Chapter 8
Approximation Algorithms

Algorithm Theory
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Approximation Ratio

An approximation algorithm is an algorithm that computes a solution for an optimization with an objective value that is provably within a bounded factor of the optimal objective value.

Formally:

- $\mathrm{OPT} \geq 0$: optimal objective value
  $\mathrm{ALG} \geq 0$: objective value achieved by the algorithm

- **Approximation Ratio α:**

$$\textbf{Minimization:}\ \alpha := \max_{\text{input instances}} \frac{\mathrm{ALG}}{\mathrm{OPT}}$$

obj.-value

$$\textbf{Maximization:}\ \alpha := \max_{\text{input instances}} \frac{\mathrm{OPT}}{\mathrm{ALG}}$$

Example: Load Balancing

We are given:

- m machines $M_1, \ldots, M_m$
- n jobs, processing time of job i is t_i

Goal:

- Assign each job to a machine such that the makespan is minimized

  makespan: largest total processing time of any machine

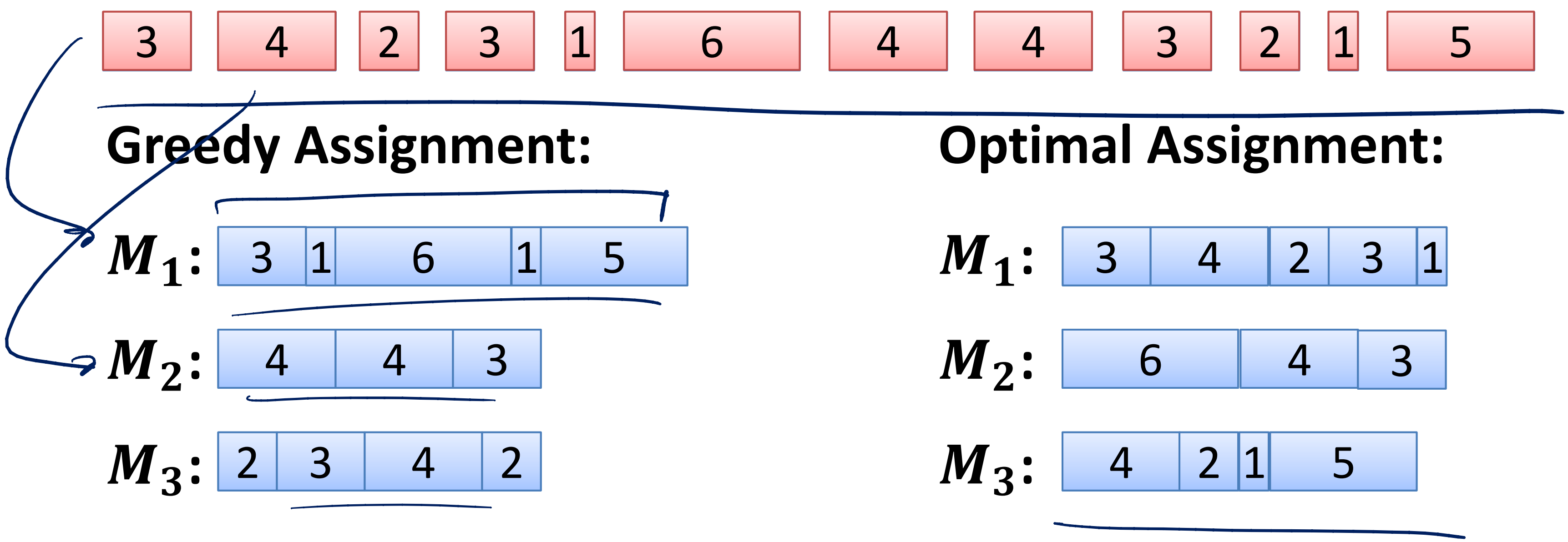
The above load balancing problem is NP-hard and we therefore want to get a good approximation for the problem.

Greedy Algorithm

There is a simple greedy algorithm:

- Go through the jobs in an arbitrary order
- When considering job i, assign the job to the machine that currently has the smallest load.

Example: 3 machines, 12 jobs



Greedy Analysis

- We will show that greedy gives a 2-approximation

- To show this, we need to compare the solution of greedy with an optimal solution (that we can't compute)

- Lower bound on the optimal makespan T^*:

$$T^* \geq \frac{1}{m} \cdot \sum_{i=1}^{n} t_i$$

- Second lower bound on optimal makespan T^*:

$$T^* \geq \max_{1 \leq i \leq n} t_i$$

Greedy Analysis

Theorem: The greedy algorithm has approximation ratio ≤ 2, i.e., for the makespan T of the greedy solution, we have $T \leq 2T^*$.

Proof:

- For machine k, let T_k be the time used by machine k
- Consider some machine M_i for which $T_i = T$
- Assume that job j is the last one schedule on M_i:

$$\boldsymbol{M_i}: \quad \underbrace{\boxed{\quad T - t_j \quad}}_{\leq T^*}\underbrace{\boxed{\quad t_j \quad}}_{\leq T^*}$$

- When job j is scheduled, M_i has the minimum load

$T - t_j$ is smallest load

$\forall k: \; T_k \geq T - t_j$ total load $\geq m \cdot \underbrace{(T - t_j)}_{\leq T^*}$

Can We Do Better?

$\underbrace{1, 1, 1, \ldots, 1}_{m(m-1)}, m$

The analysis of the greedy algorithm is almost tight:

- Example with $n = m(m-1) + 1$ jobs
- Jobs $1, \ldots, n-1 = m(m-1)$ have $t_i = 1$, job n has $t_n = m$

Greedy Schedule:

M_1: $\boxed{1}\boxed{1}\boxed{1}\boxed{1}$ … $\boxed{1}\boxed{t_n = m}$

M_2: $\boxed{1}\boxed{1}\boxed{1}\boxed{1}$ … $\boxed{1}$

M_3: $\boxed{1}\boxed{1}\boxed{1}\boxed{1}$ … $\boxed{1}$

⋮ ⋮

M_m: $\underbrace{\boxed{1}\boxed{1}\boxed{1}\boxed{1} \ldots \boxed{1}}_{m-1}$

makespan of greedy: $2m-1$

OPT

M_1: $\overbrace{\boxed{t_n}}^{m}$

M_2: $\boxed{1} \ldots \boxed{1}$

⋮

M_m: $\underbrace{\boxed{1} \ldots \boxed{1}}_{m \text{ jobs}}$

makespan: m

Improving Greedy

Bad case for the greedy algorithm:
One large job in the end can destroy everything

Idea: assign large jobs first

if $n \leq m$ problem trivial

Modified Greedy Algorithm:

$1, \ldots, m+1$

1. Sort jobs by decreasing length s.t. $t_1 \geq t_2 \geq \cdots \geq t_n$
2. Apply the greedy algorithm as before (in the sorted order)

Lemma: If $n > m$: $T^* \geq t_m + t_{m+1} \geq 2t_{m+1}$

$t_{m+1} \leq \frac{T^*}{2}$

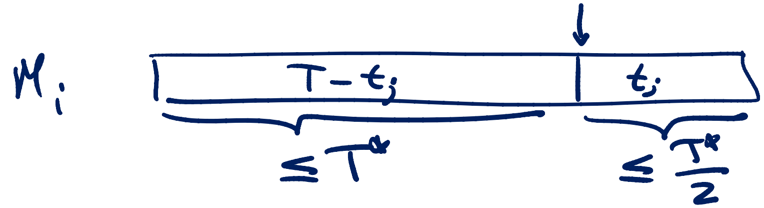
Proof:

- Two of the first $m + 1$ jobs need to be scheduled on the same machine
- Jobs m and $m + 1$ are the shortest of these jobs

Analysis of the Modified Greedy Alg.

