Chapter 1
Divide and Conquer

Algorithm Theory
WS 2016/17

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Divide-And-Conquer Principle

- Important algorithm design method

- Examples from basic alg. & data structures class (Informatik 2):
  - Sorting: Mergesort, Quicksort
  - Binary search

- Further examples
  - Median
  - Compairing orders
  - Convex hull / Delaunay triangulation / Voronoi diagram
  - Closest pairs
  - Line intersections
  - Polynomial multiplication / FFT
  - ...

Example 1: Quicksort

S | v pivot

$S_\ell \leq v$ | v | $S_r \geq v$ partition

function Quick (S: sequence): sequence;

{returns the sorted sequence S}

begin

if $\#S \leq 1$ then **return** S

else { choose pivot element v in S;

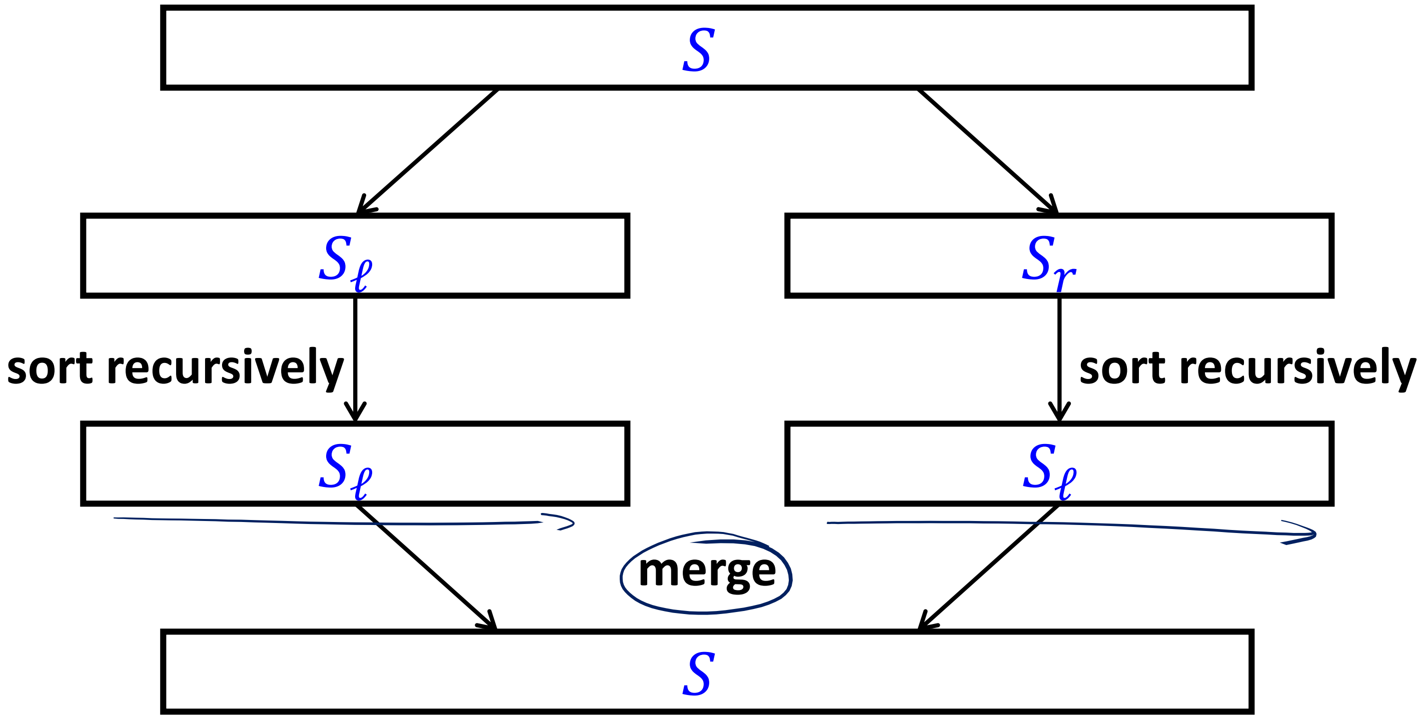
partition S into S_ℓ with elements $\geq v$,

and S_r with elements $\geq v$

return $\boxed{\text{Quick}(S_\ell) \mid v \mid \text{Quick}(S_r)}$

end;

Example 2: Mergesort



Formulation of the D&C principle

Divide-and-conquer method for solving a problem instance of size n:

QS	MS
choose pivot partition 2 subproblems	divide in the middle
recurse	recurse
	merge

1. Divide

$n \leq c$: Solve the problem directly.

