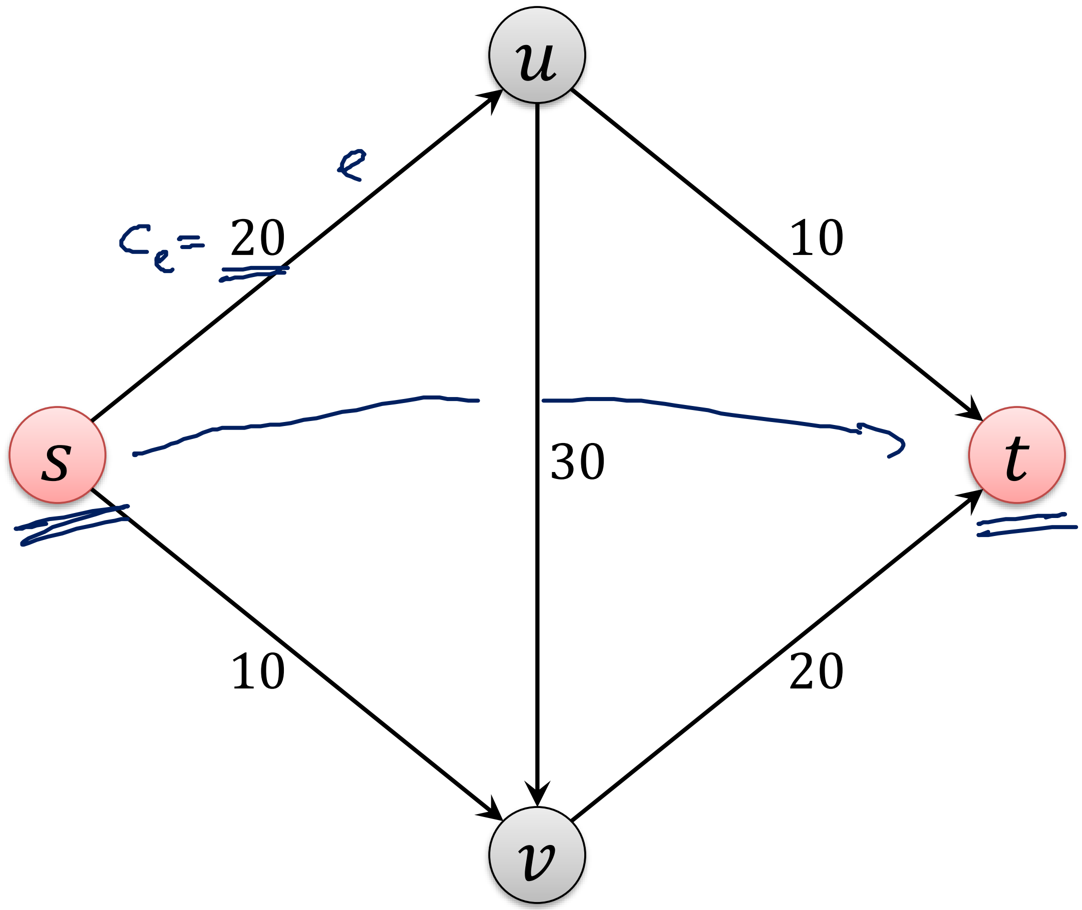
# Chapter 6
# Graph Algorithms

## Algorithm Theory
## WS 2015/16


## Fabian Kuhn

# Example: Flow Network



# Notation


**We define:**

$$f^{\text{in}}(v) := \sum_{e \text{ into } v} f(e), \qquad f^{\text{out}}(v) := \sum_{e \text{ out of } v} f(e)$$

$$0 \leq f(e) \leq c_e$$

**For a set $S \subseteq V$:**

$$f^{\text{in}}(S) := \sum_{e \text{ into } S} f(e), \qquad f^{\text{out}}(S) := \sum_{e \text{ out of } S} f(e)$$

**Flow conservation:** $\forall v \in V \setminus \{s,t\}$: $f^{\text{in}}(v) = f^{\text{out}}(v)$

**Flow value:** $|f| = f^{\text{out}}(s) = f^{\text{in}}(t)$

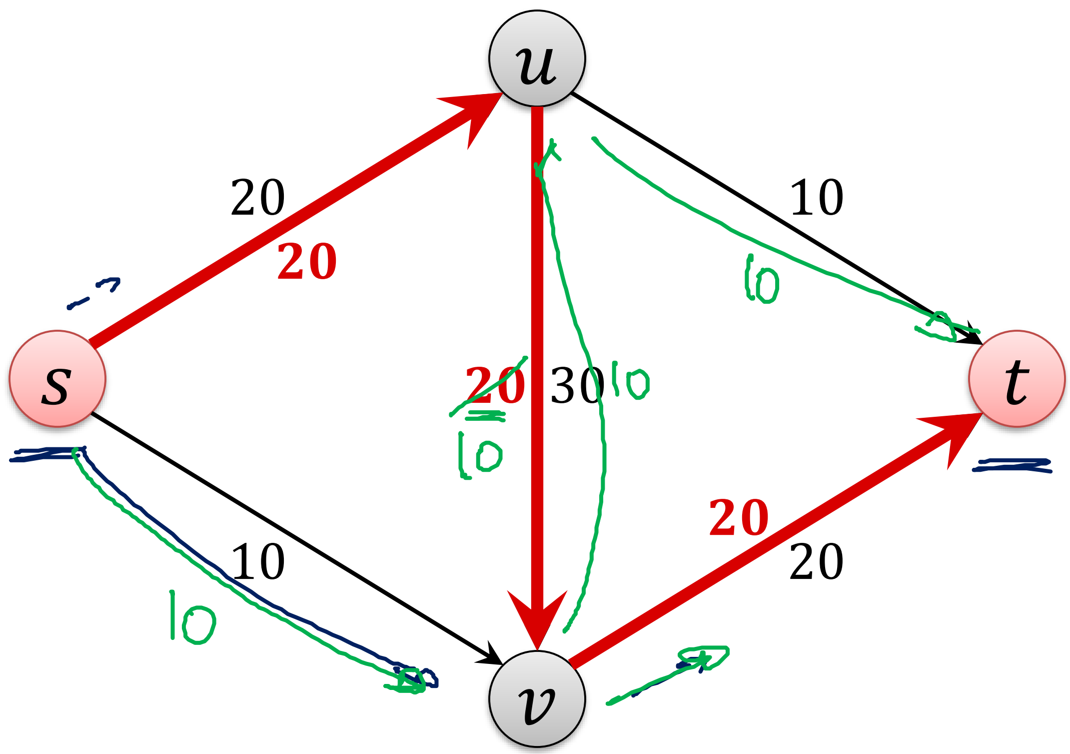
**For simplicity:** Assume that all capacities are positive integers

# Maximum Flow: Greedy?

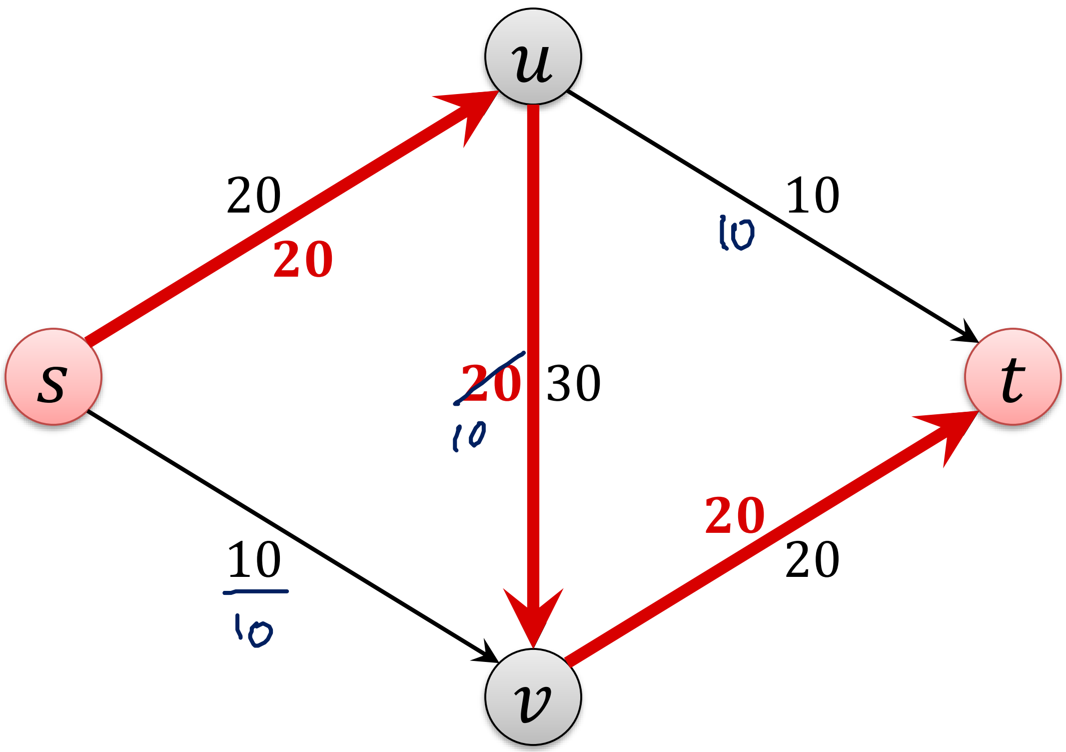
Does greedy work?