Theorem: The modified algorithm has approximation ratio $\leq {}^3/_2$.

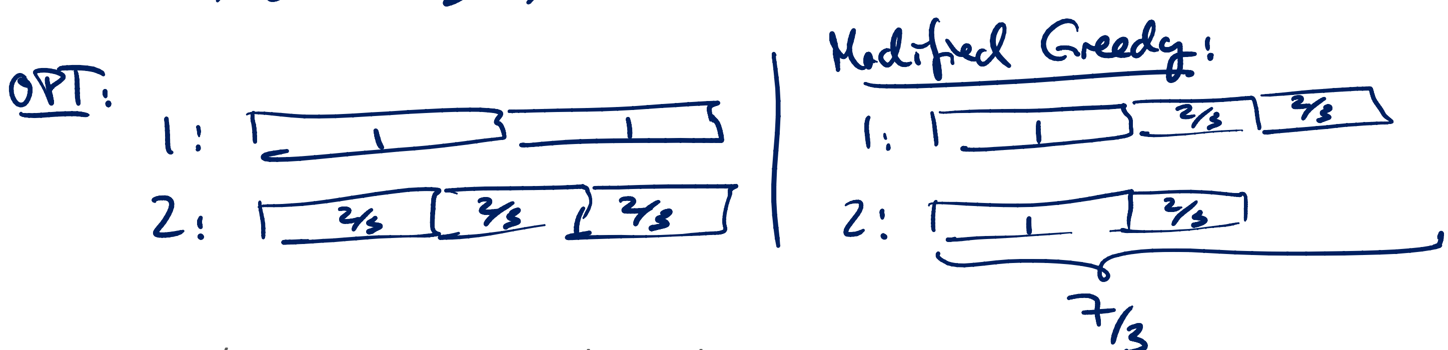
Proof:



- We show that $T \leq {}^3/_2 \cdot T^*$
- As before, we consider the machine M_i with $T_i = T$
- Job j (of length t_j) is the last one scheduled on machine M_i
- If j is the only job on M_i, we have $T = T^*$

- Otherwise, we have $j \geq m + 1$
  - The first m jobs are assigned to m distinct machines

jobs $1, 1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3},$ $m=2$

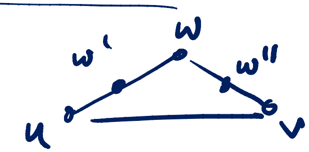


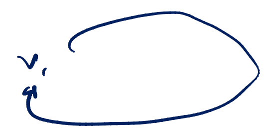
Metric TSP

Input:

- Set V of n nodes (points, cities, locations, sites)
- Distance function $d: V \times V \to \mathbb{R}$, i.e., $d(u,v)$ is dist from u to v
- Distances define a metric on V:

$$d(u,v) = d(v,u) \geq 0, \qquad d(u,v) = 0 \Leftrightarrow u = v$$
$$\forall u,v,w \in V : \underbrace{d(u,v) \leq d(u,w) + d(w,v)}_{\Delta\text{-inequality}}$$





Solution:

- Ordering/permutation $v_1, v_2, \dots, v_n$ of the vertices
- Length of TSP path: $\sum_{i=1}^{n-1} d(v_i, v_{i+1})$
- Length of TSP tour: $d(v_1, v_n) + \sum_{i=1}^{n-1} d(v_i, v_{i+1})$

Goal:

- Minimize length of TSP path or TSP tour

Metric TSP

- The problem is NP-hard
- We have seen that the greedy algorithm (always going to the nearest unvisited node) gives an $O(\log n)$-approximation
- Can we get a constant approximation ratio?
- We will see that we can…

TSP and MST

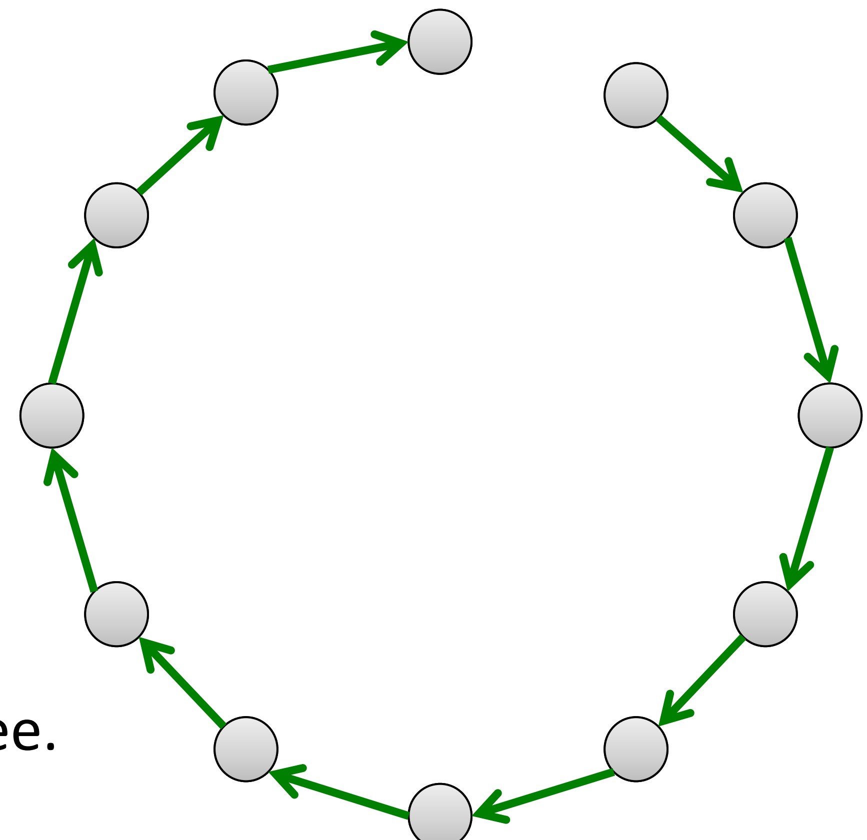
$w(MST) \leq \cancel{OPT}\ TSP_{OPT}$

Claim: The length of an optimal TSP path is lower bounded by the weight of a minimum spanning tree