$n > c$: Divide the problem into k subproblems of sizes $n_1, \dots, n_k < n$ $(k \geq 2)$. ($k=1$ possible)

2. Conquer

Solve the k subproblems in the same way (recursively).

3. Combine

Combine the partial solutions to generate a solution for the original instance.

Analysis

Recurrence relation:

- $\boldsymbol{T(n)}$: max. number of steps necessary for solving an instance of size n
- $\boldsymbol{T(n)} \leq \begin{cases} \boldsymbol{a} & \textbf{if } \boldsymbol{n \leq c} \\ \boldsymbol{T(n_1) + \cdots + T(n_k)} & \textbf{if } \boldsymbol{n > c} \\ \boldsymbol{+ \text{cost for divide and combine}} & \end{cases}$

Special case: $\boldsymbol{k = 2, n_1 = n_2 = {}^{n}/_{2}}$

$n_1 = \lfloor n/2 \rfloor$, $n_2 = \lceil n/2 \rceil$

- cost for divide and combine: $\mathrm{DC}(n)$
- $T(1) = a$
- $T(n) = 2T(n/2) + \mathrm{DC}(n)$

Mergesort

$T(n) = 2T(n/2) + O(n)$

$T(n) = O(n \log n)$

Comparing Orders

- Many web systems maintain user preferences / rankings on things like books, movies, restaurants, …

- Collaborative filtering:
  - Predict user taste by comparing rankings of different users.
  - If the system finds users with similar tastes, it can make recommendations (e.g., Amazon)

- Core issue: Compare two rankings
  - Intuitively, two rankings (of movies) are more similar, the more pairs are ordered in the same way
  - Label the first user's movies from 1 to n according to ranking
  - Order labels according to second user's ranking
  - How far is this from the ascending order (of the first user)?

Number of Inversions

Formal problem:

- **Given**: array $A = [a_1, a_2, a_3, \dots, a_n]$ of distinct elements

- **Objective**: Compute number of inversions I

$$I \coloneqq \left|\{0 \le i < j \le n \mid a_i > a_j)\}\right|$$

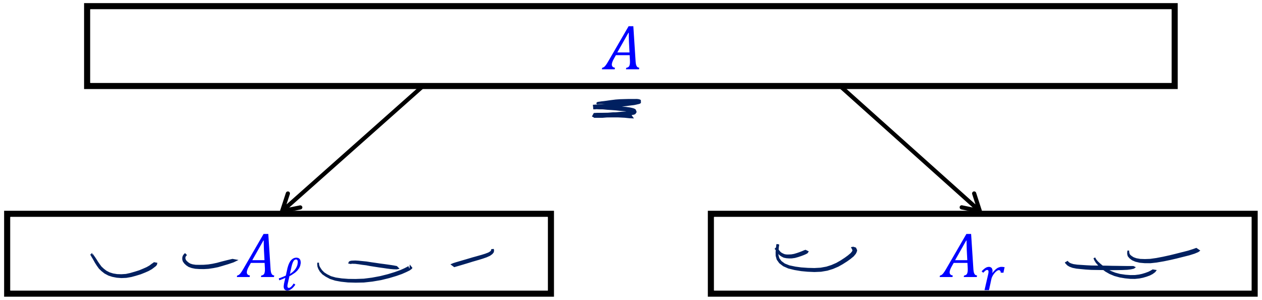
- **Example**: $A = [\ 4\ , 1\ , 5\ , 2\ , 7\ , 10\ , 6\ ]$

