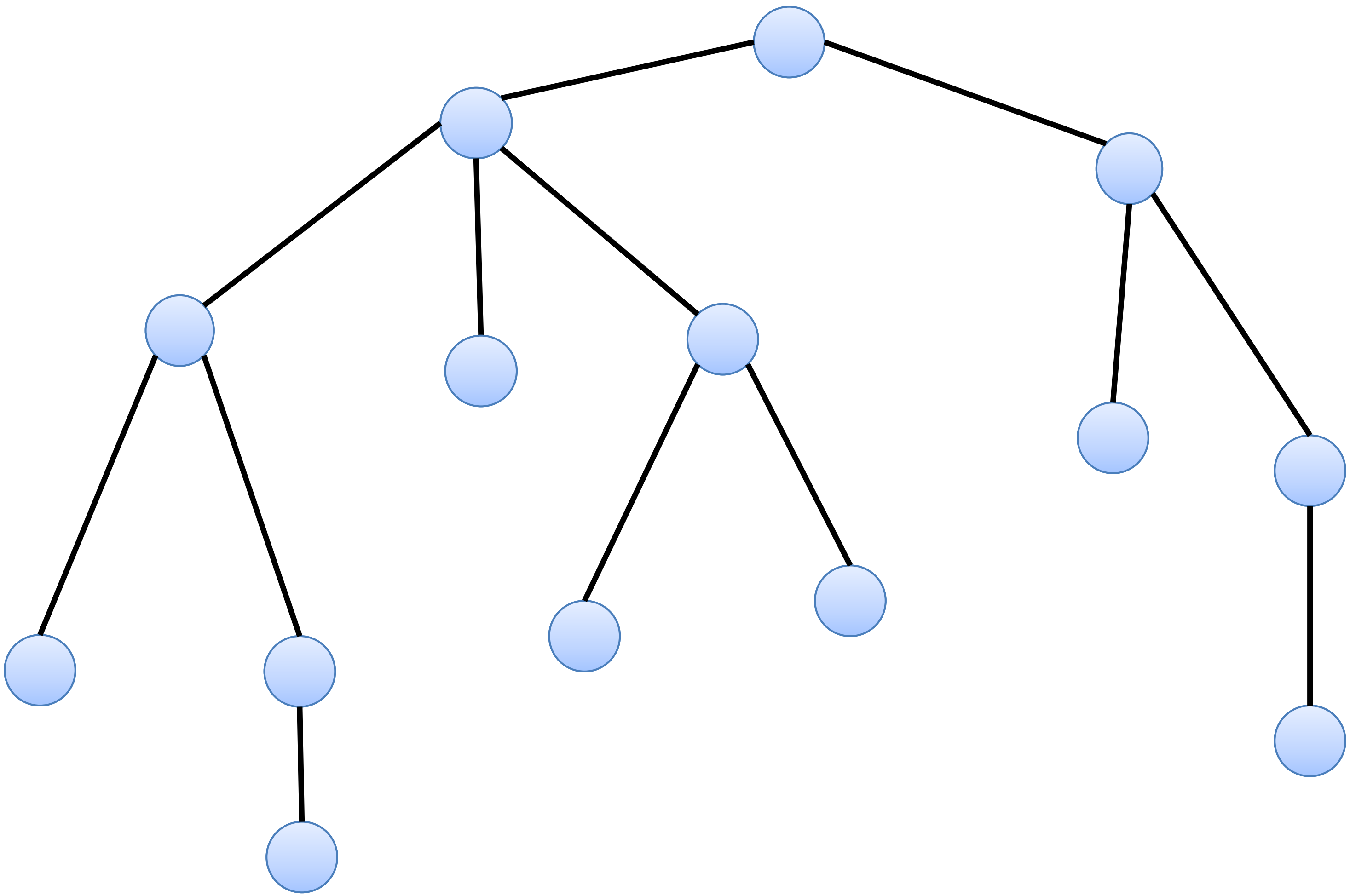
- A TSP path is a spanning tree, it's length is the weight of the tree



Corollary: Since an optimal TSP tour is longer than an optimal TSP path, the length of an optimal TSP tour is also lower bounded by the weight of a minimum spanning tree.