A natural greedy algorithm:

- As long as possible, find an $s$-$t$-path with free capacity and add as much flow as possible to the path



# Improving the Greedy Solution




- Try to push 10 units of flow on edge $(s, v)$
- Too much incoming flow at $v$: reduce flow on edge $(u, v)$
- Add that flow on edge $(u, t)$

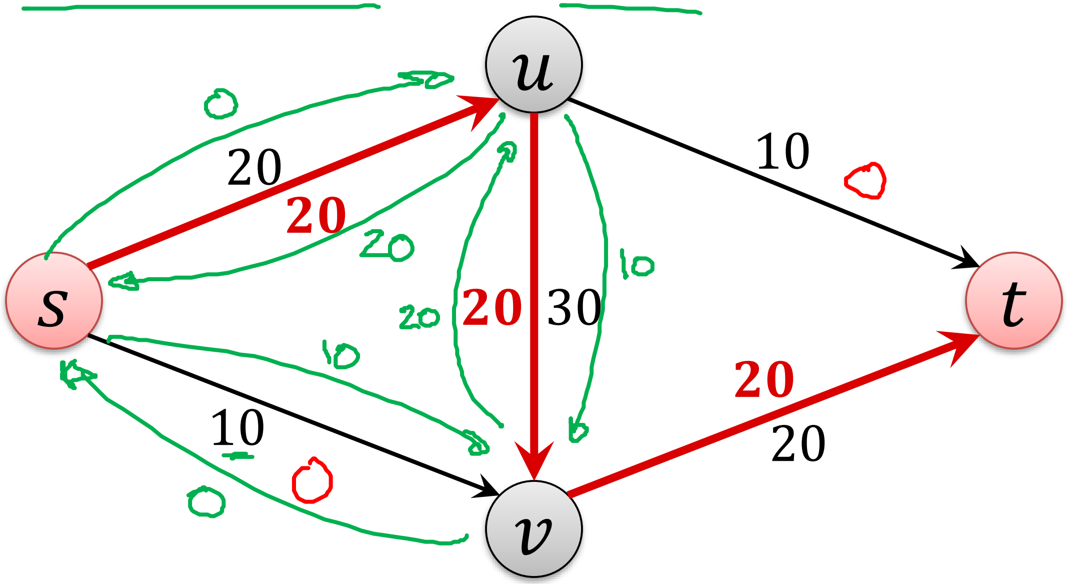
# Residual Graph

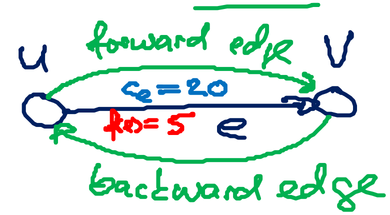
Given a flow network $G = (V, E)$ with capacities $c_e$ (for $e \in E$)

For a flow $f$ on $G$, define directed graph $G_f = (V_f, E_f)$ as follows:

residual graph

- Node set $V_f = V$
- For each edge $e = (u, v)$ in $E$, there are two edges in $E_f$:
  - forward edge $e = (u, v)$ with residual capacity $c_e - f(e)$
  - backward edge $e' = (v, u)$ with residual capacity $f(e)$