Proof:

- A TSP path is a spanning tree, it's length is the weight of the tree

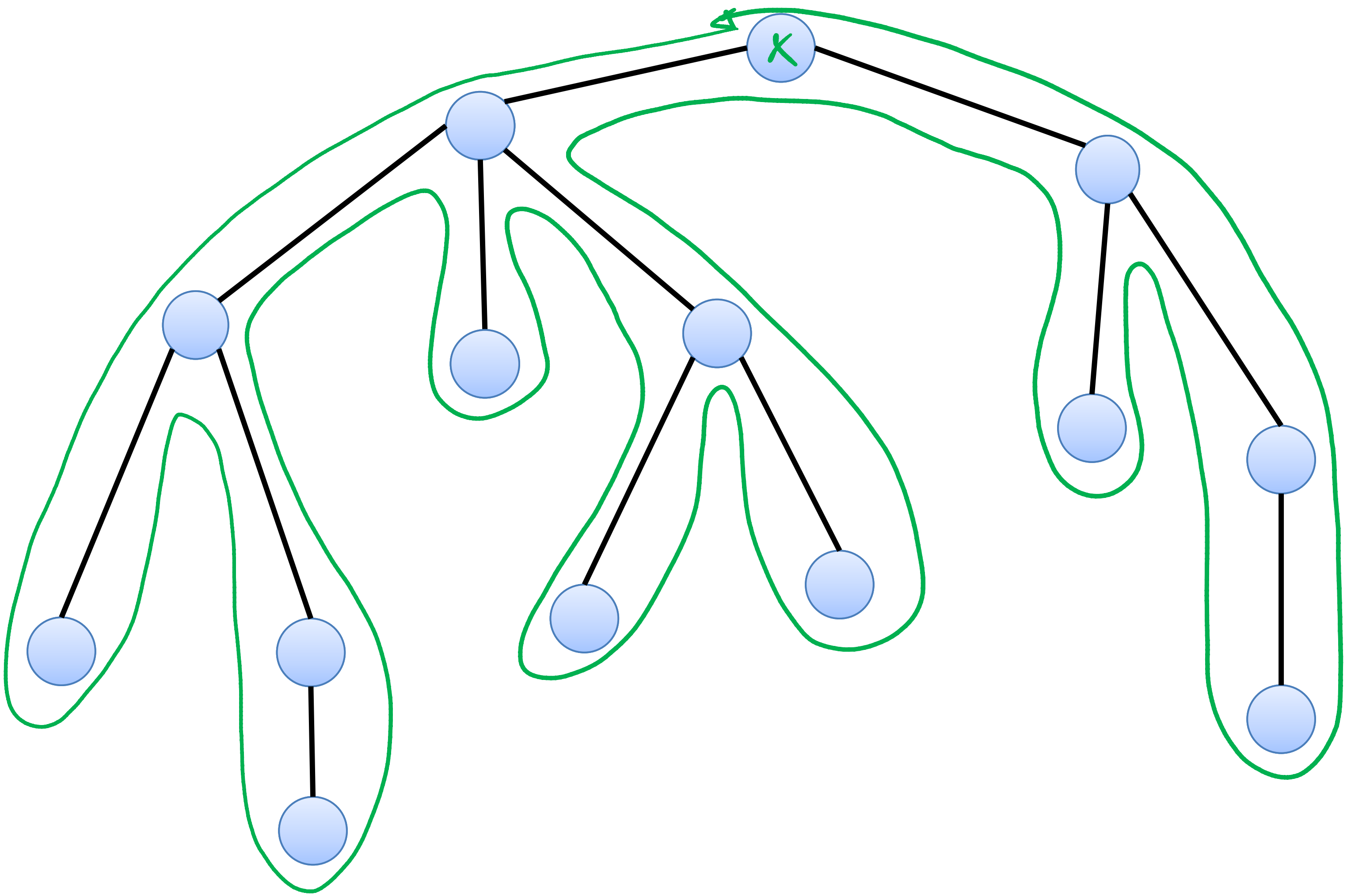
$w(MST) \leq TSP_{PATH} \leq TSP_{TOUR}$



Corollary: Since an optimal TSP tour is longer than an optimal TSP path, the length of an optimal TSP tour is also lower bounded by the weight of a minimum spanning tree.