5 inversions

- **Naïve solution**: check all the pairs

running time : $O(n^2)$

Divide and conquer



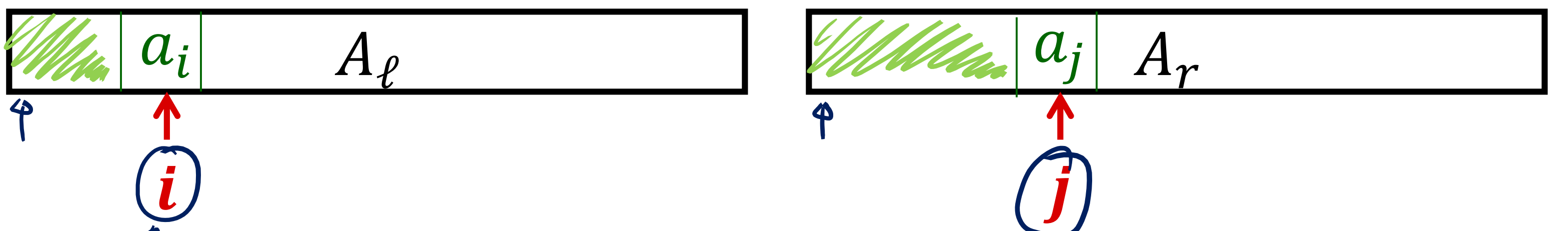
1. Divide array into 2 equal parts A_ℓ and A_r
2. Recursively compute #inversions in A_ℓ and A_r
3. Combine: add #pairs $a_i \in A_\ell, a_j \in A_r$ such that $a_i > a_j$

Combine:

count # $a_i \in A_\ell$, $a_j \in A_r$: $a_j < a_i$

Combine Step

Assume A_ℓ and A_r are sorted



Idea:

- Maintain pointers i and j to go through the sorted parts
- While going through the sorted parts, we merge the two parts into one sorted part (like in MergeSort)

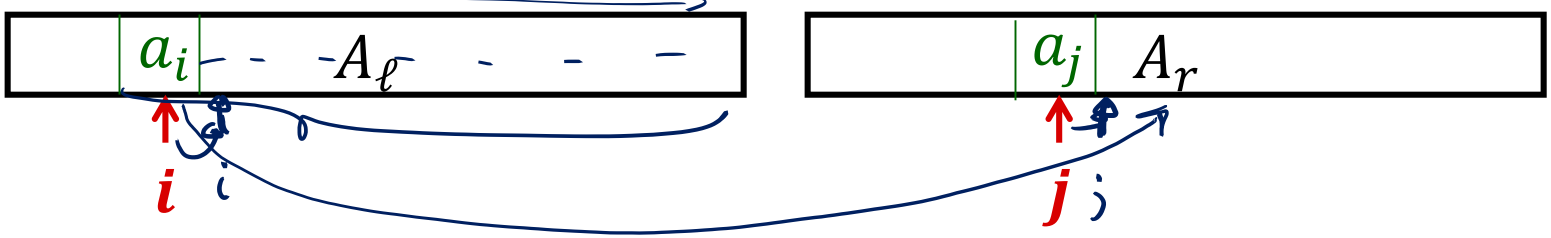
  and we count the number of inversions between the parts

Invariant:

- At each point in time, all inversions involving some element left of i (in A_ℓ) or left of j (in A_r) are counted
  - and all others still have to be counted...

Combine Step

Assume A_ℓ and A_r are sorted



- Pointers i and j, initially pointing to first elements of A_ℓ and A_r
- If $a_i < a_j$:
  - a_i is smallest among the remaining elements
  - No inversion of a_i and one of the remaining elements
  - Do not change count
- If $a_i > a_j$:
  - a_j is smallest among the remaining elements
  - a_j is smaller than all remaining elements in A_ℓ
  - Add number of remaining elements in A_ℓ to count
- Increment point, pointing to smaller element

Combine Step

- Need sub-sequences in sorted order
- Then, combine step is like merging in merge sort