The MST Tour

Walk around the MST…

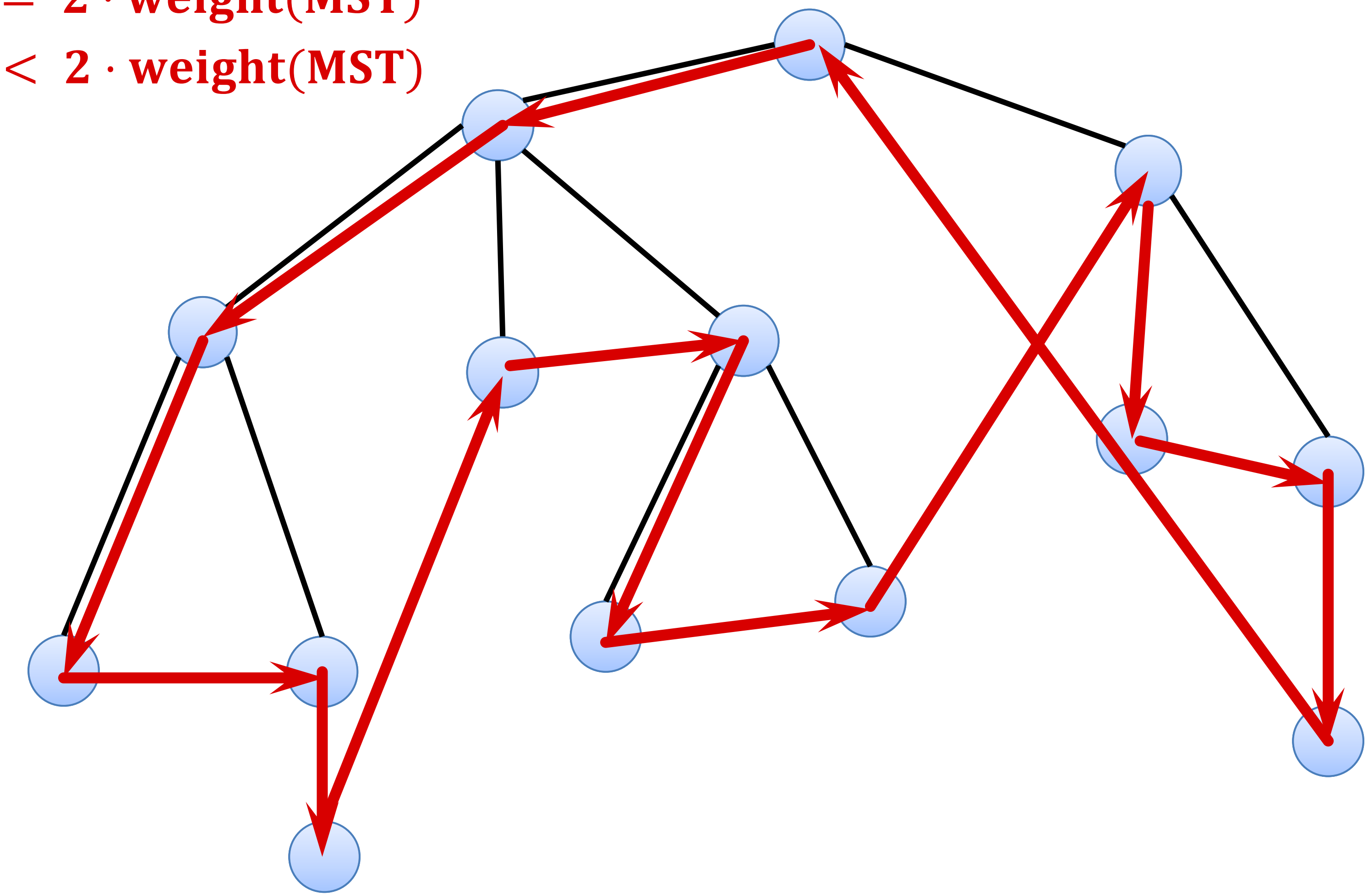


The MST Tour

Walk around the MST…

Cost (walk) $= \ 2 \cdot \textbf{weight(MST)}$

Cost (tour) $< \ 2 \cdot \textbf{weight(MST)}$



Approximation Ratio of MST Tour

Theorem: The MST TSP tour gives a 2-approximation for the metric TSP problem.

Proof:

- Triangle inequality → length of tour is at most $2 \cdot \text{weight}(\text{MST})$
- We have seen that $\text{weight}(\text{MST}) < \text{opt. tour length}$

Can we do even better?

