Chapter 7
Randomization

Algorithm Theory
WS 2015/16

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Randomized Contraction Algorithm

Algorithm:

while there are >2 nodes **do**

contract a uniformly random edge

return cut induced by the last two remaining nodes

(cut defined by the original node sets represented by the last 2 nodes)

Theorem: The random contraction algorithm returns a **specific** minimum cut with probability at least $\frac{2}{n(n-1)}$.

Theorem: The random contraction algorithm can be implemented in time $O(n^2)$.

- There are $n-2$ contractions, each can be done in time $O(n)$.

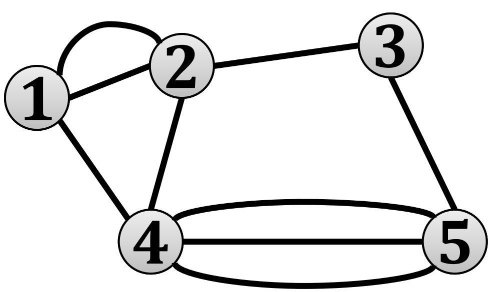
Implementing Edge Contractions

Edge Contraction:

- Given: multigraph with n nodes
  - assume that set of nodes is $\{1, \dots, n\}$
- Goal: contract edge $\{u, v\}$

Data Structure

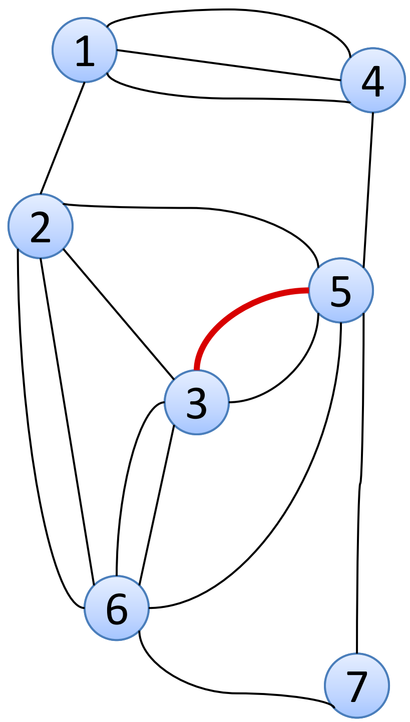
- We can use either adjacency lists or an adjacency matrix
- Entry in row i and column j: #edges between nodes i and j
- Example:



$$A = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 & 0 \end{pmatrix}$$

Contracting An Edge

Example: Contract one of the edges between 3 and 5

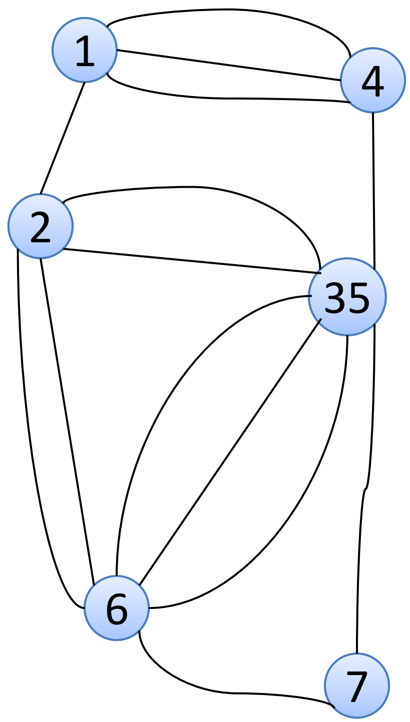


	1	2	3	4	5	6	7
1	0	1	0	3	0	0	0
2	1	0	1	0	1	2	0
3	0	1	0	0	2	2	0
4	3	0	0	0	1	0	0
5	0	1	2	1	0	1	1
6	0	2	2	0	1	0	1
7	0	0	0	0	1	1	0