- **Idea**: Solve sorting and #inversions at the same time!
  1. Partition A into two equal parts A_ℓ and A_r
  2. Recursively compute #inversions and sort A_ℓ and A_r

     and recursively sort A_ℓ & A_r

  3. Merge A_ℓ and A_r to sorted sequence, at the same time, compute number of inversions between elements a_i in A_ℓ and a_j in A_r

     combine costs $O(n)$

Combine Step: Example

- Assume A_ℓ and A_r are sorted

3	5	8	13	14	18	24	25	30

i

6	7	9	19	21	23	28	32	33

j

3	5	6	7	8	9	13	14	18									

$0 + 7 + 7 + 6$

Number of Inversion: Analysis

Recurrence relation:

$$T(n) \leq \underbrace{2 \cdot T(n/2)}_{\text{cost of recursion}} + \underbrace{cn}_{\text{cost of combine}}, \qquad T(1) \leq a$$

Guess the solution by repeated substitution:

$$\begin{aligned}
T(n) &\leq 2T(n/2) + cn \\
&\leq 2\left(2T(n/4) + c\tfrac{n}{2}\right) + cn = 4T(n/4) + 2cn \\
&\leq 4\left(2T(n/8) + c\tfrac{n}{4}\right) + 2cn = 8T(n/8) + 3cn \\
&\;\;\vdots \\
&\leq 2^k T(n/2^k) + kcn \qquad \text{set } 2^k = n \\
&\underset{(n=2^k)}{\leq} n\overset{a}{\cancel{T(1)}} + cn\log_2 n = n(a + c\log_2 n)
\end{aligned}$$

Number of Inversions: Analysis

Recurrence relation:

$$T(n) \leq 2 \cdot T(n/2) + cn, \qquad T(1) \leq a$$

Verify by induction: Guess: $T(n) \leq n(a + c\log n)$

Base: $n = 1$: $T(1) \leq a$ ✓

Step: guess is true for instances of size $< n$

$$T(n) \leq 2T(n/2) + cn$$

$$\overset{(IH)}{\leq} 2 \cdot \frac{n}{2}\left(a + c\log\frac{n}{2}\right) + cn$$

$$= n\left(a + c + c\underbrace{\log\frac{n}{2}}_{\log n - 1}\right) = n(a + c\log n)$$