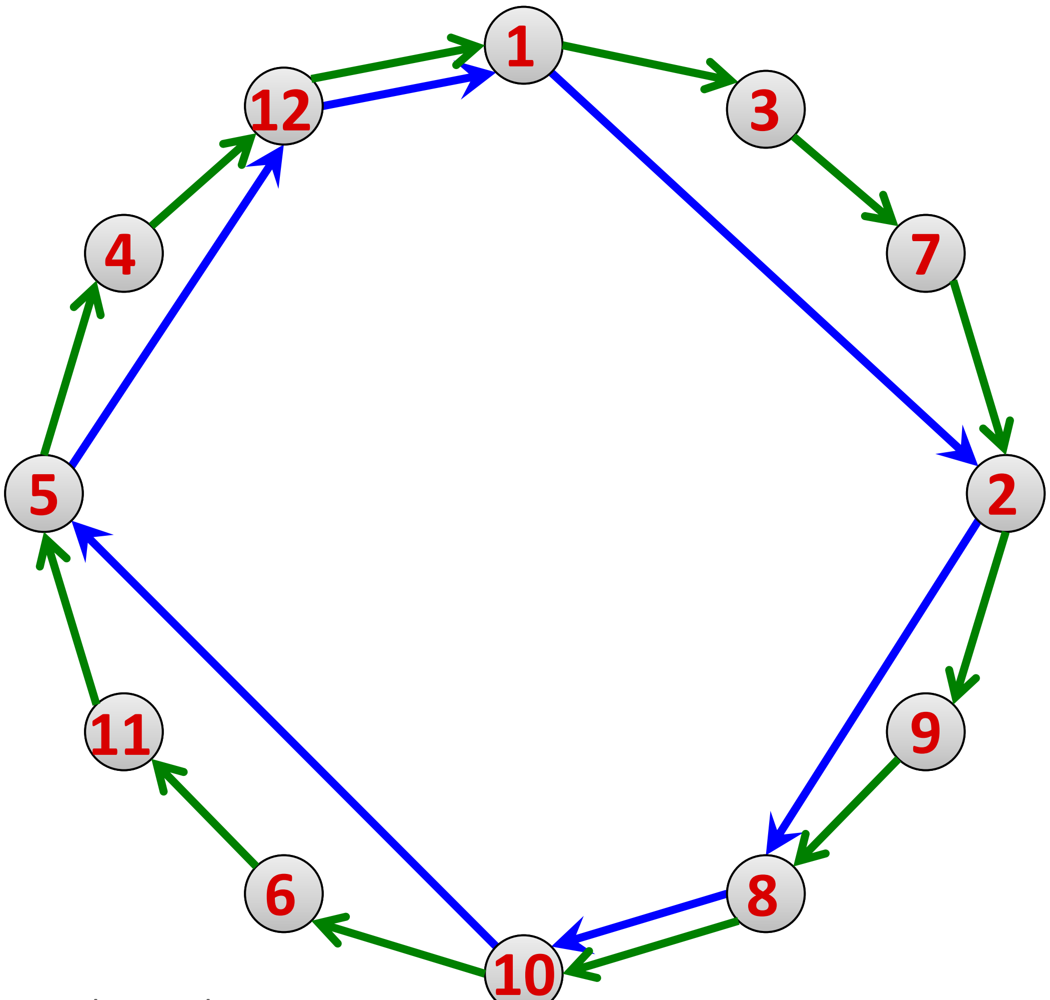
Metric TSP Subproblems

Claim: Given a metric (V, d) and (V', d) for $V' \subseteq V$, the optimal TSP path/tour of (V', d) is at most as large as the optimal TSP path/tour of (V, d).