The MST Tour

Walk around the MST…

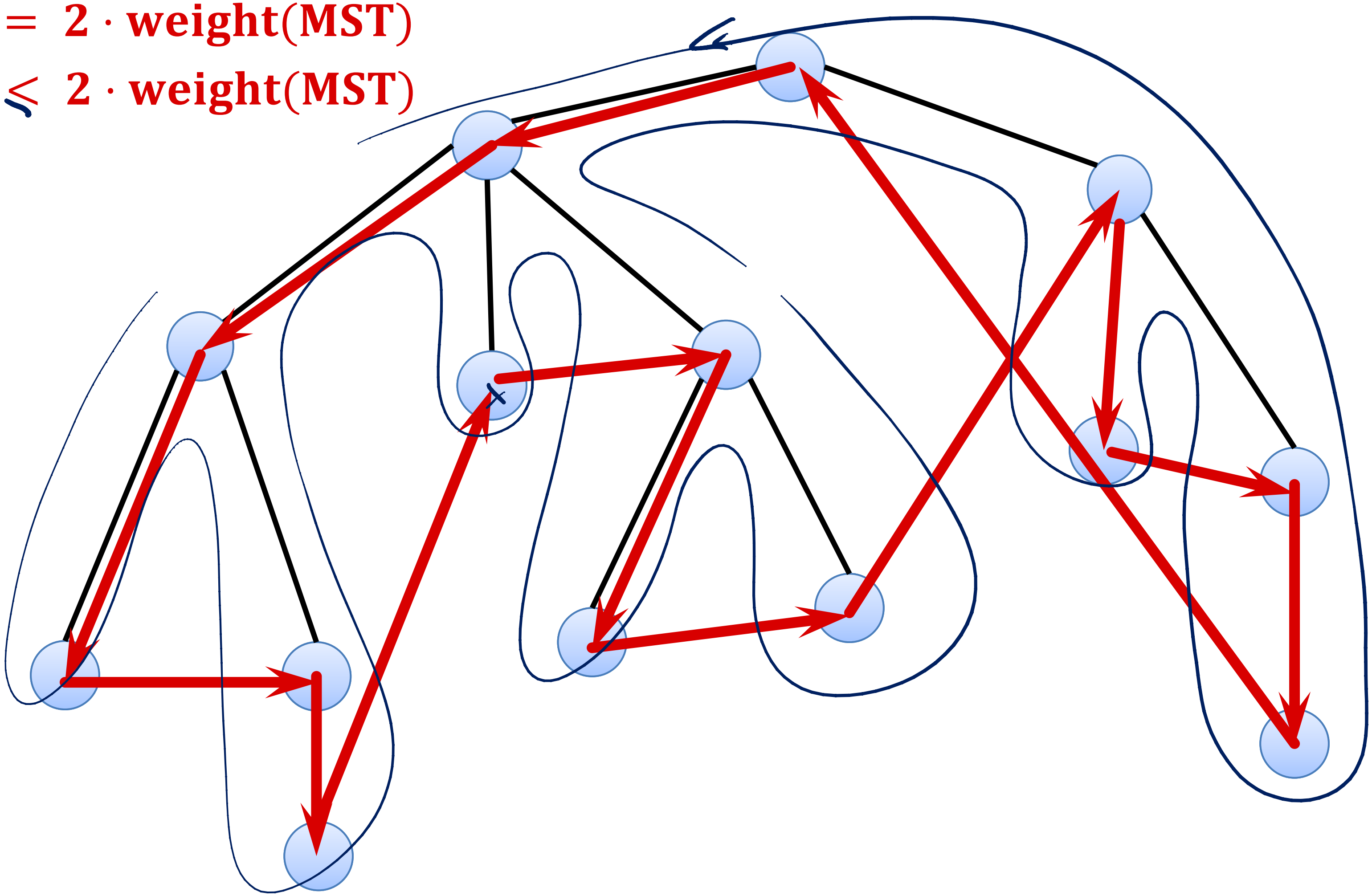


The MST Tour

Walk around the MST…

Cost (walk) $= 2 \cdot \mathbf{weight(MST)}$

Cost (tour) $\leq 2 \cdot \mathbf{weight(MST)}$



Approximation Ratio of MST Tour

Theorem: The MST TSP tour gives a 2-approximation for the metric TSP problem.

Proof:

- Triangle inequality → length of tour is at most $2 \cdot \text{weight(MST)}$
- We have seen that $\underline{\text{weight(MST)}} < \underline{\text{opt. tour length}}$

Can we do even better?

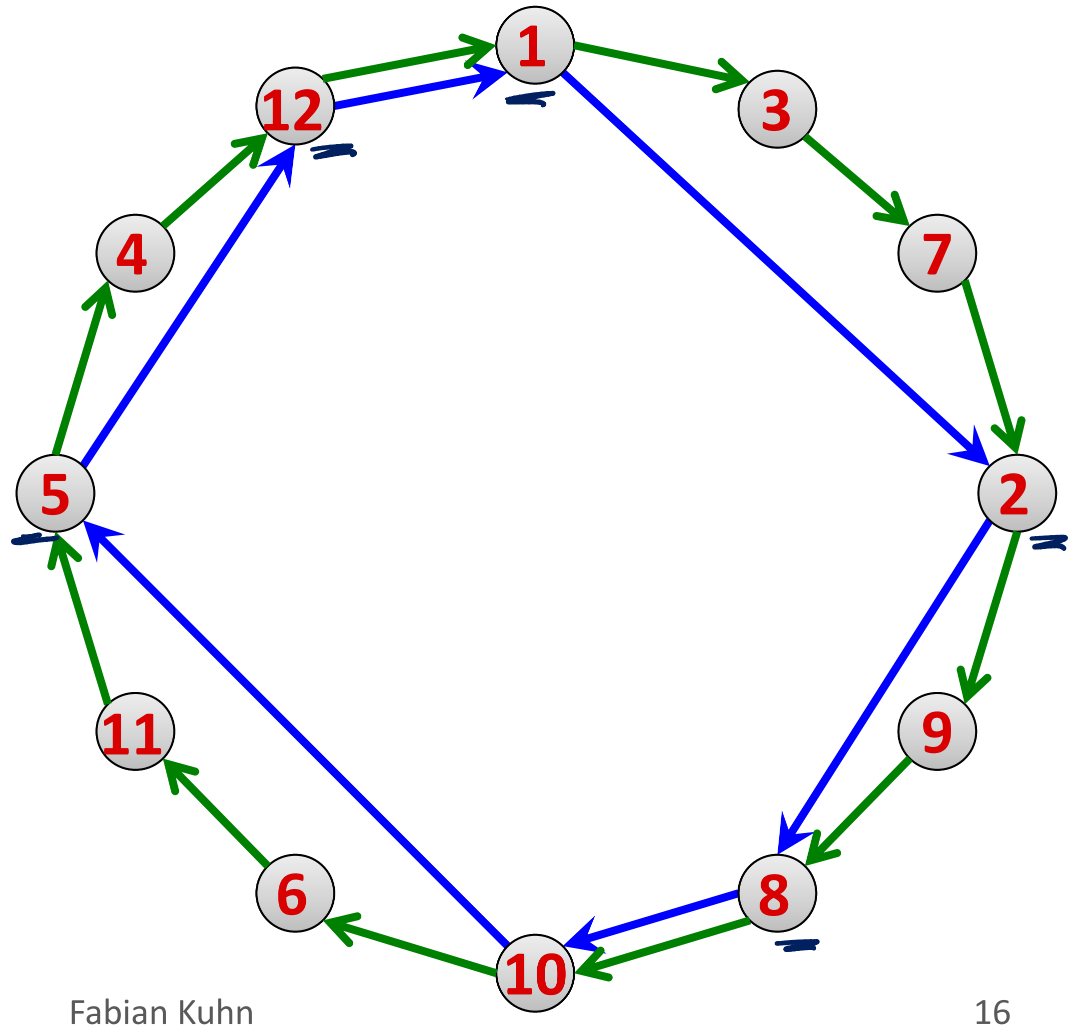
Metric TSP Subproblems

Claim: Given a metric (V, d) and (V', d) for $V' \subseteq V$, the optimal TSP path/tour of (V', d) is at most as large as the optimal TSP path/tour of (V, d).

Optimal TSP tour of nodes $1, 2, \dots, 12$

Induced TSP tour for nodes $1, 2, 5, 8, 10, 12$

blue tour $\leq$ green tour



TSP and Matching

- Consider a metric TSP instance (V, d) with an even number of nodes $|V|$

- Recall that a perfect matching is a matching $M \subseteq V \times V$ such that every node of V is incident to an edge of M.

- Because $|V|$ is even and because in a metric TSP, there is an edge between any two nodes $u, v \in V$, any partition of V into $|V|/2$ pairs is a perfect matching.