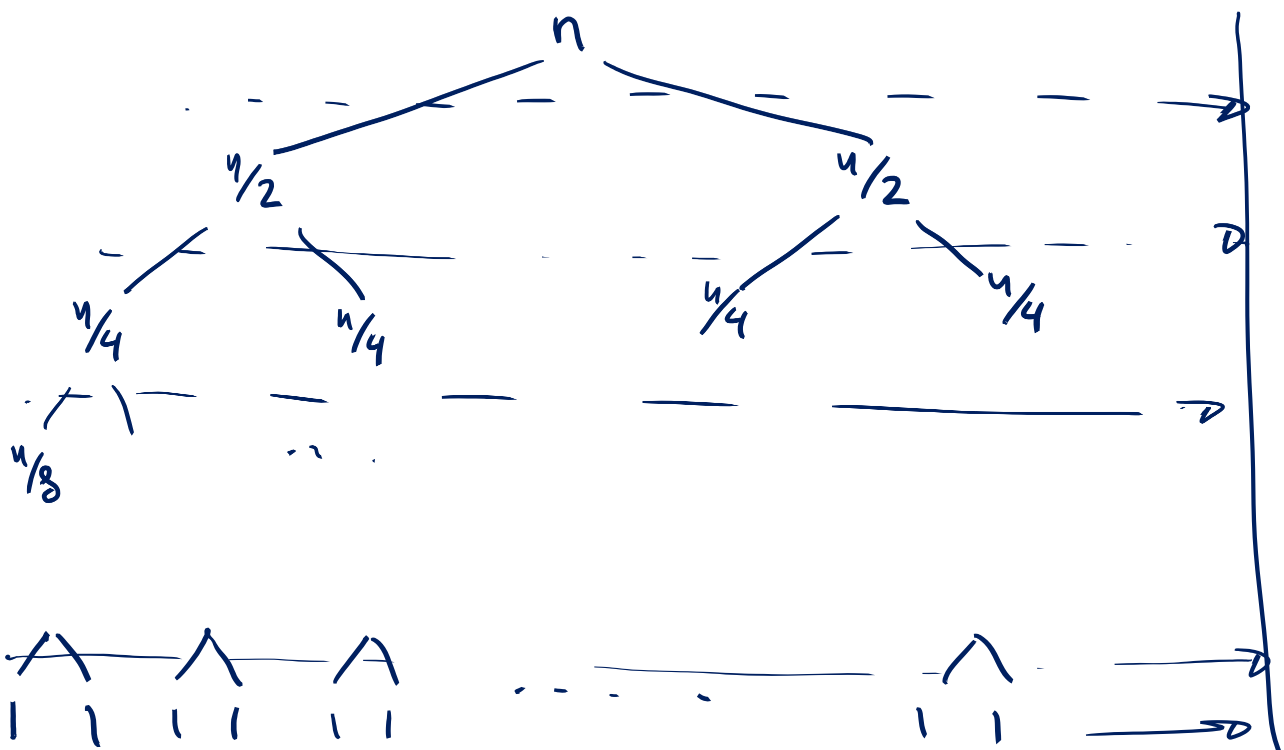
$\square$

Number of Inversions: Analysis

Recurrence relation:

$$T(n) \leq 2 \cdot T(n/2) + cn, \qquad T(1) \leq a$$

Guess the solution by drawing the recursion tree:



divide & combine

cn

$2 \cdot c\frac{n}{2} = cn$

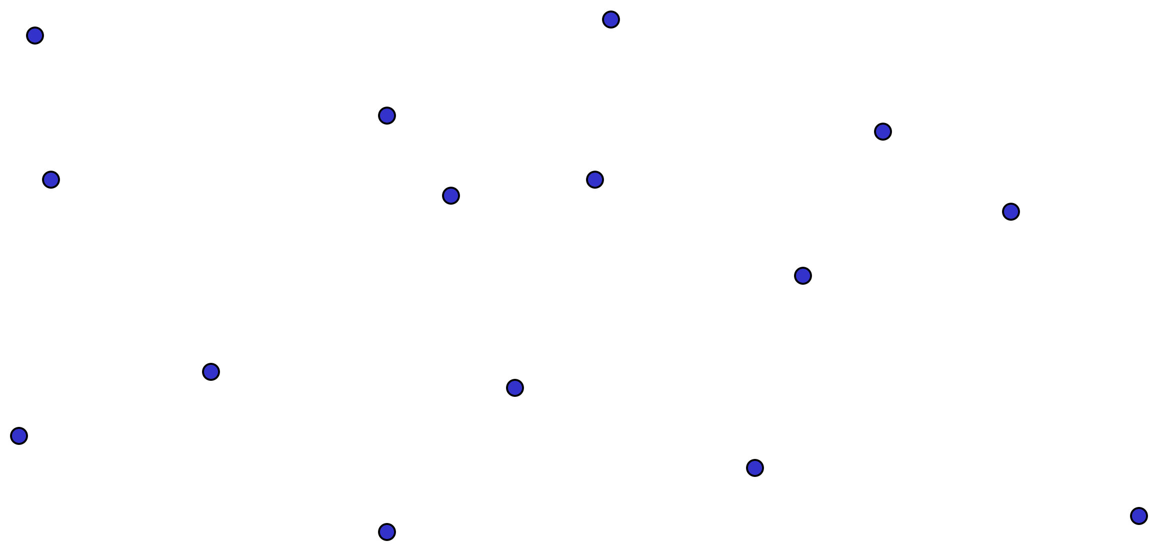
$4 \cdot c\frac{n}{4} = cn$

$\log_2 n$ levels

$n \cdot T(1) = n \cdot a$

Geometric divide-and-conquer

Closest Pair Problem: Given a set S of n points, find a pair of points with the smallest distance.