Optimal TSP tour of nodes $1, 2, \dots, 12$

Induced TSP tour for nodes $1, 2, 5, 8, 10, 12$

blue tour $\leq$ green tour



TSP and Matching

- Consider a metric TSP instance (V, d) with an even number of nodes $|V|$
- Recall that a perfect matching is a matching $M \subseteq V \times V$ such that every node of V is incident to an edge of M.
- Because $|V|$ is even and because in a metric TSP, there is an edge between any two nodes $u, v \in V$, any partition of V into $|V|/2$ pairs is a perfect matching.
- The weight of a matching M is the sum of the distances represented by all edges in M:

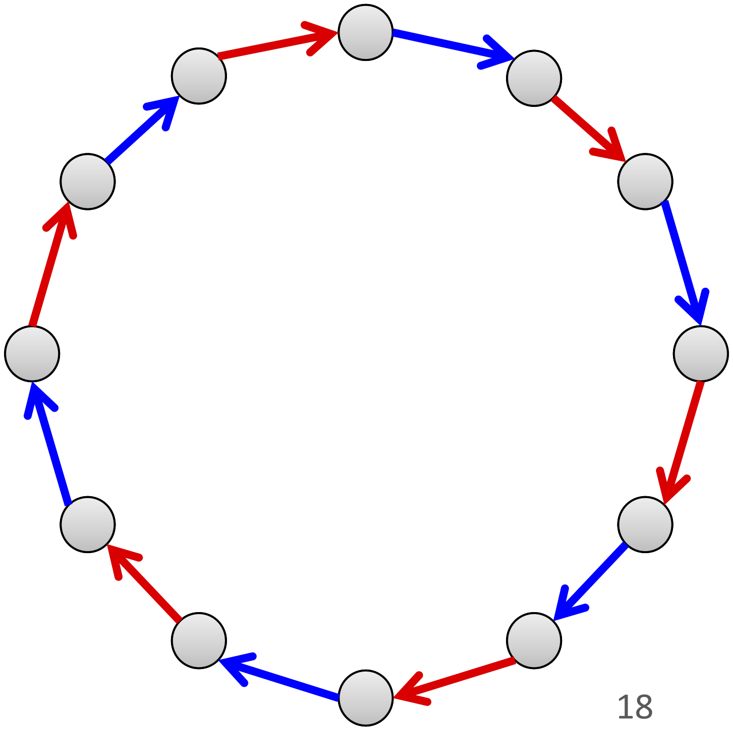
$$w(M) = \sum_{\{u,v\} \in M} d(u, v)$$

TSP and Matching

Lemma: Assume we are given a TSP instance (V, d) with an even number of nodes. The length of an optimal TSP tour of (V, d) is at least twice the weight of a minimum weight perfect matching of (V, d).

Proof:

- The edges of a TSP tour can be partitioned into 2 perfect matchings



Minimum Weight Perfect Matching

Claim: If $|V|$ is even, a minimum weight perfect matching of (V, d) can be computed in polynomial time

Proof Sketch:

- We have seen that a maximum matching in an unweighted graph can be computed in polynomial time
- With a more complicated algorithm, also a maximum weighted matching can be computed in polynomial time
- In a complete graph, a maximum weighted matching is also a (maximum weight) perfect matching
- Define weight $w(u, v) := D - d(u, v)$
- A maximum weight perfect matching for (V, w) is a minimum weight perfect matching for (V, d)

Algorithm Outline

Problem of MST algorithm:

- Every edge has to be visited twice

Goal:

- Get a graph on which every edge only has to be visited once (and where still the total edge weight is small compared to an optimal TSP tour)

Euler Tours:

- A tour that visits each edge of a graph exactly once is called an Euler tour
- An Euler tour in a (multi-)graph exists if and only if every node of the graph has even degree
- That’s definitely not true for a tree, but can we modify our MST suitably?

Euler Tour

Theorem: A connected (multi-)graph G has an Euler tour if and only if every node of G has even degree.