- The weight of a matching M is the sum of the distances represented by all edges in M:

$$w(M) = \sum_{\{u,v\} \in M} d(u, v)$$

TSP and Matching

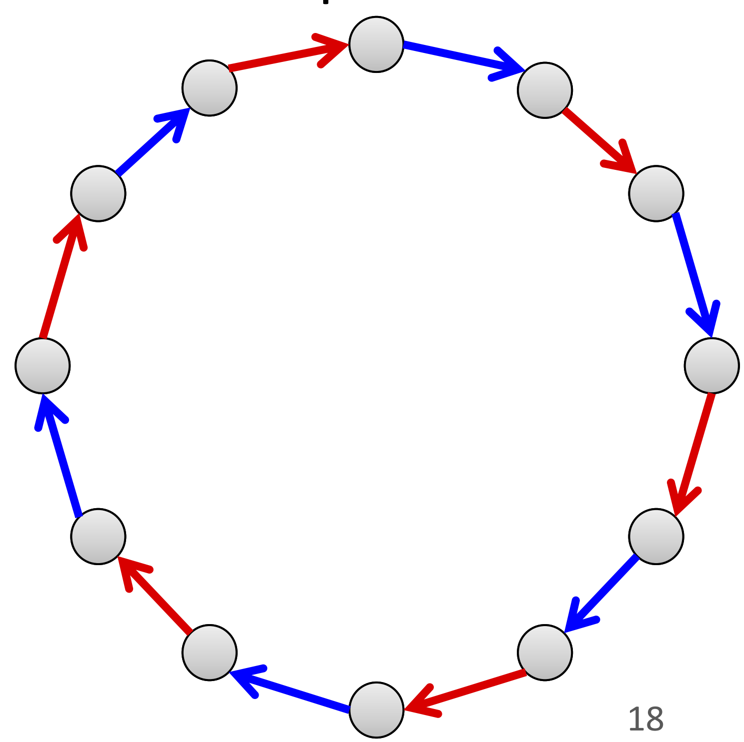
$w(M) \leq \frac{1}{2} TSP_{OPT}$

Lemma: Assume we are given a TSP instance (V, d) with an even number of nodes. The length of an optimal TSP tour of (V, d) is at least twice the weight of a minimum weight perfect matching of (V, d).

Proof:

- The edges of a TSP tour can be partitioned into 2 perfect matchings

$TSP_{OPT} = \underbrace{red}_{\text{pf. matching}} + \underbrace{blue}_{\text{perf. matching}}$



Minimum Weight Perfect Matching

Claim: If $|V|$ is even, a minimum weight perfect matching of (V, d) can be computed in polynomial time

Proof Sketch:

- We have seen that a maximum matching in an unweighted graph can be computed in polynomial time
- With a more complicated algorithm, also a maximum weighted matching can be computed in polynomial time
- In a complete graph, a maximum weighted matching is also a (maximum weight) perfect matching
- Define weight $w(u, v) := D - d(u, v)$ → longest $d(u,v)$

  $\max_M \sum_{e \in M} (D - d(e))$

  $\max_M \frac{1}{2} \cdot D - \sum_{e \in M} d(e)$
- A maximum weight perfect matching for (V, w) is a minimum weight perfect matching for (V, d)

Algorithm Outline

Problem of MST algorithm:

- Every edge has to be visited twice

Goal:

- Get a graph on which every edge only has to be visited once (and where still the total edge weight is small compared to an optimal TSP tour)

Euler Tours:

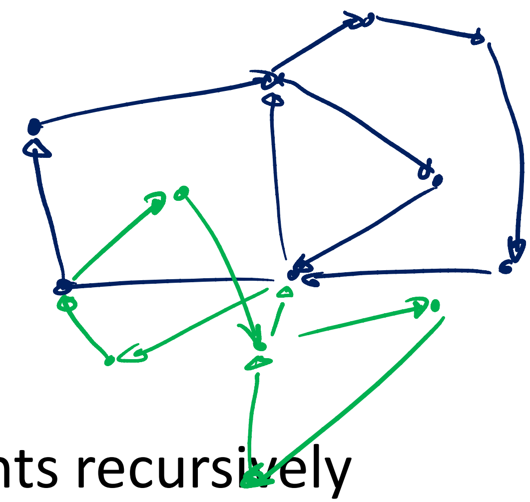
- A tour that visits each edge of a graph exactly once is called an Euler tour
- An Euler tour in a (multi-)graph exists if and only if every node of the graph has even degree
- That's definitely not true for a tree, but can we modify our MST suitably?

Euler Tour

Theorem: A connected (multi-)graph G has an Euler tour if and only if every node of G has even degree.

Proof:

- If G has an odd degree node, it clearly cannot have an Euler tour
- If G has only even degree nodes, a tour can be found recursively:



1. Start at some node
2. As long as possible, follow an unvisited edge
   - Gives a partial tour, the remaining graph still has even degree
3. Solve problem on remaining components recursively
4. Merge the obtained tours into one tour that visits all edges