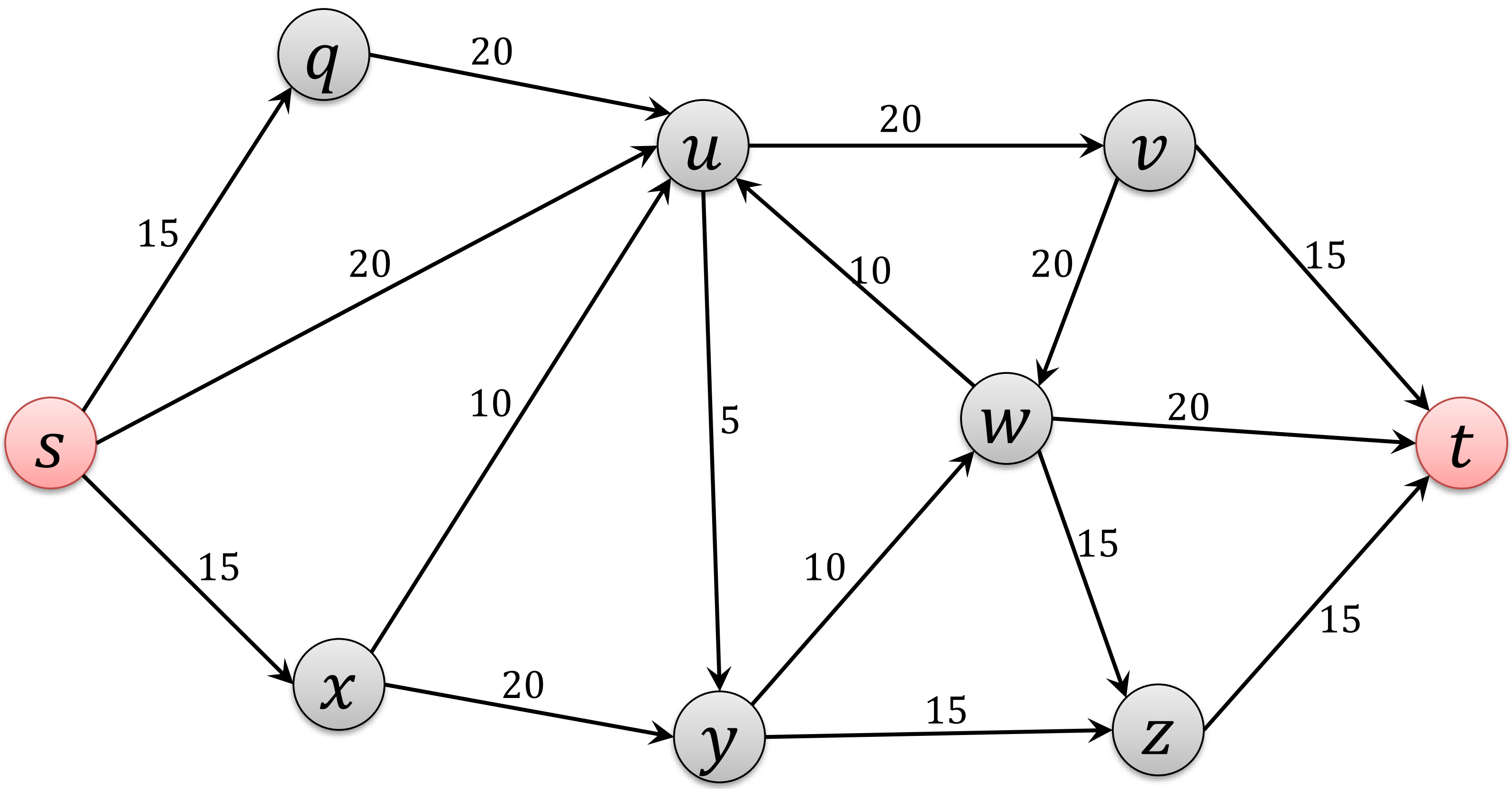



forw. edge: cap = 15

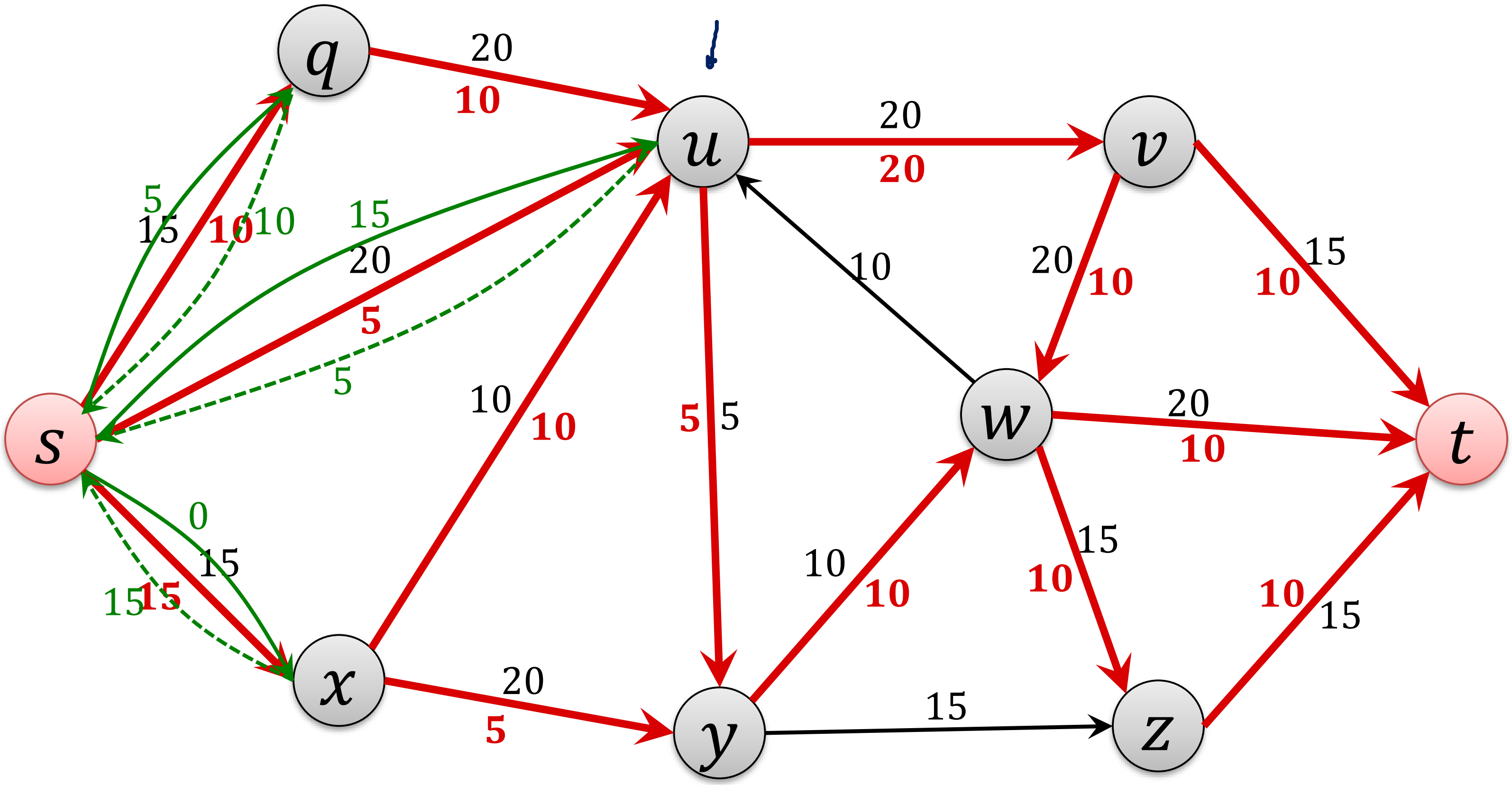
backw. edge: cap = 5

# Residual Graph: Example



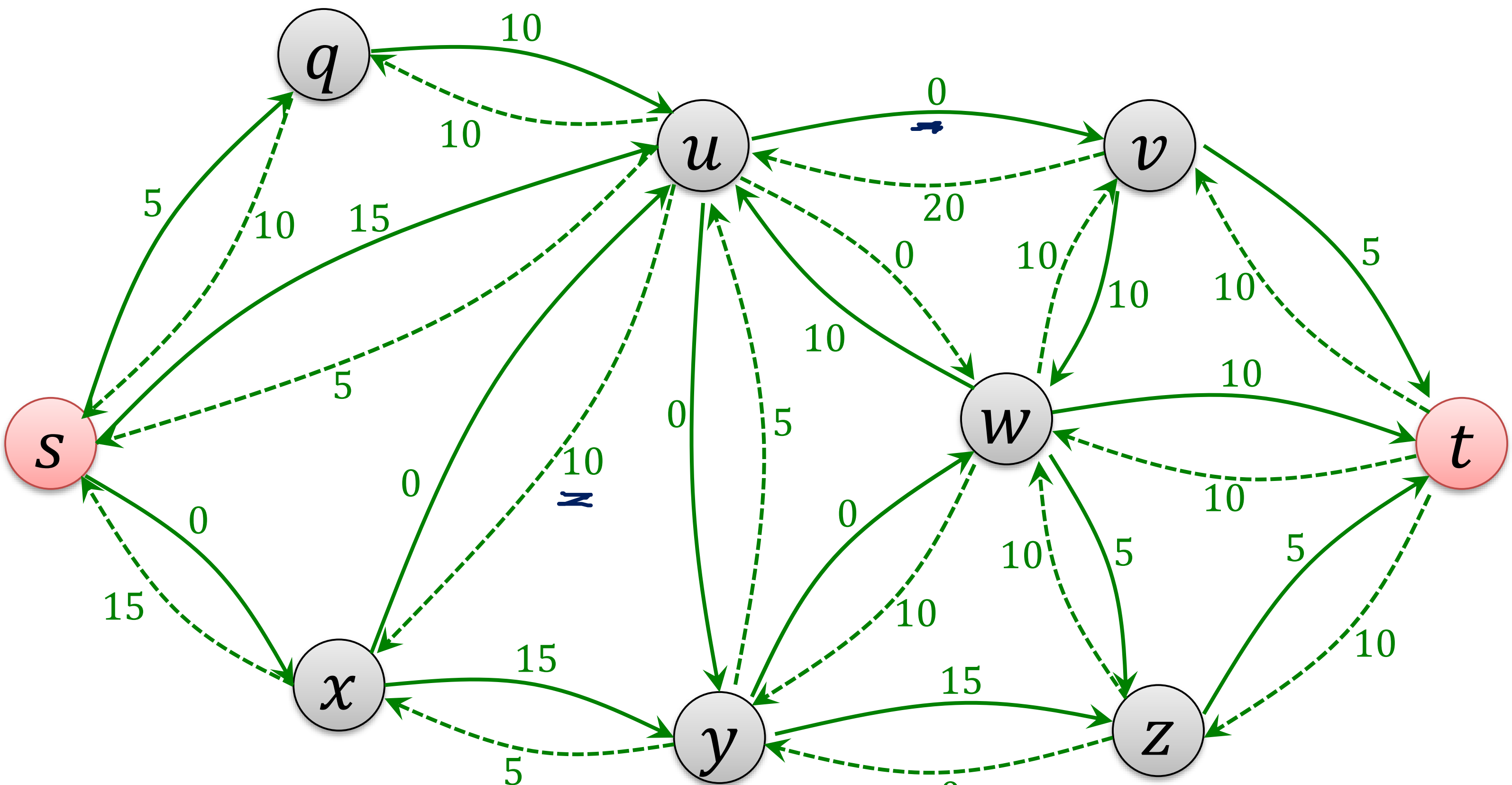
# Residual Graph: Example

## Flow $f$



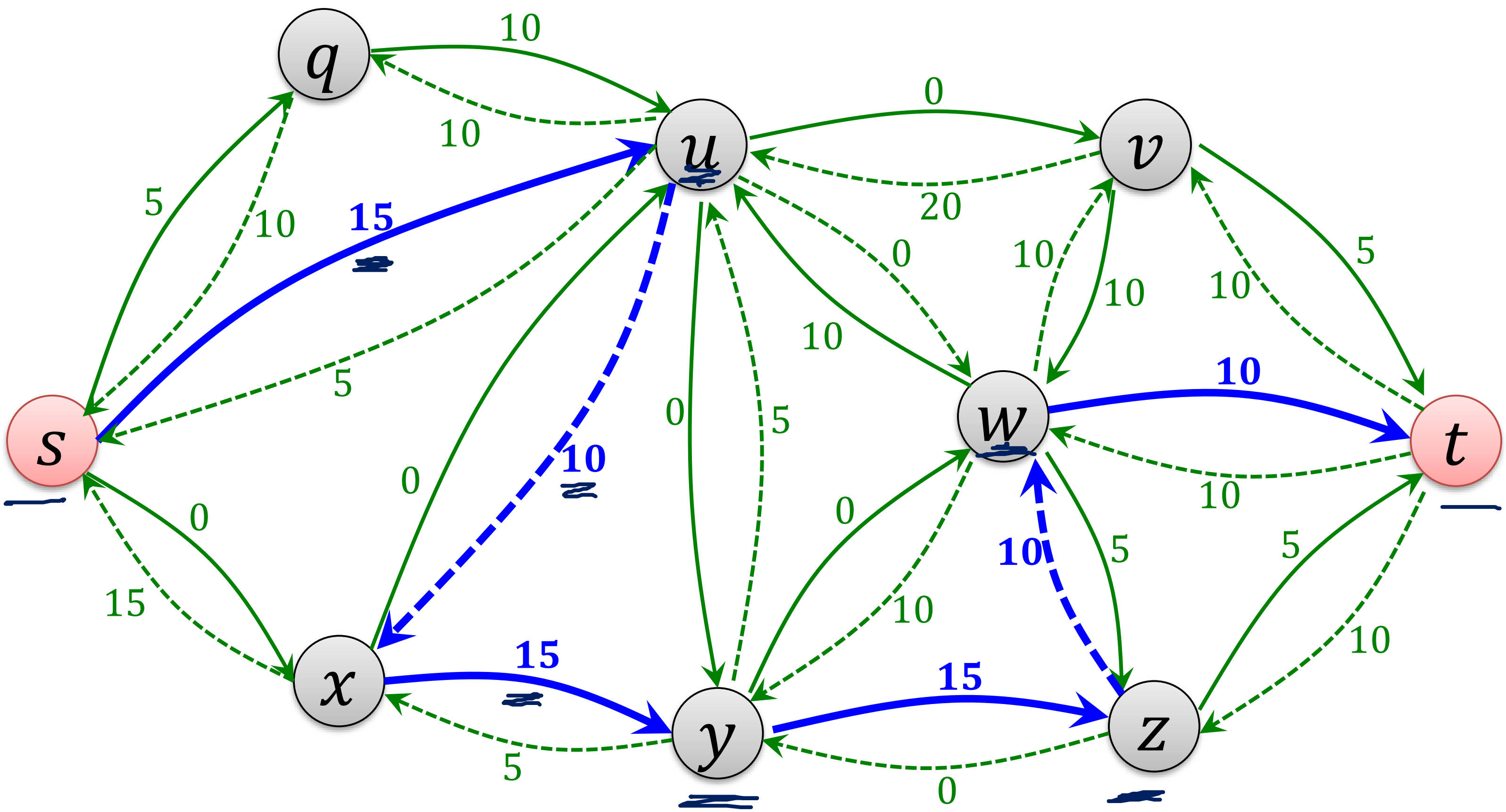
# Residual Graph: Example

## Residual Graph $G_f$



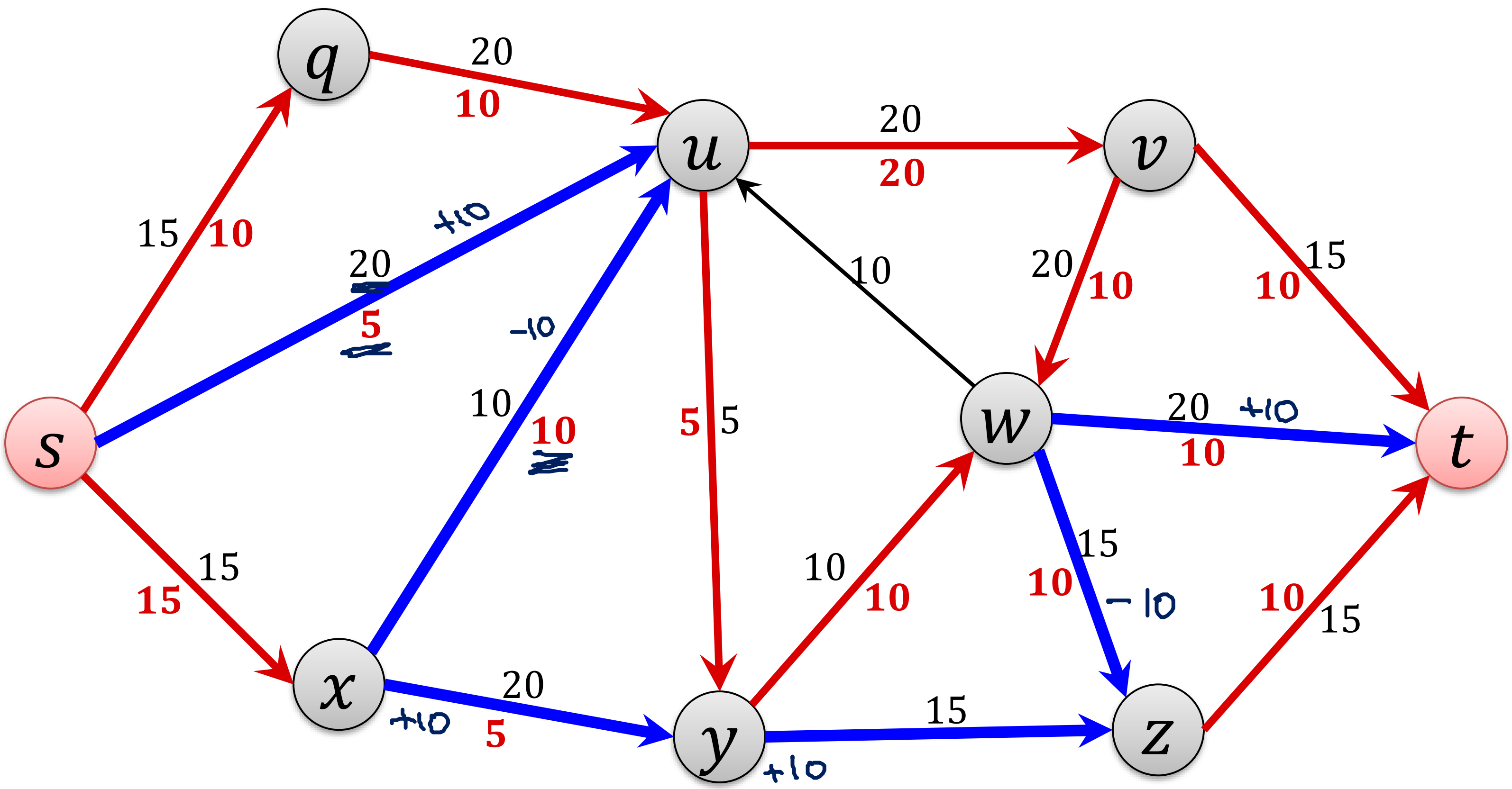