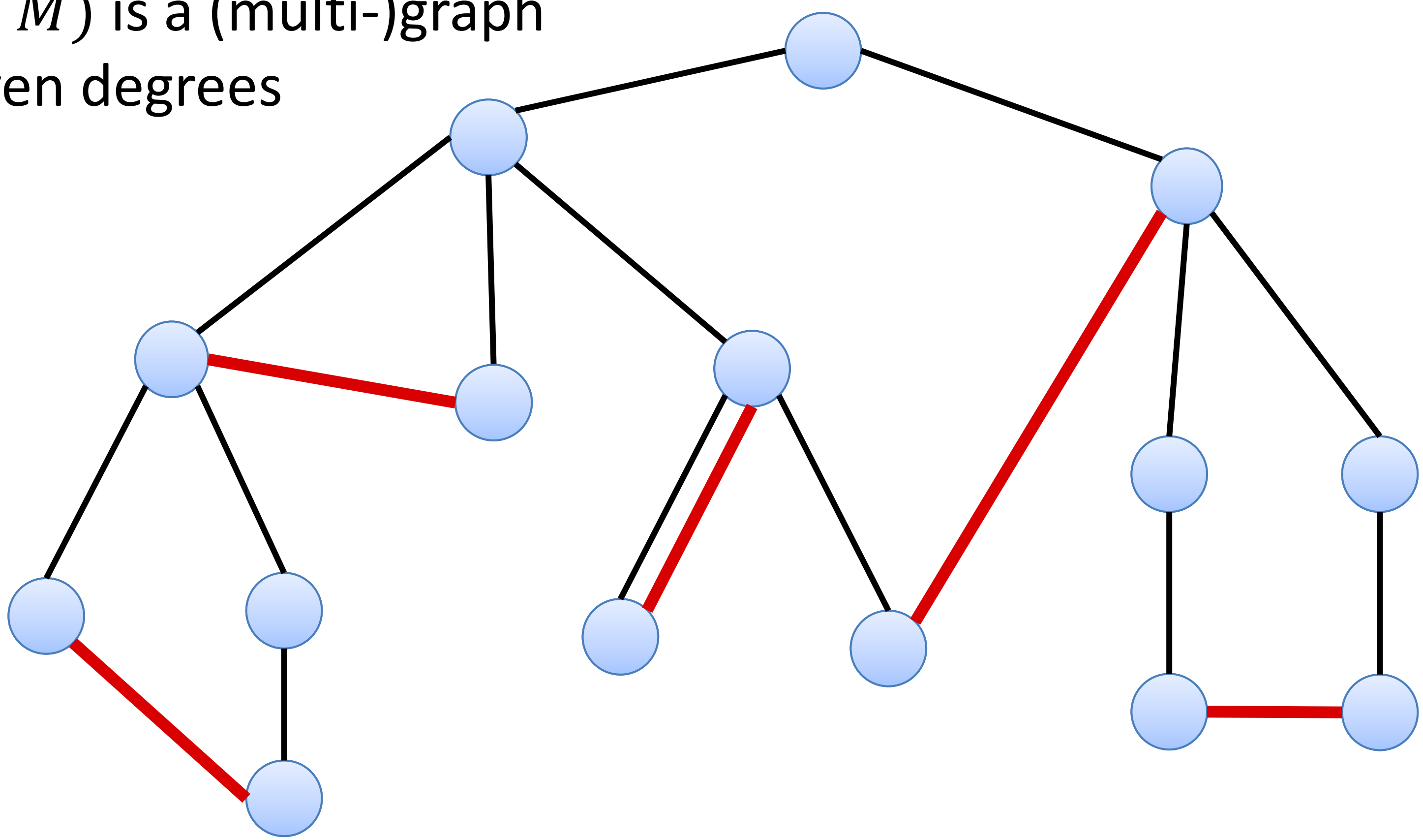
Proof:

- If G has an odd degree node, it clearly cannot have an Euler tour
- If G has only even degree nodes, a tour can be found recursively:

1. Start at some node
2. As long as possible, follow an unvisited edge
   - Gives a partial tour, the remaining graph still has even degree
3. Solve problem on remaining components recursively
4. Merge the obtained tours into one tour that visits all edges