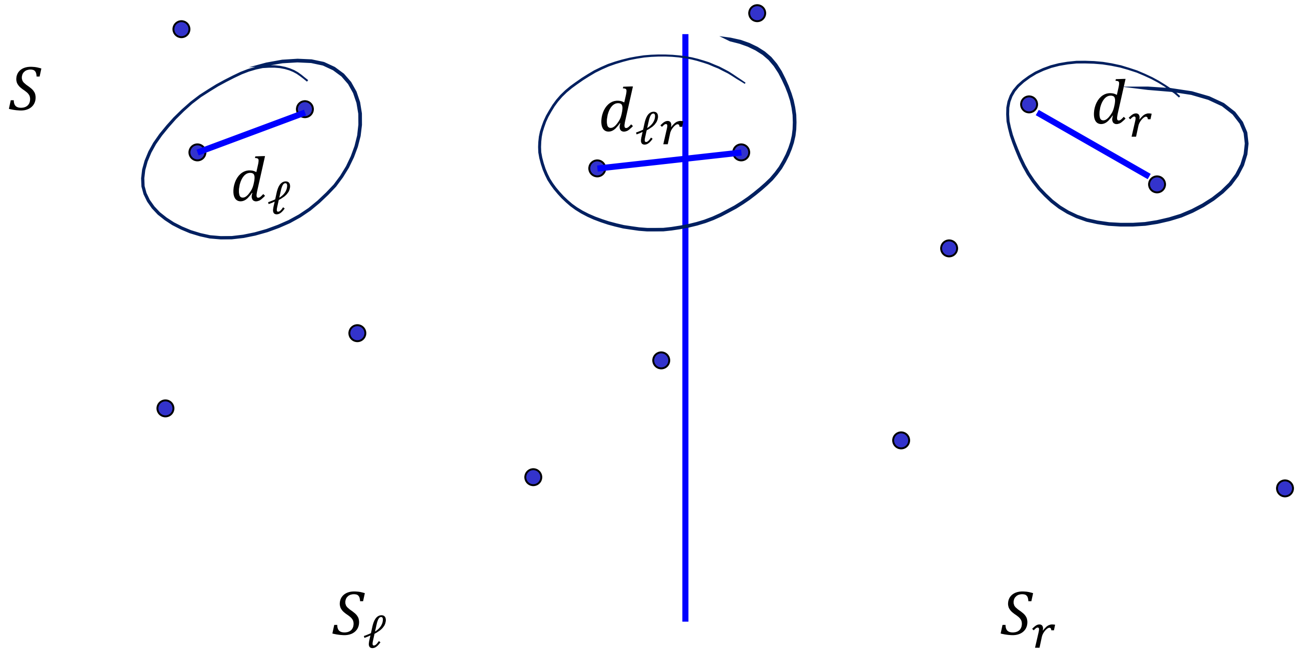
Naïve solution: check all pairs

$O(n^2)$

Divide-and-conquer solution

0. sort by x-coordinate : $O(n \log n)$

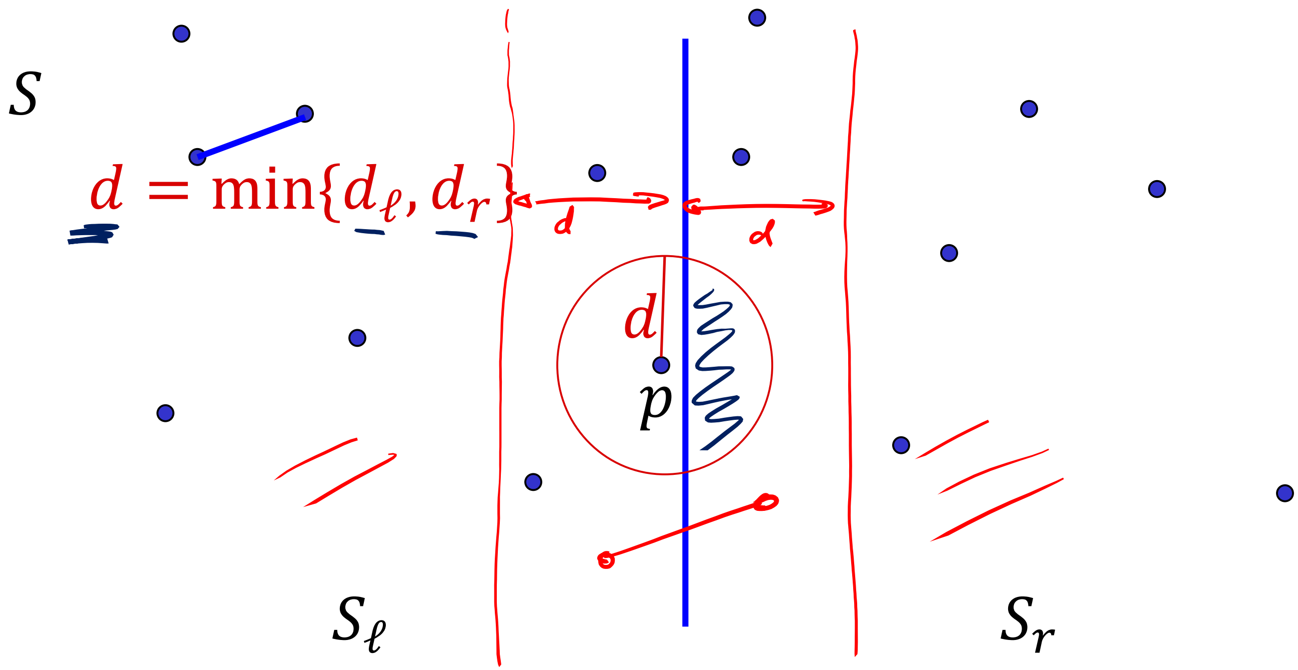
1. **Divide:** Divide S into two equal sized sets S_ℓ und S_r.
2. **Conquer:** $d_\ell = \text{mindist}(S_\ell)$ $d_r = \text{mindist}(S_r)$
3. **Combine:** $d_{\ell r} = \min\{d(p_\ell, p_r) \mid p_\ell \in S_\ell, p_r \in S_r\}$
   return $\min\{d_\ell, d_r, d_{\ell r}\}$