{3,5}							

Contracting An Edge

Example: Contract one of the edges between 3 and 5

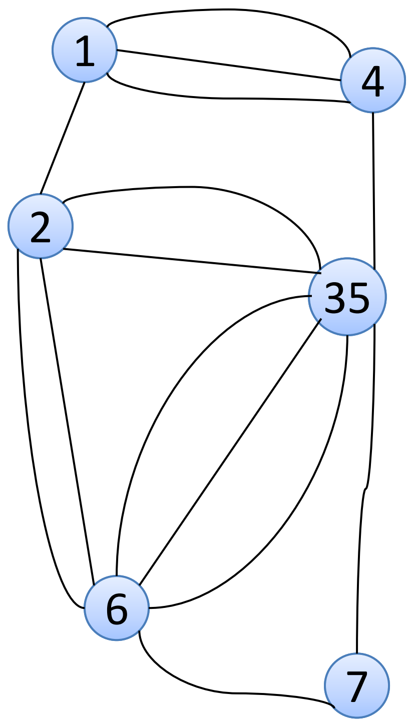


	1	2	3	4	5	6	7
1	0	1	0	3	0	0	0
2	1	0	1	0	1	2	0
3	0	1	0	0	2	2	0
4	3	0	0	0	1	0	0
5	0	1	2	1	0	1	1
6	0	2	2	0	1	0	1
7	0	0	0	0	1	1	0

{3,5}	0	2		1		3	1

Contracting An Edge

Example: Contract one of the edges between 3 and 5



	1	2	35	4		6	7
1	0	1	0	3		0	0
2	1	0	2	0		2	0
35	0	2	0	1		3	1
4	3	0	1	0		0	0
6	0	2	3	0		0	1
7	0	0	1	0		1	0

{3,5}	0	2		1		3	1

Contracting an Edge

Claim: Given the adjacency matrix of an n-node multigraph and an edge $\{u, v\}$, one can contract the edge $\{u, v\}$ in time $O(n)$.

- Row/column of combined node $\{u, v\}$ is sum of rows/columns of u and v

- Row/column of u can be replaced by new row/column of combined node $\{u, v\}$

- Swap row/column of v with last row/column in order to have the new $(n-1)$-node multigraph as a contiguous $(n-1) \times (n-1)$ submatrix

Finding a Random Edge

- We need to contract a uniformly random edge
- How to find a uniformly random edge in a multigraph?
  - Finding a random non-zero entry (with the right probability) in an adjacency matrix costs $O(n^2)$.

Idea for more efficient algorithm:

- First choose a random node u
  - with probability proportional to the degree (#edges) of u
- Pick a random edge of u
  - only need to look at one row → time $O(n)$

Choose a Random Node

Edge Sampling:

1. Choose a node $u \in V$ with probability
$$\frac{\deg(u)}{\sum_{v \in V} \deg(v)} = \frac{\deg(u)}{2m}$$
2. Choose a uniformly random edge of u

Choose a Random Node

- We need to choose a random node u with probability $\frac{\deg(u)}{2m}$
- Keep track of the number of edges m and maintain an array with the degrees of all the nodes
  - Can be done with essentially no extra cost when doing edge contractions

Choose a random node:

degsum = 0;

for all nodes $u \in V$:

    with probability $\frac{\deg(u)}{2m - \text{degsum}}$:

        pick node u; terminate

    else

        degsum += $\deg(u)$

Randomized Min Cut Algorithm

Theorem: If the contraction algorithm is repeated $O(n^2 \log n)$ times, one of the $O(n^2 \log n)$ instances returns a min. cut w.h.p.

Corollary: The contraction algorithm allows to compute a minimum cut in $O(n^4 \log n)$ time w.h.p.

- One instance consists of $n-2$ edge contractions
- Each edge contraction can be carried out in time $O(n)$
  - Actually: $O(\text{current \#nodes})$
- Time per instance of the contraction algorithm: $O(n^2)$

Can We Do Better?