# Augmenting Path

## Residual Graph $G_f$



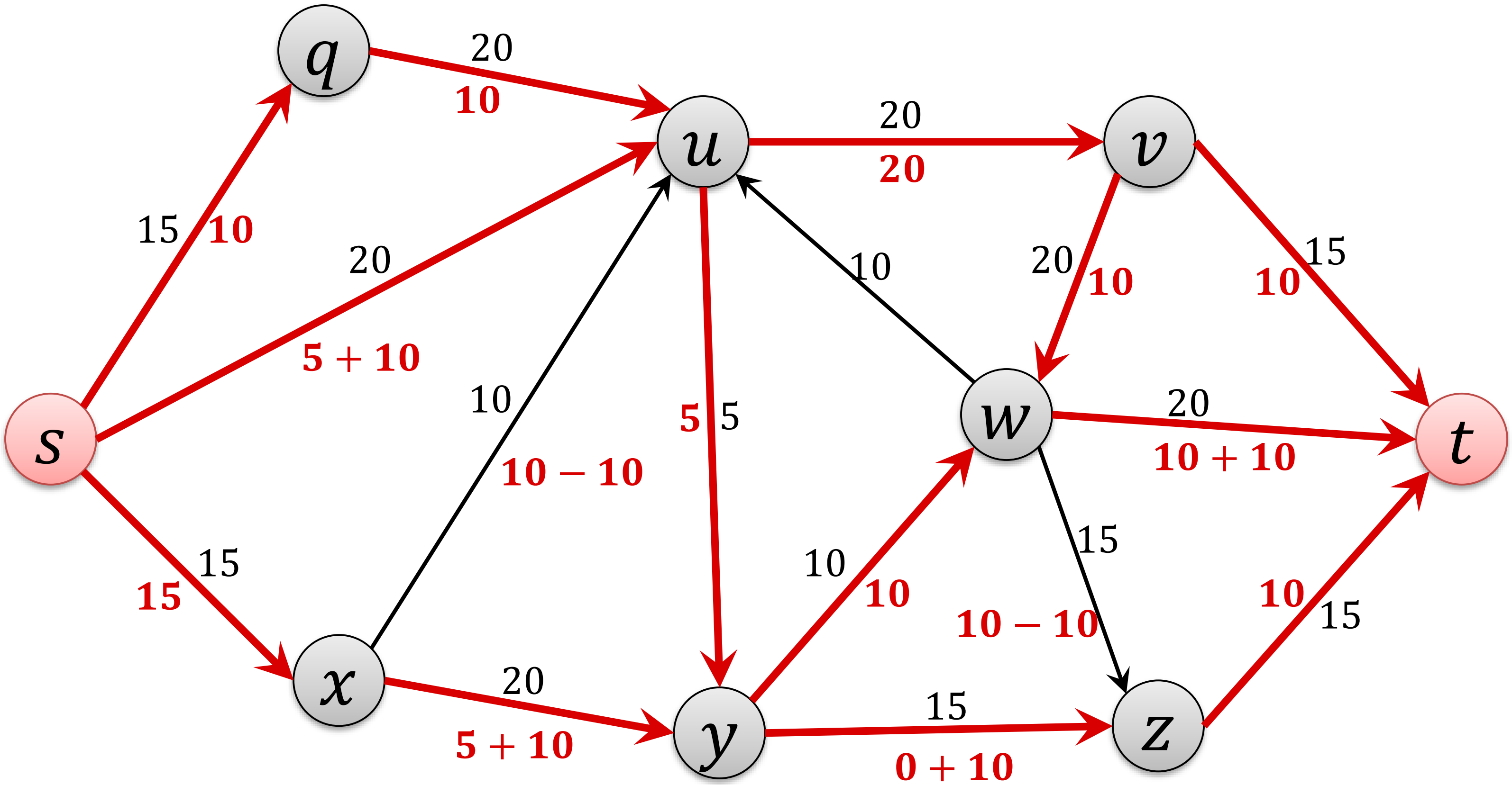
# Augmenting Path

## Augmenting Path



# Augmenting Path

## New Flow



# Augmenting Path

**Definition:**

An augmenting path $P$ is a (simple) $s$-$t$-path on the residual graph $G_f$ on which each edge has residual capacity $> 0$.