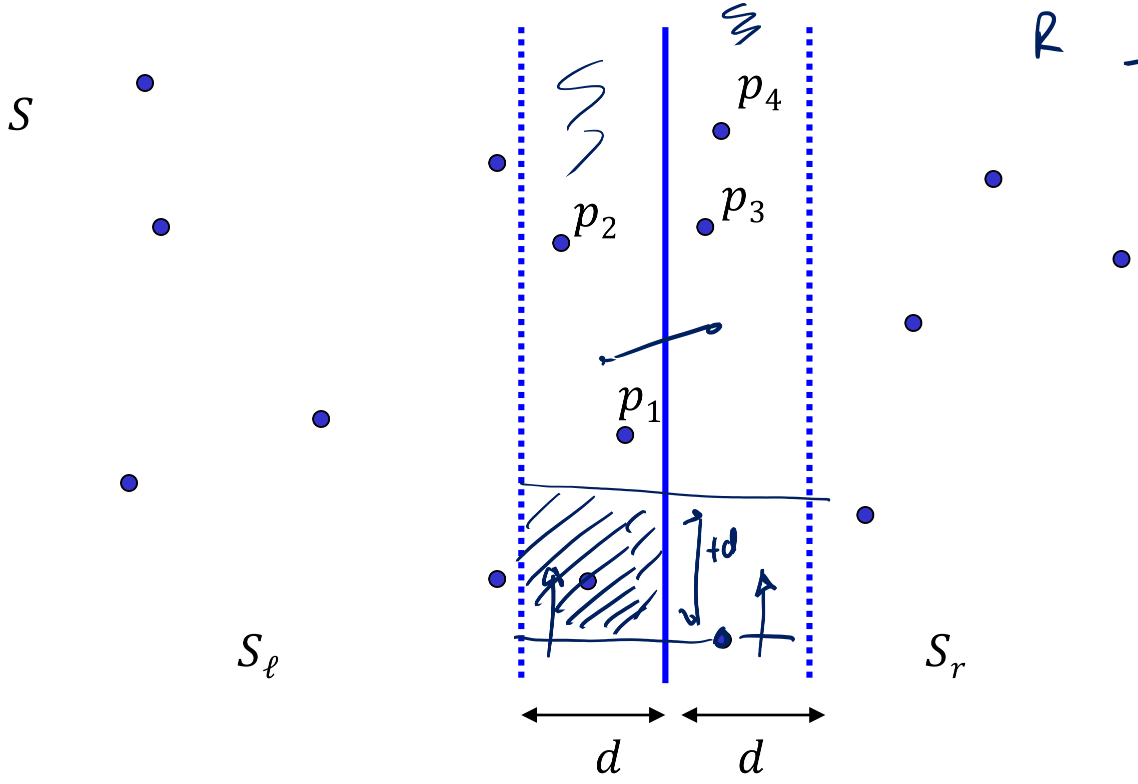
Divide-and-conquer solution

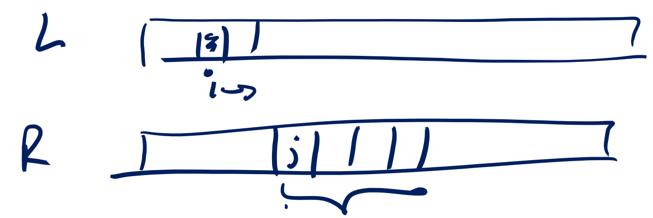
1. **Divide:** Divide S into two equal sized sets S_ℓ und S_r.
2. **Conquer:** $d_\ell = \mathrm{mindist}(S_\ell)$ $\quad d_r = \mathrm{mindist}(S_r)$
3. **Combine:** $d_{\ell r} = \min\{d(p_\ell, p_r) \mid p_\ell \in S_\ell, p_r \in S_r\}$
   return $\min\{d_\ell, d_r, d_{\ell r}\}$ ← Obs: only need $d_{\ell r}$ if $d_{\ell r} \le d$

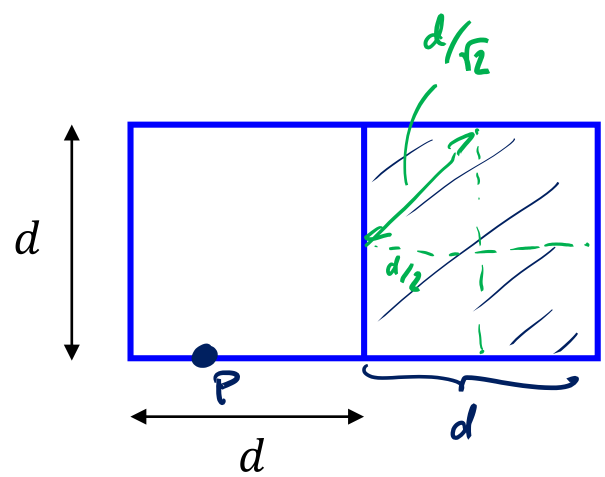
Computation of $d_{\ell r}$:



Combine step







$$d = \min\{d_\ell, dr\}$$

Combine step

1. Consider only points within distance $\leq d$ of the bisection line, in the order of increasing y-coordinates.

2. For each point p consider all points q on the other side which are within y-distance less than d

3. There are at most 4 such points.

Implementation

- Initially sort the points in S in order of increasing x-coordinates
- While computing closest pair, also sort S according to y-coord.
  - Partition S into S_ℓ and S_r, solve and sort sub-problems recursively
  - Merge to get sorted S according to y-coordinates
    combine + merge in $O(n)$ time
  - Center points: points within x-distance $d = \min\{d_\ell, d_r\}$ of center
  - Go through center points in S in order of incr. y-coordinates

Running Time

Recurrence relation:

$$T(n) = 2 \cdot T(n/2) + c \cdot n, \qquad T(1) = a$$

Solution:

- Same as for computing number of number of inversions, merge sort (and many others…)

$$T(n) = O(n \cdot \log n)$$