- Time $O(n^4 \log n)$ is not very spectacular, a simple max flow based implementation has time $O(n^4)$.

However, we will see that the contraction algorithm is nevertheless very interesting because:

1. The algorithm can be improved to beat every known deterministic algorithm.

1. It allows to obtain strong statements about the distribution of cuts in graphs.

Better Randomized Algorithm

Recall:

- Consider a fixed min cut (A, B), assume (A, B) has size k
- The algorithm outputs (A, B) iff none of the k edges crossing (A, B) gets contracted.
- Throughout the algorithm, the edge connectivity is at least k and therefore each node has degree $\geq k$
- Before contraction i, there are $n + 1 - i$ nodes and thus at least $(n + 1 - i)k/2$ edges
- If no edge crossing (A, B) is contracted before, the probability to contract an edge crossing (A, B) in step i is at most

$$\frac{k}{\frac{(n+1-i)k}{2}} = \frac{2}{n+1-i}.$$

Improving the Contraction Algorithm

- For a specific min cut (A, B), if (A, B) survives the first i contractions,

$$\mathbb{P}(\text{edge crossing } (A, B) \text{ in contraction } i + 1) \leq \frac{2}{n - i}.$$

- **Observation:** The probability only gets large for large i

- **Idea:** The early steps are much safer than the late steps.

  Maybe we can repeat the late steps more often than the early ones.

Safe Contraction Phase

Lemma: A given min cut (A, B) of an n-node graph G survives the first $n - \left\lceil {}^{n}/_{\sqrt{2}} + 1 \right\rceil$ contractions, with probability $> {}^{1}/_{2}$.

Proof:

- Event $\mathcal{E}_i$: cut (A, B) survives contraction i
- Probability that (A, B) survives the first $n - t$ contractions:

Better Randomized Algorithm

Let's simplify a bit:

- Pretend that $n/\sqrt{2}$ is an integer (for all n we will need it).
- Assume that a given min cut survives the first $n - {}^{n}/_{\sqrt{2}}$ contractions with probability $\geq {}^{1}/_{2}$.

$\mathbf{contract}(\boldsymbol{G}, \boldsymbol{t})$:

- Starting with n-node graph G, perform $n - t$ edge contractions such that the new graph has t nodes.

$\mathbf{mincut}(\boldsymbol{G})$:

1. $X_1 \coloneqq \mathrm{mincut}\left(\mathrm{contract}\left(G, n/\sqrt{2}\right)\right)$;
2. $X_2 \coloneqq \mathrm{mincut}\left(\mathrm{contract}\left(G, n/\sqrt{2}\right)\right)$;
3. **return** $\min\{X_1, X_2\}$;

Success Probability

$\mathbf{mincut}(\boldsymbol{G})$:

1. $X_1 \coloneqq \mathrm{mincut}\left(\mathrm{contract}\left(G, n/\sqrt{2}\right)\right);$
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3. **return** $\min\{X_1, X_2\};$

$\boldsymbol{P}(\boldsymbol{n})$: probability that the above algorithm returns a min cut when applied to a graph with n nodes.

- Probability that X_1 is a min cut $\geq$

Recursion:

Success Probability

Theorem: The recursive randomized min cut algorithm returns a minimum cut with probability at least $1/\log_2 n$.

Proof (by induction on n):

$$P(n) = P\left(\frac{n}{\sqrt{2}}\right) - \frac{1}{4} \cdot P\left(\frac{n}{\sqrt{2}}\right)^2, \qquad P(2) = 1$$

Running Time

1. $X_1 \coloneqq \mathrm{mincut}\left(\mathrm{contract}\left(G, n/\sqrt{2}\right)\right);$
2. $X_2 \coloneqq \mathrm{mincut}\left(\mathrm{contract}\left(G, n/\sqrt{2}\right)\right);$
3. **return** $\min\{X_1, X_2\};$

Recursion:

- $T(n)$: time to apply algorithm to n-node graphs
- Recursive calls: $2T\left({}^{n}/_{\sqrt{2}}\right)$
- Number of contractions to get to ${}^{n}/_{\sqrt{2}}$ nodes: $O(n)$

$$T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2), \qquad T(2) = O(1)$$

Running Time

Theorem: The running time of the recursive, randomized min cut algorithm is $O(n^2 \log n)$.