$\mathrm{bottleneck}(P, f)$: minimum residual capacity on any edge of the augmenting path $P$

**Augment flow $f$ to get flow $f'$:**

- For every forward edge $(u, v)$ on $P$:

$$f'\big((u,v)\big) := f\big((u,v)\big) + \mathbf{bottleneck}(P, f)$$

- For every backward edge $(u, v)$ on $P$:

$$f'\big((v,u)\big) := f\big((v,u)\big) - \mathbf{bottleneck}(P, f)$$

# Augmented Flow

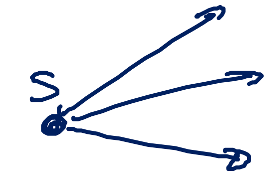
**Lemma:** Given a flow $f$ and an augmenting path $P$, the resulting augmented flow $f'$ is legal and its value is

$$|f'| = |f| + \underbrace{\textbf{bottleneck}(P, f)}_{b}.$$

✓

**Proof:**

$$|f'| = |f| + \text{bottleneck}(P, f)$$




$f'$ is legal:

$\forall e \in E \qquad 0 \le f'(e) \le c_e \qquad \text{(I)}$

$\forall v \in V \setminus \{s,t\} \qquad f'^{\,in}(v) = f'^{\,out}(v) \qquad \text{(II)}$

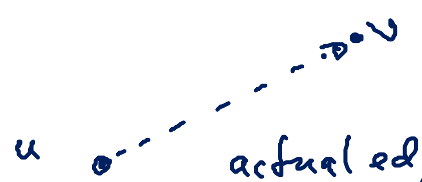
(I): ✓

fwd. edge


$$f'(e) = f(e) + \underbrace{\text{bottleneck}(P,f)}_{b \le c_e - f(e)}$$

$$0 \le f'(e) \le c_e$$

bwd. edge




actual edge $e = (v,u)$

$$f'(e) = f(e) - b$$

$$\underline{\underline{b \le f(e)}}$$

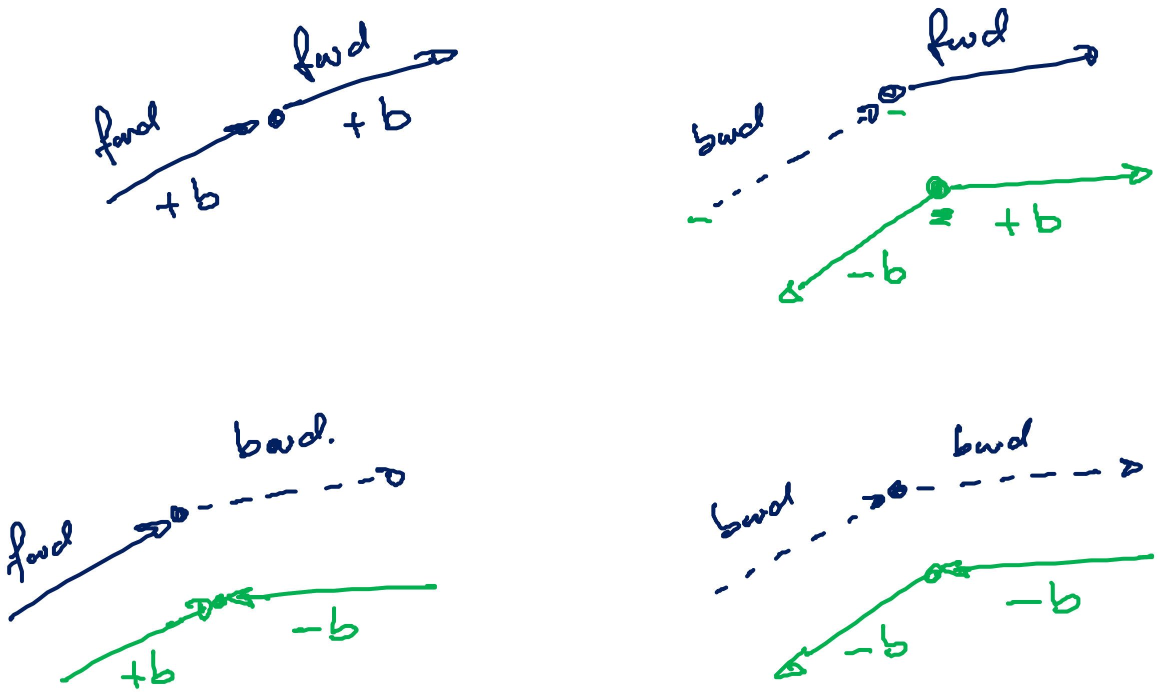
# Augmented Flow

**Lemma:** Given a flow $f$ and an augmenting path $P$, the resulting augmented flow $f'$ is legal and its value is

$$|f'| = |f| + \mathbf{bottleneck}(P, f).$$

**Proof:**

flow cons.



# Ford-Fulkerson Algorithm

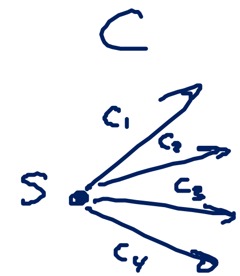
- Improve flow using an augmenting path as long as possible:

1. Initially, $f(e) = 0$ for all edges $e \in E$, $G_f = G$
2. **while** there is an augmenting $s$-$t$-path $P$ in $G_f$ **do**
3. Let $P$ be an augmenting $s$-$t$-path in $G_f$;
4. $f' \coloneqq \text{augment}(f, P)$; bottleneck$(P, f) > 0$
5. update $f$ to be $f'$;
6. update the residual graph $G_{f'}$
7. **end**;

# Ford-Fulkerson Running Time

**Theorem:** If all edge capacities are integers, the Ford-Fulkerson algorithm terminates after at most $C$ iterations, where

$$C = \sum_{e \text{ out of } s} c_e .$$




**Proof:**

At all times, for each $e \in E$ : $f(e)$ is an integer

initially: $f(e) = 0$

in one iter. : augm. path $P$ : residual cap. are integers

bottleneck$(P, f) > 0$ (it also is an int)

$\hookrightarrow \geq 1$

$\rightarrow$ new flow values are integers

$\rightarrow$ new flow value larger by $\geq 1$

every flow value $\leq C$

# Ford-Fulkerson Running Time

**Theorem:** If all edge capacities are integers, the Ford-Fulkerson algorithm can be implemented to run in $O(mC)$ time.

#edges

**Proof:**

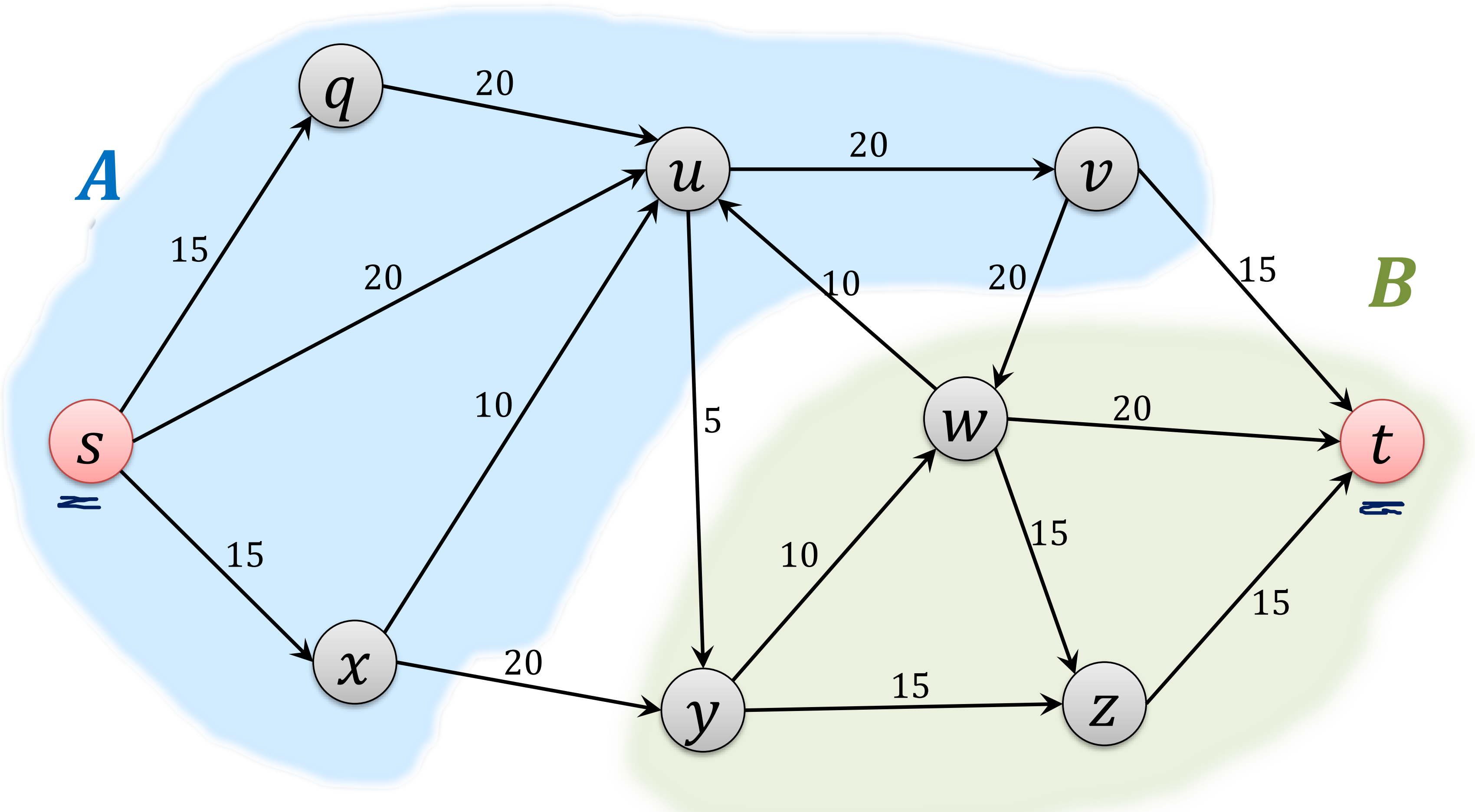
Claim: one iter. can be computed in $O(m)$ time