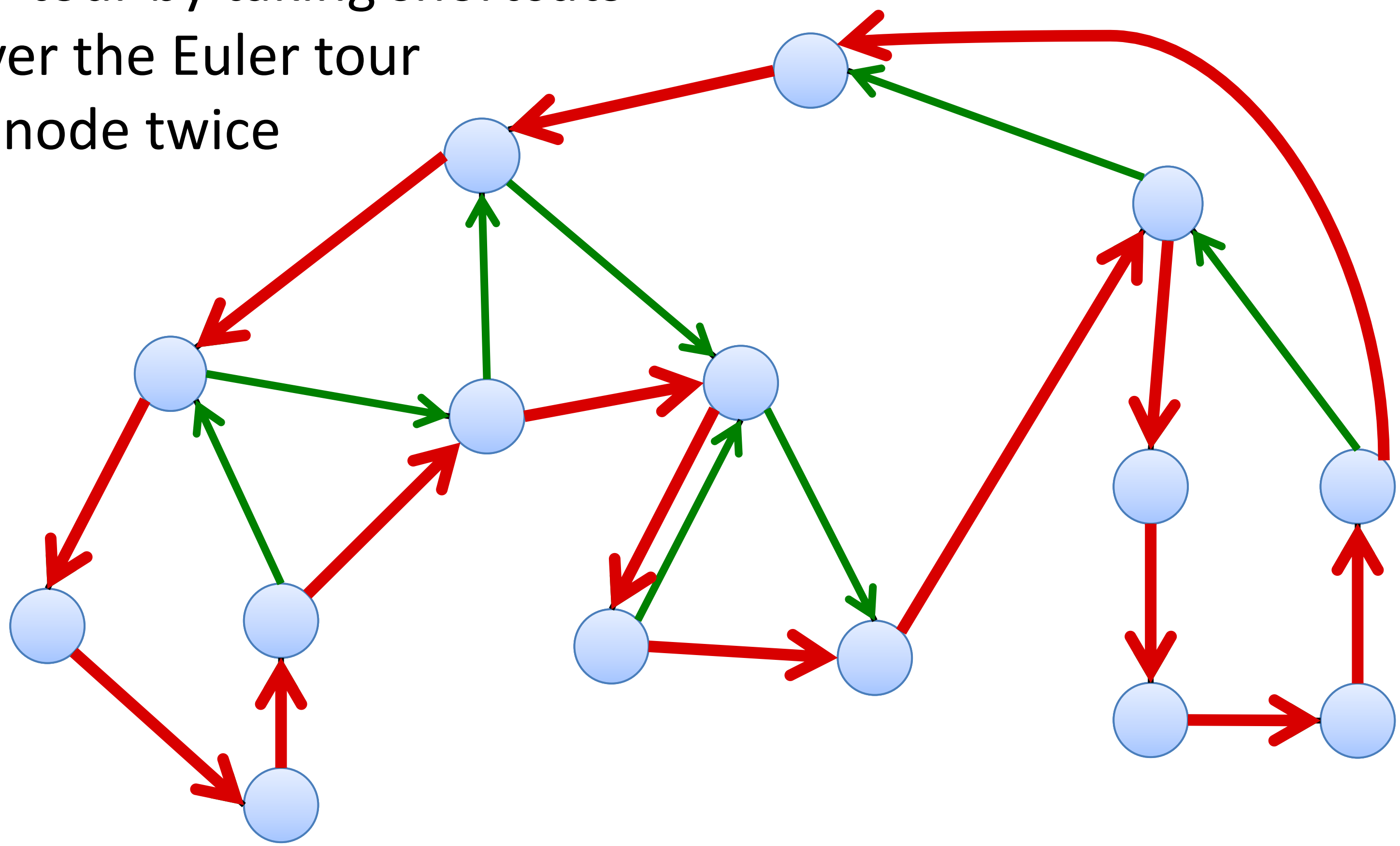
TSP Algorithm

1. Compute MST T
2. V_{odd}: nodes that have an odd degree in T ($|V_{\mathrm{odd}}|$ is even)
3. Compute min weight perfect matching M of (V_{odd}, d)
4. $(V, T \cup M)$ is a (multi-)graph with even degrees



TSP Algorithm

5. Compute Euler tour on $(V, T \cup M)$
6. Total length of Euler tour $\leq \frac{3}{2} \cdot \mathbf{TSP_{OPT}}$
7. Get TSP tour by taking shortcuts wherever the Euler tour visits a node twice



TSP Algorithm

- The described algorithm is by Christofides

Theorem: The Christofides algorithm achieves an approximation ratio of at most ${}^{3}/_{2}$.

Proof:

- The length of the Euler tour is $\leq {}^{3}/_{2} \cdot \mathrm{TSP}_{\mathrm{OPT}}$
- Because of the triangle inequality, taking shortcuts can only make the tour shorter

Set Cover

Input:

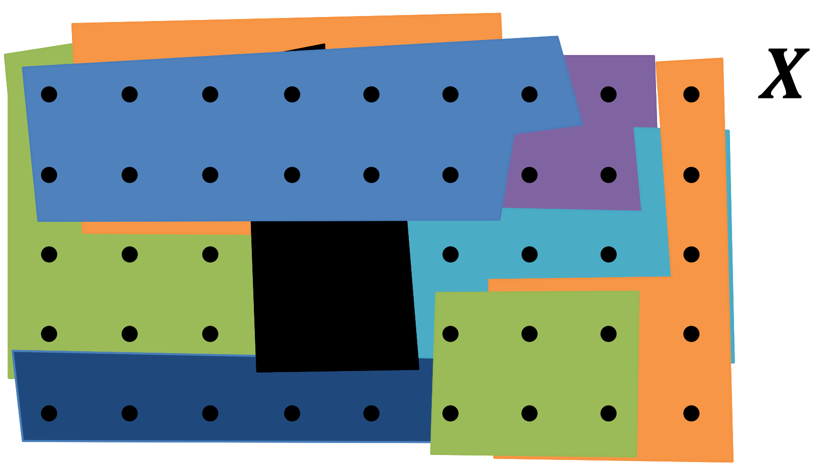
- A set of elements X and a collection $\mathcal{S}$ of subsets X, i.e., $\mathcal{S} \subseteq 2^X$
  - such that $\bigcup_{S \in \mathcal{S}} S = X$

Set Cover:

- A set cover $\mathcal{C}$ of $(X, \mathcal{S})$ is a subset of the sets $\mathcal{S}$ which covers X:

$$\bigcup_{S \in \mathcal{C}} S = X$$

Example:



Minimum (Weighted) Set Cover

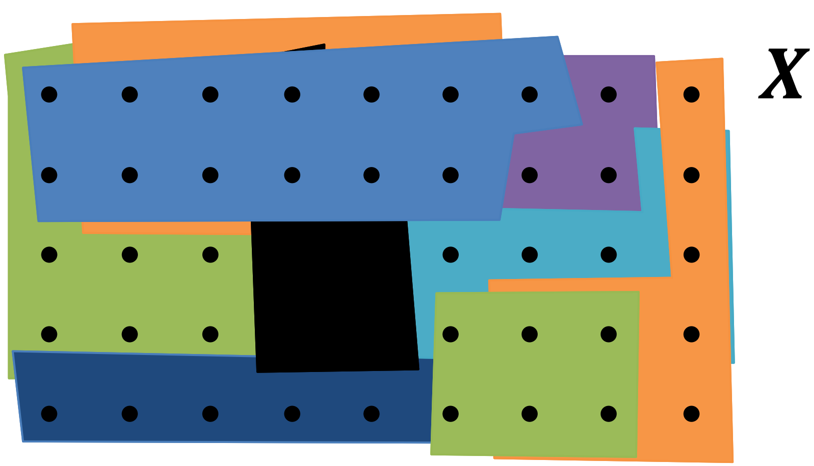
Minimum Set Cover:

- **Goal:** Find a set cover $\mathcal{C}$ of smallest possible size
  - i.e., over X with as few sets as possible

Minimum Weighted Set Cover:

- Each set $S \in \mathcal{S}$ has a weight $w_S > 0$
- **Goal:** Find a set cover $\mathcal{C}$ of minimum weight

Example:

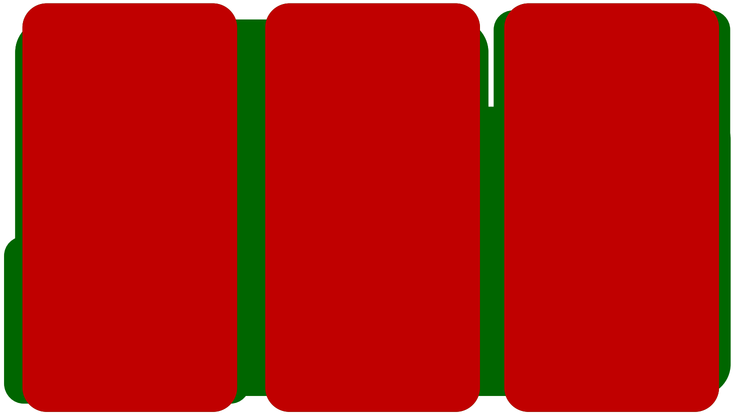


Minimum Set Cover: Greedy Algorithm

Greedy Set Cover Algorithm:

- Start with $\mathcal{C} = \emptyset$
- In each step, add set $S \in \mathcal{S} \setminus \mathcal{C}$ to $\mathcal{C}$ s.t. S covers as many uncovered elements as possible

Example:



Weighted Set Cover: Greedy Algorithm

Greedy Weighted Set Cover Algorithm:

- Start with $\mathcal{C} = \emptyset$
- In each step, add set $S \in \mathcal{S} \setminus \mathcal{C}$ with the best weight per newly covered element ratio (set with best efficiency):

$$S = \arg\min_{S \in \mathcal{S} \setminus \mathcal{C}} \frac{w_S}{\left|S \setminus \bigcup_{T \in \mathcal{C}} T\right|}$$

Analysis of Greedy Algorithm:

- Assign a price $p(x)$ to each element $x \in X$:
  The efficiency of the set when covering the element
- If covering x with set S, if partial cover is $\mathcal{C}$ before adding S:

$$p(e) = \frac{w_S}{\left|S \setminus \bigcup_{T \in \mathcal{C}} T\right|}$$

Weighted Set Cover: Greedy Algorithm

Example:

- Universe $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- Sets $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5, S_6\}$

$$S_1 = \{1, 2, 3, 4, 5\}, \qquad w_{S_1} = 4$$
$$S_2 = \{2, 6, 7\}, \qquad w_{S_2} = 1$$
$$S_3 = \{1, 6, 7, 8, 9\}, \qquad w_{S_3} = 4$$
$$S_4 = \{2, 4, 7, 9, 10\}, \qquad w_{S_4} = 6$$
$$S_5 = \{1, 3, 5, 6, 7, 8, 9, 10\}, \qquad w_{S_5} = 9$$
$$S_6 = \{9, 10\}, \qquad w_{S_6} = 3$$

Weighted Set Cover: Greedy Algorithm

Lemma: Consider a set $S = \{x_1, x_2, \dots, x_k\} \in \mathcal{S}$ be a set and assume that the elements are covered in the order $x_1, x_2, \dots, x_k$ by the greedy algorithm (ties broken arbitrarily).

Then, the price of element x_i is at most $p(x_i) \leq \frac{w_S}{k-i+1}$

Weighted Set Cover: Greedy Algorithm

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Then, the price of element x_i is at most $p(x_i) \leq \frac{w_S}{k-i+1}$

Corollary: The total price of a set $S \in \mathcal{S}$ of size $|S| = k$ is

$$\sum_{x \in S} p(x) \leq w_S \cdot H_k, \qquad \text{where } H_k = \sum_{i=1}^{k} \frac{1}{i} \leq 1 + \ln k$$

Weighted Set Cover: Greedy Algorithm

Corollary: The total price of a set $S \in \mathcal{S}$ of size $|S| = k$ is

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Theorem: The approximation ratio of the greedy minimum (weighted) set cover algorithm is at most $\boldsymbol{H_s \leq 1 + \ln s}$, where s is the cardinality of the largest set ($s = \max_{S \in \mathcal{S}} |S|$).

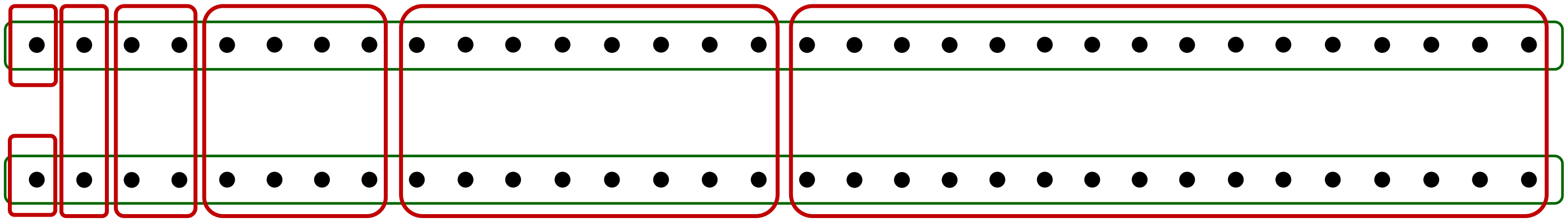
Set Cover Greedy Algorithm

Can we improve this analysis?

No! Even for the unweighted minimum set cover problem, the approximation ratio of the greedy algorithm is $\geq \big(1 - o(1)\big) \cdot \ln s$.

- if s is the size of the largest set... (s can be linear in n)

Let's show that the approximation ratio is at least $\Omega(\log n)$...



$\mathbf{OPT = 2}$

$\mathbf{GREEDY \geq \log_2 n}$