TSP Algorithm

$\sum_{v \in V} \deg(v) = 2 \cdot m$

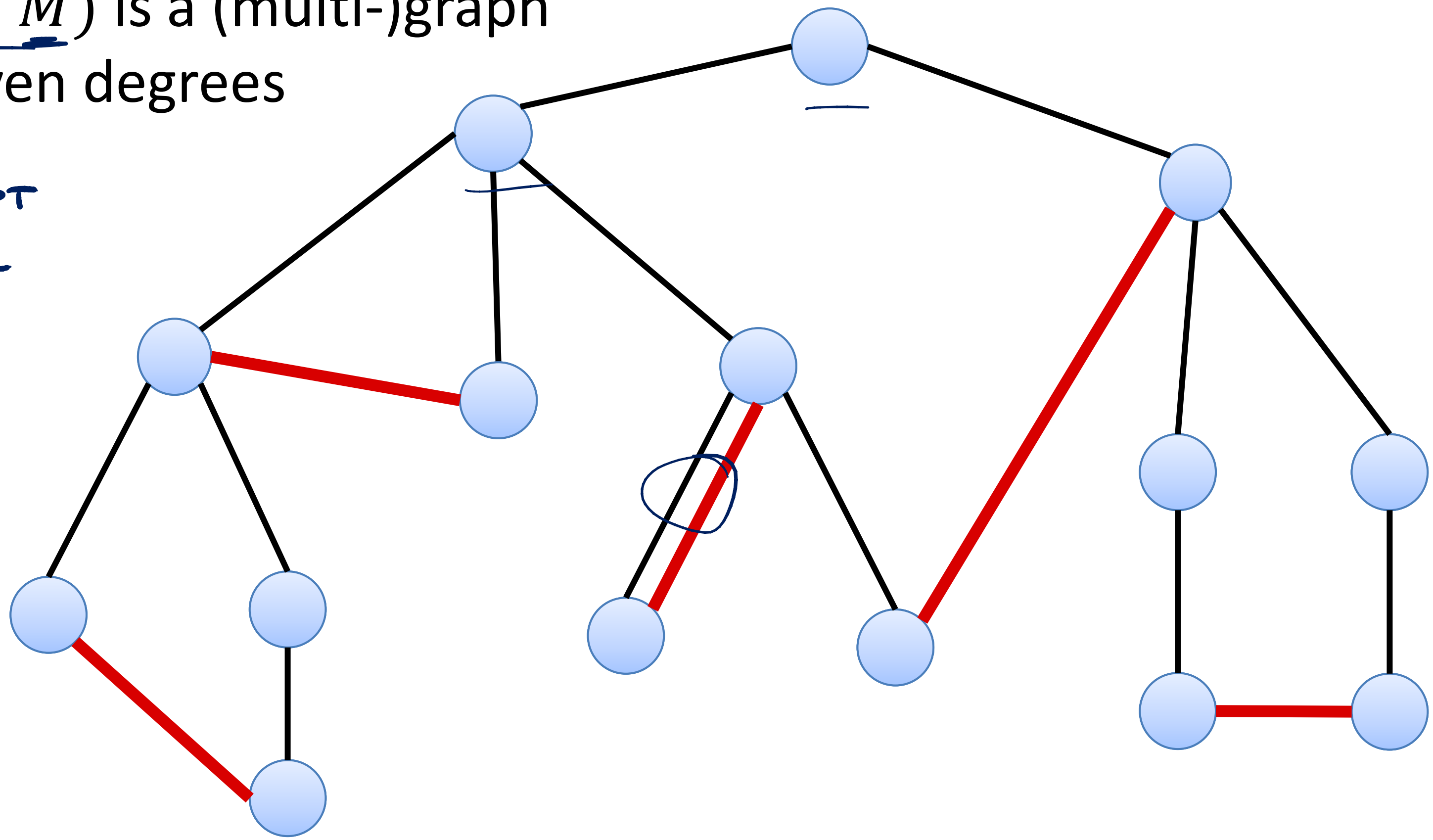
even number

$\sum_{v \in V_{odd}} \deg(v)$ is even

1. Compute MST T
2. V_{odd}: nodes that have an odd degree in T ($|V_{\text{odd}}|$ is even)
3. Compute min weight perfect matching M of (V_{odd}, d)
4. $(V, T \cup M)$ is a (multi-)graph with even degrees

$w(MST) \leq TSP_{OPT}$

$w(M) \leq \frac{1}{2} TSP_{OPT}$

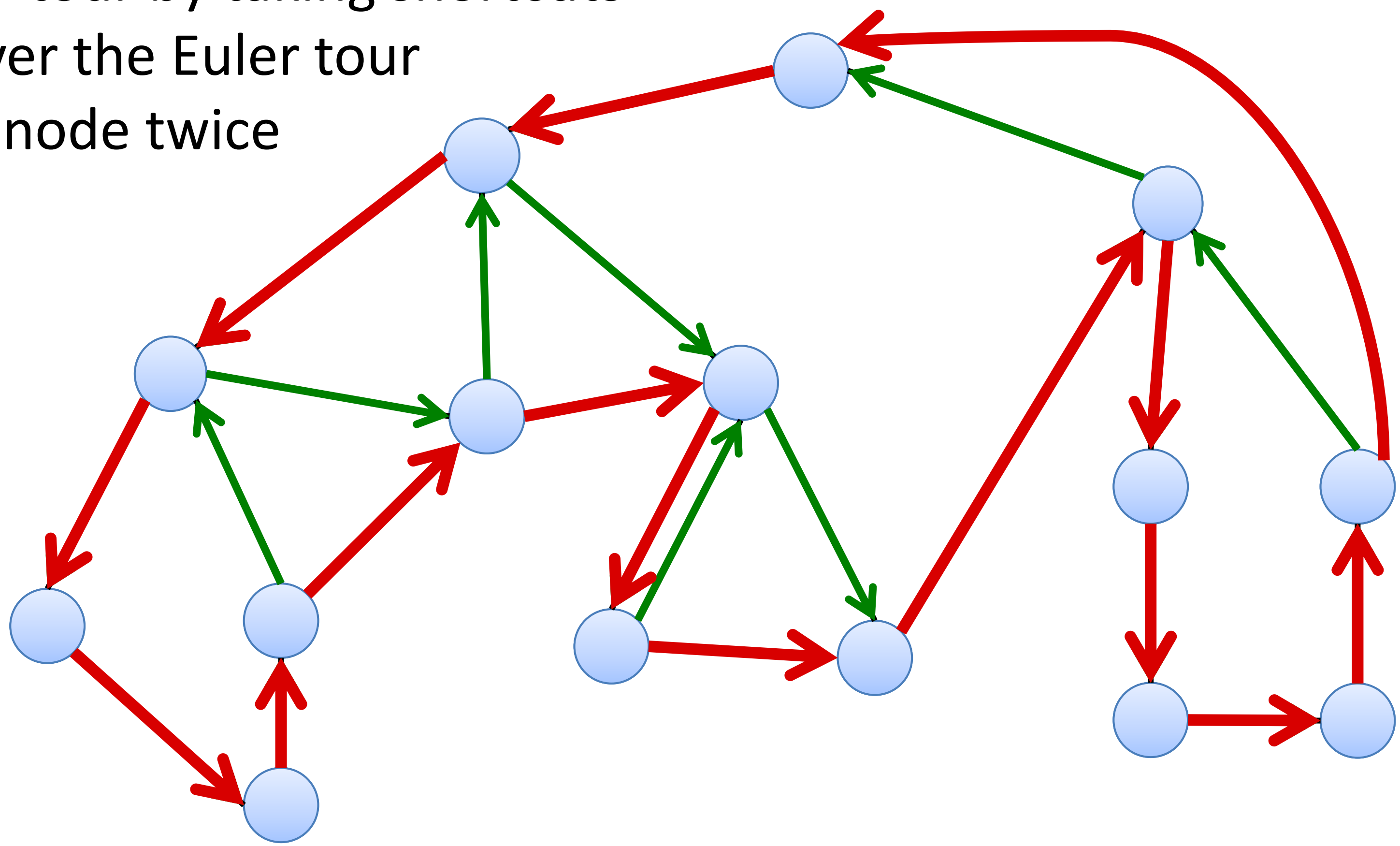


TSP Algorithm

$w(T) \leq TSP_{OPT}$

$w(M) \leq \frac{1}{2} TSP_{OPT}$

5. Compute Euler tour on $(V, T \cup M)$
6. Total length of Euler tour $\leq \frac{3}{2} \cdot \mathbf{TSP_{OPT}}$
7. Get TSP tour by taking shortcuts wherever the Euler tour visits a node twice



TSP Algorithm

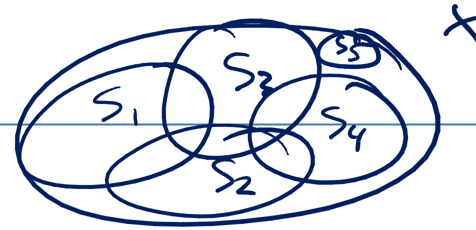
- The described algorithm is by Christofides

Theorem: The Christofides algorithm achieves an approximation ratio of at most ${}^{3}/_{2}$.