1. compute/update residual graph $G_f$ → first iter: $O(m)$ → later iter: $O(n)$
2. find augm. path / conclude there is no augm. path
   → s-t path in $G_f$ with res. cap. $> 0$
   → graph traversal (DFS/BFS): $O(m)$ time
3. update flow values: $O(n)$ time

# $s$-$t$ Cuts

**Definition:**

An $s$-$t$ cut is a partition $(A, B)$ of the vertex set such that $s \in A$ and $t \in B$

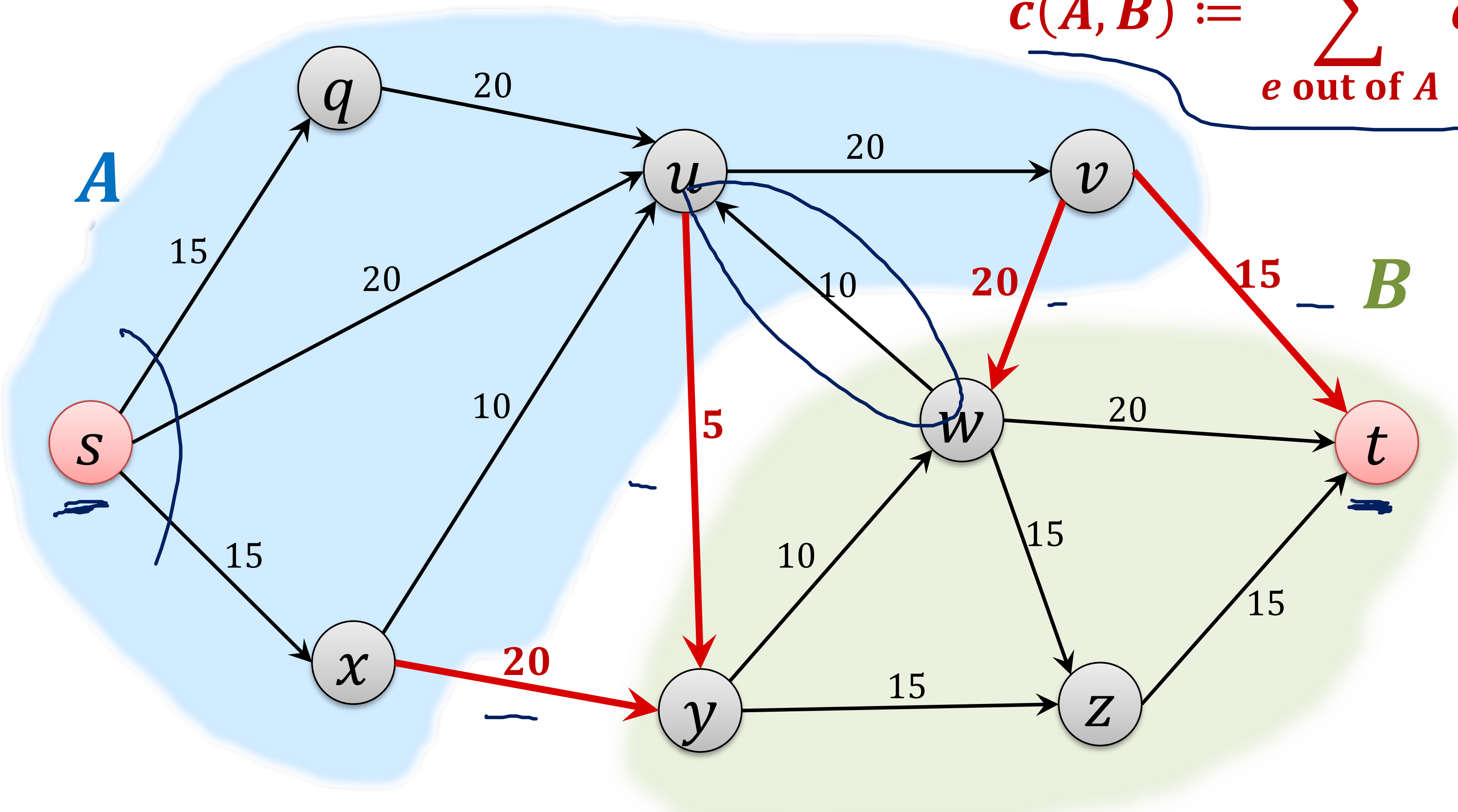


# Cut Capacity

**Definition:**

The capacity $c(A, B)$ of an $s$-$t$-cut $(A, B)$ is defined as

$$c(A, B) := \sum_{e \text{ out of } A} c_e.$$



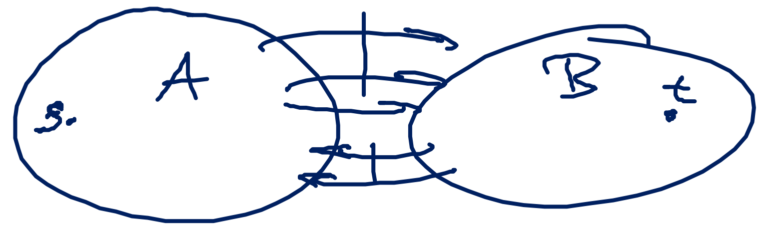
# Cuts and Flow Value

**Lemma:** Let $f$ be any $s$-$t$ flow, and $(A, B)$ any $s$-$t$ cut. Then,

$$|f| = f^{\text{out}}(A) - f^{\text{in}}(A).$$

**Proof:**

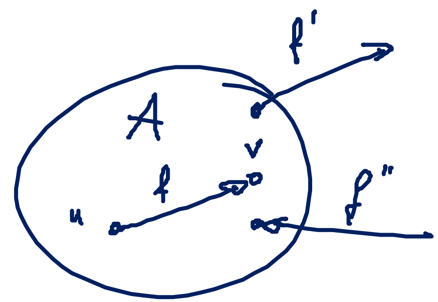
$$|f| = f^{\text{out}}(s) \quad \left(= f^{\text{in}}(t)\right)$$




$$|f| = f^{\text{out}}(s) - \underbrace{f^{\text{in}}(s)}_{=0}$$

$$= \sum_{v \in A} \underbrace{\left(f^{\text{out}}(v) - f^{\text{in}}(v)\right)}_{=0 \text{ except for } v = s} \qquad \left(\forall v \in A \setminus \{s\} : f^{\text{out}}(v) = f^{\text{in}}(v)\right)$$

$$= f^{\text{out}}(A) - f^{\text{in}}(A)$$



# Cuts and Flow Value

**Lemma:** Let $f$ be any $s$-$t$ flow, and $(A, B)$ any $s$-$t$ cut. Then,

$$|\boldsymbol{f}| = \boldsymbol{f}^{\text{out}}(\boldsymbol{A}) - \boldsymbol{f}^{\text{in}}(\boldsymbol{A}).$$

**Lemma:** Let $f$ be any $s$-$t$ flow, and $(A, B)$ any $s$-$t$ cut. Then,

$$|\boldsymbol{f}| = \underline{\boldsymbol{f}^{\text{in}}}(\boldsymbol{B}) - \underline{\boldsymbol{f}^{\text{out}}}(\boldsymbol{B}).$$

**Proof:**

symmetric

or observe

$$f^{out}(A) = f^{in}(B)$$

$$f^{in}(A) = f^{out}(B)$$

# Upper Bound on Flow Value

**Lemma:**

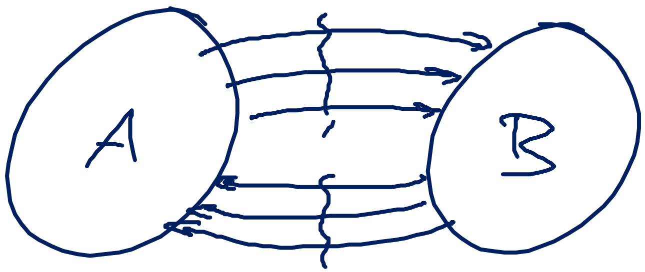
Let $f$ be any $s$-$t$ flow and $(A, B)$ any $s$-$t$ cut. Then $|f| \leq c(A, B)$.