Proof:

- The length of the Euler tour is $\leq {}^{3}/_{2} \cdot \mathrm{TSP}_{\mathrm{OPT}}$
- Because of the triangle inequality, taking shortcuts can only make the tour shorter

Set Cover



Input:

universe

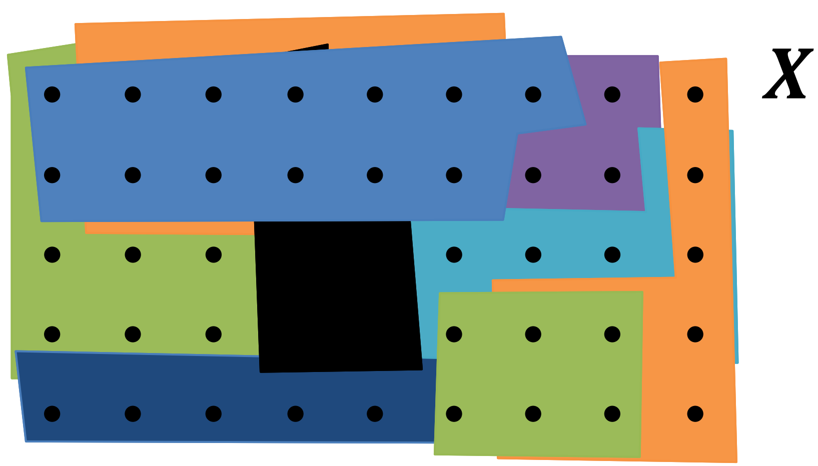
- A set of elements X and a collection $\mathcal{S}$ of subsets X, i.e., $\mathcal{S} \subseteq 2^X$
  - such that $\bigcup_{S \in \mathcal{S}} S = X$

Set Cover:

- A set cover $\mathcal{C}$ of $(X, \mathcal{S})$ is a subset of the sets $\mathcal{S}$ which covers X:

$$\bigcup_{S \in \mathcal{C}} S = X$$

Example:



Minimum (Weighted) Set Cover

Minimum Set Cover:

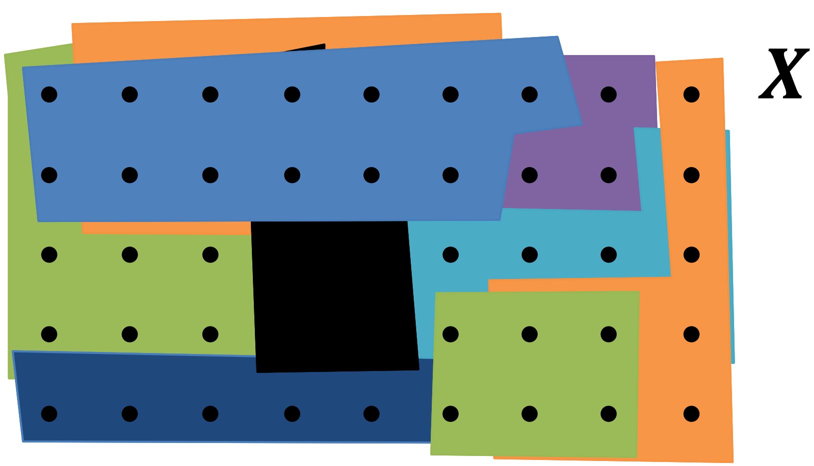
cardinality

- **Goal:** Find a set cover $\mathcal{C}$ of smallest possible size
  - i.e., cover X with as few sets as possible

Minimum Weighted Set Cover:

- Each set $S \in \mathcal{S}$ has a weight $w_S > 0$
- **Goal:** Find a set cover $\mathcal{C}$ of minimum weight

Example:



Minimum Set Cover: Greedy Algorithm

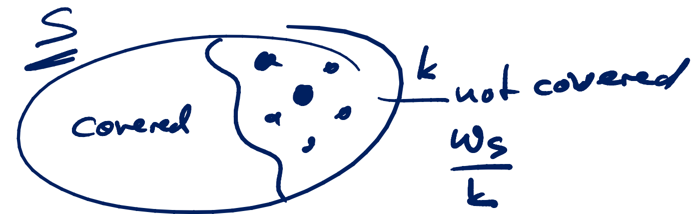
Greedy Set Cover Algorithm:

- Start with $\mathcal{C} = \emptyset$
- In each step, add set $S \in \mathcal{S} \setminus \mathcal{C}$ to $\mathcal{C}$ s.t. S covers as many uncovered elements as possible

Example:

Weighted Set Cover: Greedy Algorithm

Greedy Weighted Set Cover Algorithm:



$\frac{w_S}{k}$

- Start with $\mathcal{C} = \emptyset$
- In each step, add set $S \in \mathcal{S} \setminus \mathcal{C}$ with the best weight per newly covered element ratio (set with best efficiency):

$$S = \arg\min_{S \in \mathcal{S} \setminus \mathcal{C}} \frac{w_S}{\left|S \setminus \bigcup_{T \in \mathcal{C}} T\right|}$$

\# of newly covered elements

Analysis of Greedy Algorithm:

- Assign a price $p(x)$ to each element $x \in X$:
  The efficiency of the set when covering the element
- If covering x with set S, if partial cover is $\mathcal{C}$ before adding S:

$$p(x) = \frac{w_S}{\left|S \setminus \bigcup_{T \in \mathcal{C}} T\right|}$$

in the end

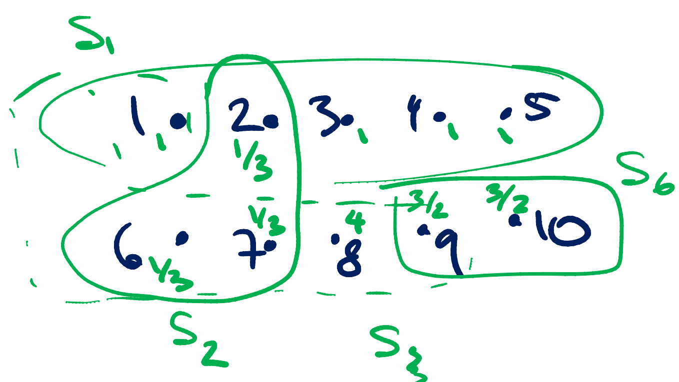
$$\sum_{x \in X} p(x) = \sum_{S \in \mathcal{C}} w_S$$

weight of set cover

Example:

- Universe $X = \{1,2,3,4,5,6,7,8,9,10\}$
- Sets $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5, S_6\}$

$$
\begin{aligned}
S_1 &= \{1,\cancel{2},3,4,5\}, & w_{S_1} &= 4\\
&\cancel{S_2 = \{2,6,7\},} & w_{S_2} &= 1\\
S_3 &= \{1,\cancel{6},\cancel{7},8,9\}, & w_{S_3} &= 4\\
S_4 &= \{\cancel{2},4,\cancel{7},9,10\}, & w_{S_4} &= 6\\
S_5 &= \{1,3,5,\cancel{6},\cancel{7},8,9,10\}, & w_{S_5} &= 9\\
S_6 &= \{9,10\}, & w_{S_6} &= 3
\end{aligned}
$$

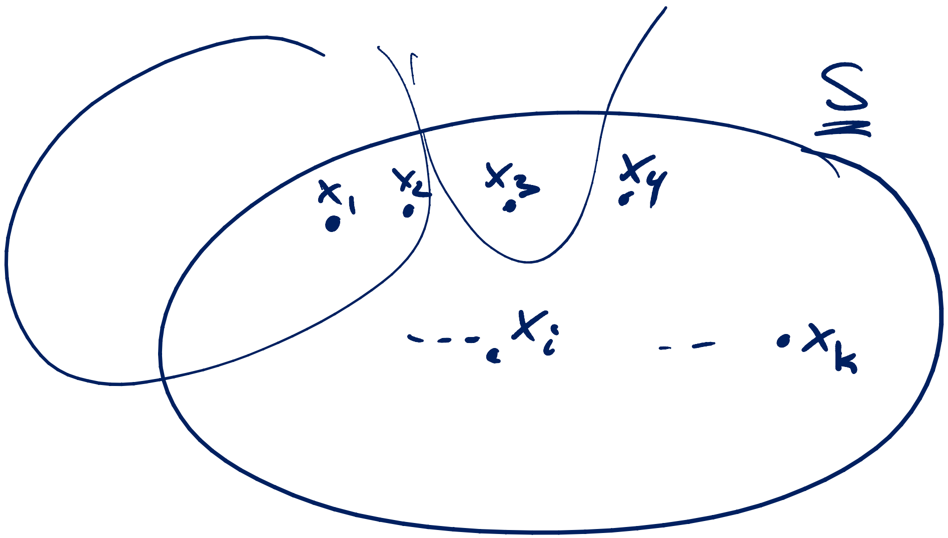


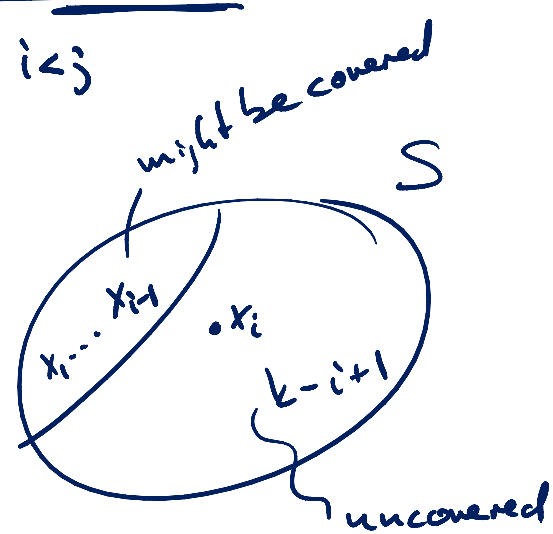
Weighted Set Cover: Greedy Algorithm

Lemma: Consider a set $S = \{x_1, x_2, \dots, x_k\} \in \mathcal{S}$ be a set and assume that the elements are covered in the order $x_1, x_2, \dots, x_k$ by the greedy algorithm (ties broken arbitrarily).

Then, the price of element x_i is at most $p(x_i) \le \frac{w_S}{k-i+1}$

$i < j$





$p(x_1) \le \frac{w_S}{k}$ $\quad$ $p(x_2) \le \frac{w_S}{k-1}$

$$\frac{w_S}{k-i+1}$$

Weighted Set Cover: Greedy Algorithm

Lemma: Consider a set $S = \{x_1, x_2, \dots, x_k\} \in \mathcal{S}$ be a set and assume that the elements are covered in the order $x_1, x_2, \dots, x_k$ by the greedy algorithm (ties broken arbitrarily).

Then, the price of element x_i is at most $p(x_i) \leq \frac{w_S}{k-i+1}$

Corollary: The total price of a set $S \in \mathcal{S}$ of size $|S| = k$ is

$$\sum_{x \in S} p(x) \leq w_S \cdot H_k, \qquad \text{where } H_k = \sum_{i=1}^{k} \frac{1}{i} \leq 1 + \ln k$$

$$= p(x_1) + p(x_2) + \dots + p(x_k) \underset{\text{Lemma}}{\leq} w_S \Big(\underbrace{\frac{1}{k} + \frac{1}{k-1} + \frac{1}{k-2} + \dots \frac{1}{1}}_{H_k} \Big)$$

Weighted Set Cover: Greedy Algorithm

Corollary: The total price of a set $S \in \mathcal{S}$ of size $|S| = k$ is

$$\sum_{x \in S} p(x) \leq w_S \cdot H_k, \qquad \text{where } H_k = \sum_{i=1}^{k} \frac{1}{i} \leq 1 + \ln k$$

Theorem: The approximation ratio of the greedy minimum (weighted) set cover algorithm is at most $\boldsymbol{H_s \leq 1 + \ln s}$, where s is the cardinality of the largest set ($s = \max_{S \in \mathcal{S}} |S|$). $\leq H_n \leq 1 + \ln n$

OPT set cover C^*

$OPT = w_1 + \ldots + w_5$

total price $\leq (w_1 + \ldots + w_5) \cdot H_s$

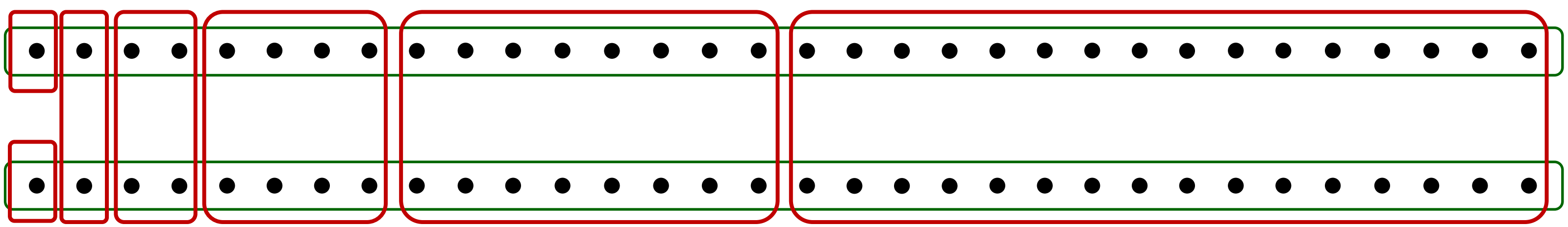
Set Cover Greedy Algorithm

Can we improve this analysis? $\ln s - \ln\ln s$

No! Even for the unweighted minimum set cover problem, the approximation ratio of the greedy algorithm is $\geq \big(1 - o(1)\big) \cdot \ln s$.

- if s is the size of the largest set... (s can be linear in n)

Let's show that the approximation ratio is at least $\Omega(\log n)$...



$\frac{\log_2 n}{2}$

$\mathbf{OPT = 2}$

$\mathbf{GREEDY \geq \log_2 n}$