**Proof:**

$$|f| = f^{out}(A) - f^{in}(A)$$

$$\leq c(A,B) - 0$$

$f^{out}(A) \leq c(A,B)$




$f^{in}(A) \geq 0$

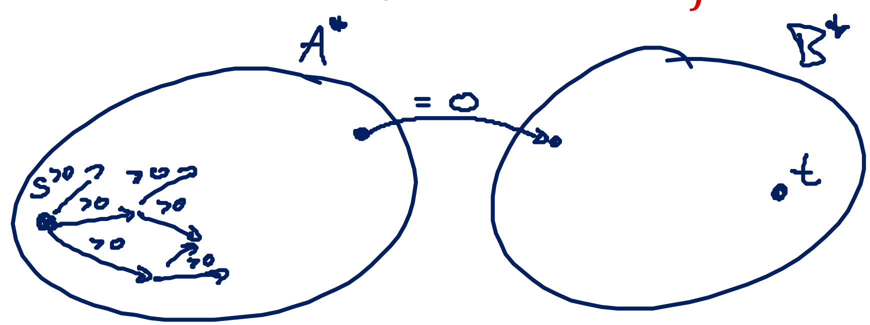
# Ford-Fulkerson Gives Optimal Solution

**Lemma:** If $f$ is an $s$-$t$ flow such that there is no augmenting path in $G_f$, then there is an $s$-$t$ cut $(A^*, B^*)$ in $G$ for which

$$|f| = c(A^*, B^*).$$

**Proof:**

- Define $A^*$: set of nodes that can be reached from $s$ on a path with positive residual capacities in $G_f$:



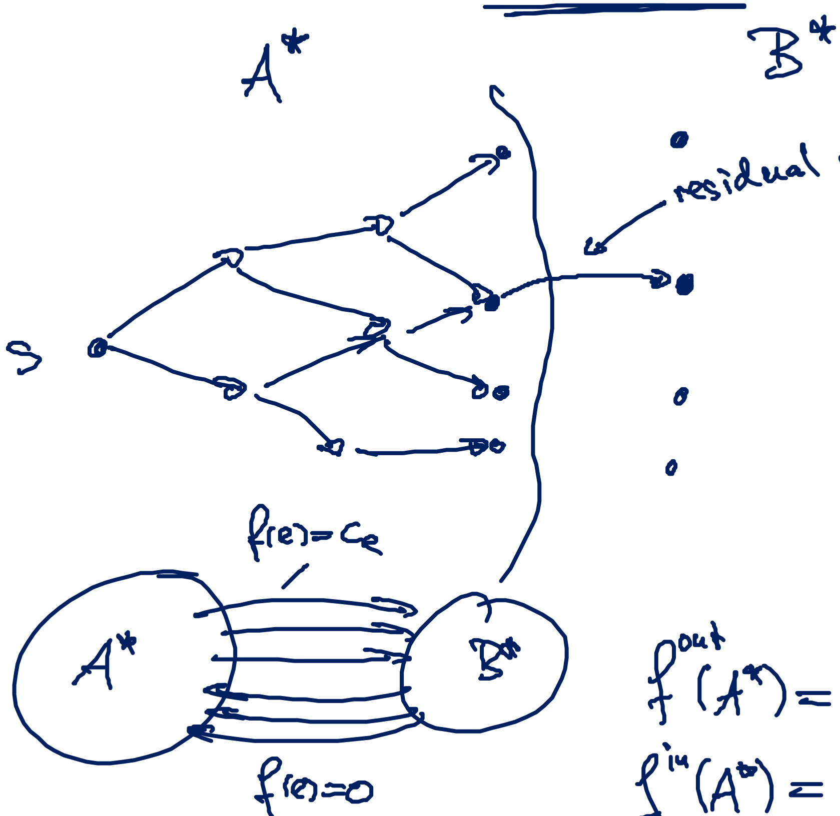

- For $B^* = V \setminus A^*$, $(A^*, B^*)$ is an $s$-$t$ cut
  - By definition $s \in A^*$ and $t \notin A^*$ ← because there is no augm. path

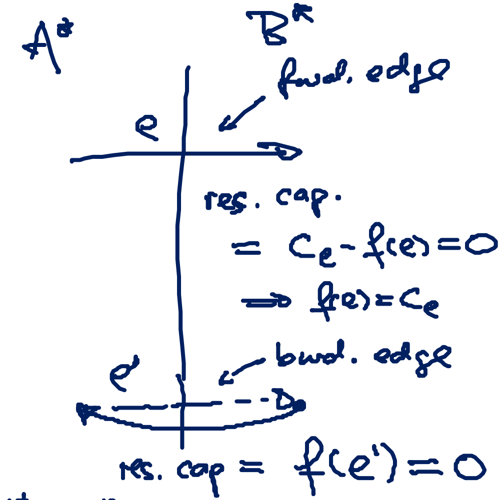
# Ford-Fulkerson Gives Optimal Solution

**Lemma:** If $f$ is an $s$-$t$ flow such that there is no augmenting path in $G_f$, then there is an $s$-$t$ cut $(A^*, B^*)$ in $G$ for which

$$\underline{|f| = c(A^*, B^*)}.$$

**Proof:**




$f^{out}(A^*) = cap(A^*, B^*)$

$f^{in}(A^*) = 0$

# Ford-Fulkerson Gives Optimal Solution

**Lemma:** If $f$ is an $s$-$t$ flow such that there is no augmenting path in $G_f$, then there is an $s$-$t$ cut $(A^*, B^*)$ in $G$ for which

$$|\boldsymbol{f}| = \boldsymbol{c}(\boldsymbol{A}^*, \boldsymbol{B}^*).$$

**Proof:**

# Ford-Fulkerson Gives Optimal Solution

**Theorem:** The flow returned by the Ford-Fulkerson algorithm is a maximum flow.

**Proof:**

$f^*$: flow returned by FF

$\hookrightarrow$ cut $(A^*, B^*)$

s.t. $|f^*| = c(A^*, B^*)$

for every flow $f$: $|f| \leq c(A^*, B^*)$

# Min-Cut Algorithm

Ford-Fulkerson also gives a min-cut algorithm:

**Theorem:** Given a flow $f$ of maximum value, we can compute an $s$-$t$ cut of minimum capacity in $O(m)$ time.

**Proof:**

$f$ maximum $\longrightarrow$ augm. path

can find cut $(A^*, B^*)$ s.t. $|f| = c(A^*, B^*)$

↳ as before: DFS / BFS on res. graph (from s)

↳ all nodes reachable from s

↳ $A^*$ (set of nodes reachable from s)

$\longrightarrow$ $A^*$ can be computed in $O(m)$ time

$(A^*, B^*)$ is an s-t cut with min. capacity

because: for every other s-t cut $(A, B)$, we have $|f| \le c(A, B)$

$$|f| = c(A^*, B^*) \le c(A, B)$$

# Max-Flow Min-Cut Theorem

**Theorem:** **(Max-Flow Min-Cut Theorem)**

In every flow network, the maximum value of an $s$-$t$ flow is equal to the minimum capacity of an $s$-$t$ cut.

**Proof:**

FF gives max flow $f^*$ and min $s$-$t$ cut $(A^*, B^*)$

s.t. $|f^*| = c(A^*, B^*)$

# Integer Capacities

**Theorem:** **(Integer-Valued Flows)**

If all capacities in the flow network are integers, then there is a maximum flow $f$ for which the flow $f(e)$ of every edge $e$ is an integer.

**Proof:**

FF gives an integer flow

# Non-Integer Capacities

What if capacities are not integers?

$O(m C)$

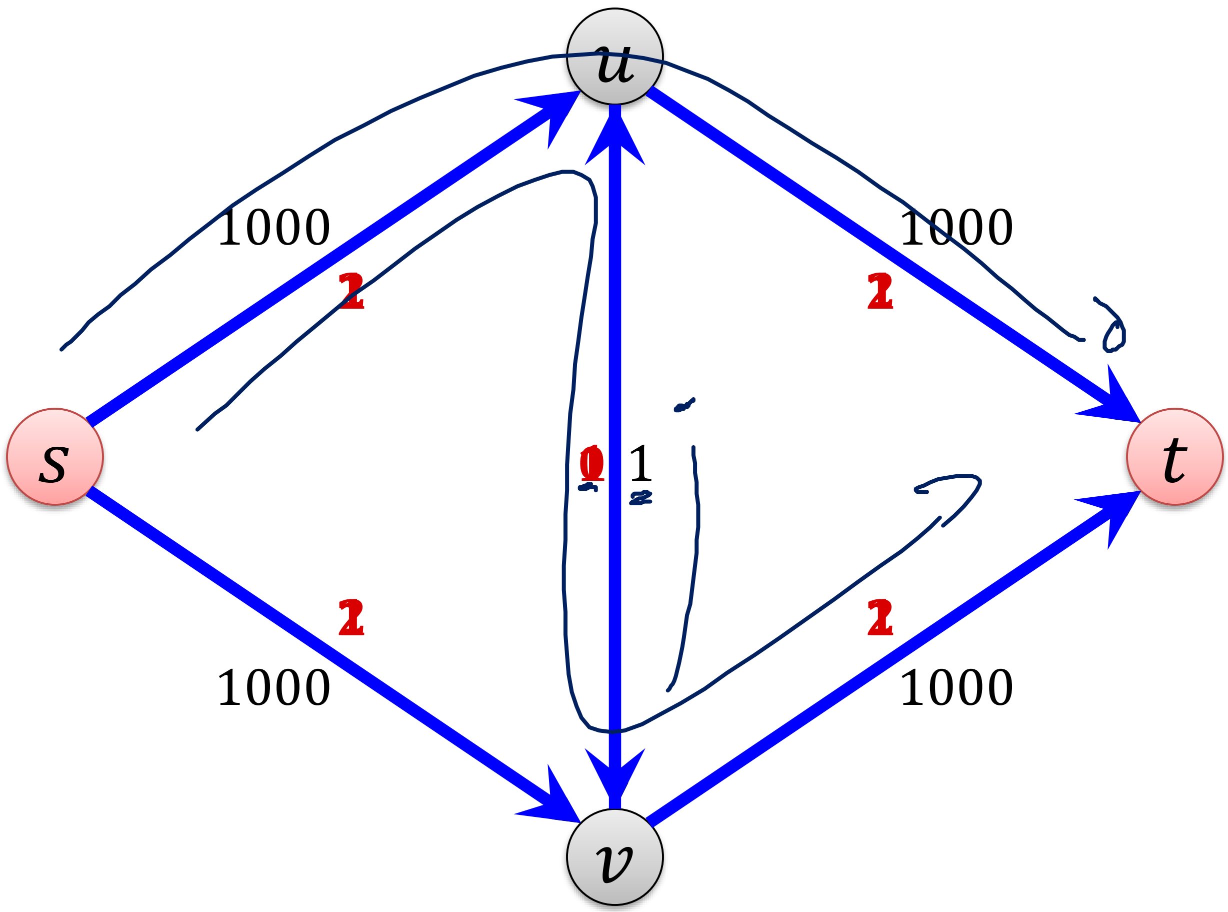
- rational capacities: $c_e \in \mathbb{Q}$
  - can be turned into integers by multiplying them with large enough integer
  - algorithm still works correctly

- real (non-rational) capacities:
  - not clear whether the algorithm always terminates

- even for integer capacities, time can linearly depend on the value of the maximum flow

$C \longrightarrow \log C$

# Slow Execution




- Number of iterations: 2000 (value of max. flow)

# Improved Algorithm

**Idea:** Find the best augmenting path in each step

- best: path $P$ with maximum $\text{bottleneck}(P, f)$

- Best path might be rather expensive to find
  → find almost best path

$\Delta$ always a power of 2

- **Scaling parameter Δ:**
  (initially, $\Delta =$ "$\max c_e$ rounded down to next power of 2")

- As long as there is an augmenting path that improves the flow by at least $\Delta$, augment using such a path

- If there is no such path: $\Delta := \Delta/_2$

# Scaling Parameter Analysis

**Lemma:** If all capacities are integers, number of different scaling parameters used is $\leq 1 + \lfloor \log_2 C \rfloor$.

$C_{max}$: max. edge cap.

for all $e$: $c_e \leq C$

initially

$\Delta = 2^{\lfloor \log_2 C_{max} \rfloor}$

largest $\Delta$

# of scaling param: $\leq \lfloor \log_2 C_{max} \rfloor + 1$

- **Δ-scaling phase:** Time during which scaling parameter is Δ

running time:

$\underbrace{\#\text{phases}}_{O(\log C)} \cdot \underbrace{\#\text{iter. per phase}}_{?} \cdot O(m)$

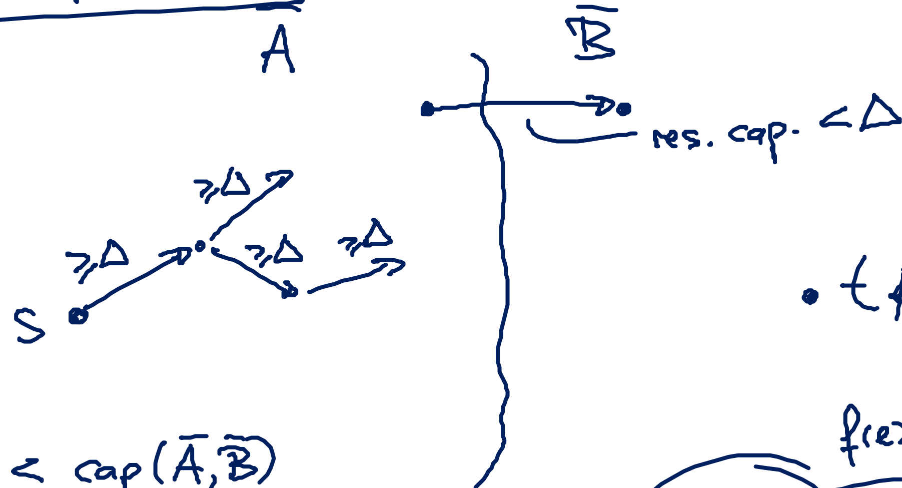
find one path

# Length of a Scaling Phase

**Lemma:** If $f$ is the flow at the end of the $\Delta$-scaling phase, the maximum flow in the network has value at most $|f| + m\Delta$.

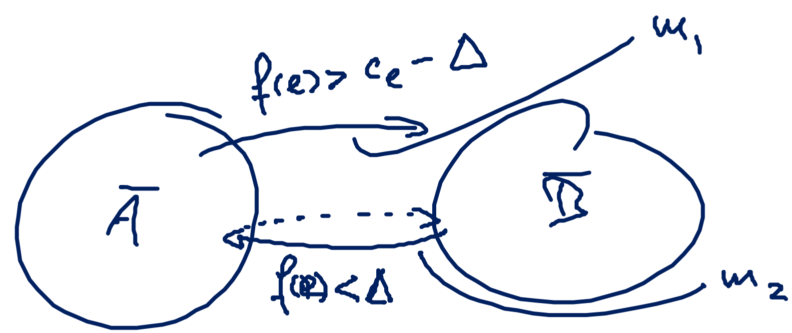
$|f^*| < |f| + m\Delta$

define s-t cut $(\bar{A}, \bar{B})$




$|f| + m\Delta < cap(\bar{A}, \bar{B})$

$|f^*| \leq cap(\bar{A}, \bar{B})$




$$|f| = f^{out}(\bar{A}) - f^{in}(\bar{A}) < cap(\bar{A}, \bar{B}) - m_1\Delta - m_2\Delta$$

$$\leq cap(\bar{A}, \bar{B}) - m\Delta$$

# Length of a Scaling Phase

**Lemma:** The number of augmentation in each scaling phase is at most $2m$.

at the beginning of the $\Delta$-scaling phase

↳ at the ~~...~~ end of the $2\Delta$-scaling phase

$\Longrightarrow |f^*| < |f| + \underline{\underline{2m\Delta}}$ (prev. lemma)

each augm path improves $|f|$ by $\geq \Delta$

$\square$

<u>running time:</u> $O(\log C) \cdot O(m) \cdot O(m) = O(m^2 \log C)$

# Running Time: Scaling Max Flow Alg.

**Theorem:** The number of augmentations of the algorithm with scaling parameter and integer capacities is at most $O(m \log C)$. The algorithm can be implemented in time $O(m^2 \log C)$.

# Strongly Polynomial Algorithm

- Time of regular Ford-Fulkerson algorithm with integer capacities:

$$O(mC)$$

- Time of algorithm with scaling parameter:

$$O(m^2 \log C)$$

- $O(\log C)$ is polynomial in the size of the input, but not in $n$
- Can we get an algorithm that runs in time polynomial in $n$?
- Always picking a shortest augmenting path leads to running time

$$O(m^2 n)$$

works if cap. are reals

# Other Algorithms

- There are many other algorithms to solve the maximum flow problem, for example:

- **Preflow-push algorithm:**
  - Maintains a preflow (∀ nodes: inflow ≥ outflow)
  - Alg. guarantees: As soon as we have a flow, it is optimal
  - Detailed discussion in 2012/13 ~~last year's~~ lecture
  - Running time of basic algorithm: $O(m \cdot n^2)$
  - Doing steps in the "right" order: $O(n^3)$

- **Current best known complexity: $\boldsymbol{O(m \cdot n)}$**
  - For graphs with $m \geq n^{1+\epsilon}$ (for every constant $\epsilon > 0$) [King,Rao,Tarjan 1992/1994]
  - For sparse graphs with $m \leq n^{16/15-\delta}$ [Orlin, 2013]

max. flow in undirected networks (1+ε)-approx. max flow. $O(m \cdot n^{o(1)})$ necessary

# Maximum Flow Applications

- Maximum flow has many applications

- Reducing a problem to a max flow problem can even be seen as an important algorithmic technique

- Examples:
  - related network flow problems
  - computation of small cuts
  - computation of matchings
  - computing disjoint paths
  - scheduling problems
  - assignment problems with some side constraints
  - …