Proof:

- Can be shown in the usual way, by induction on n

Remark:

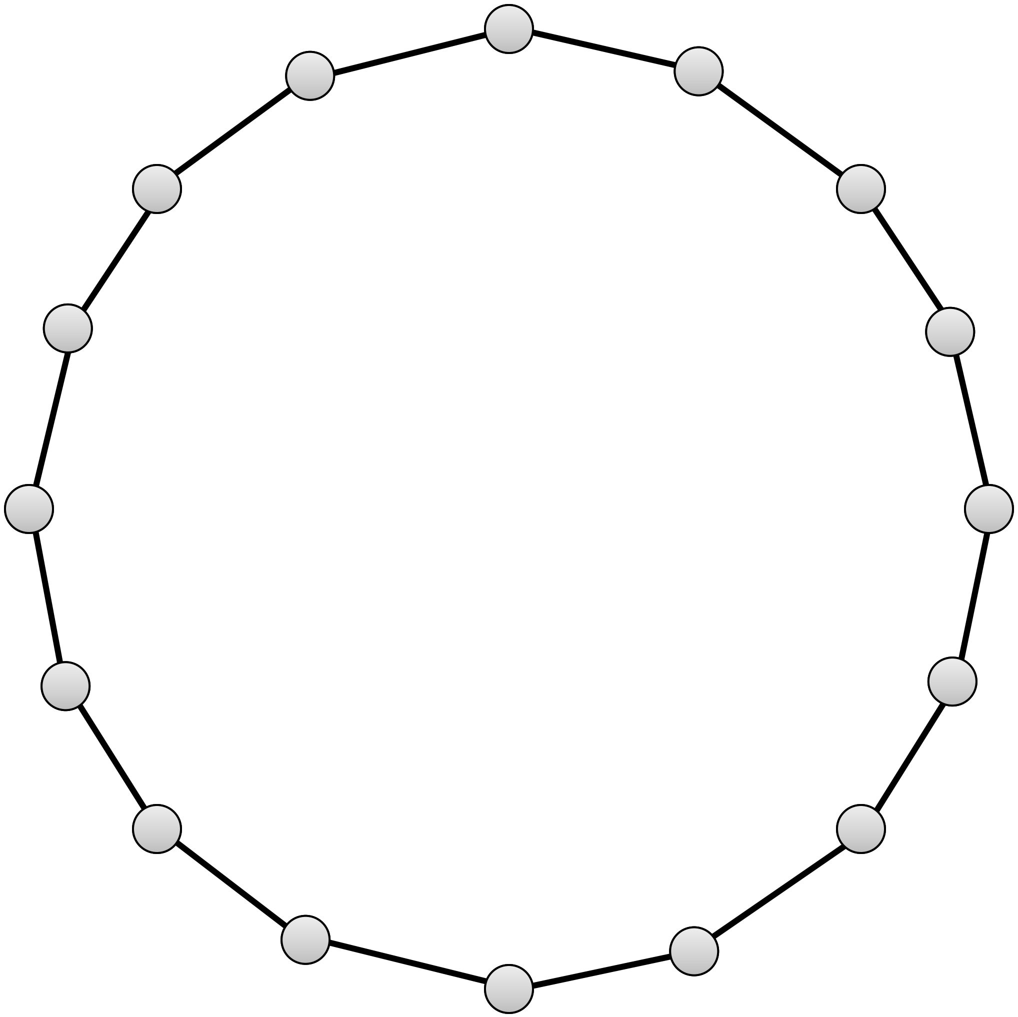
- The running time is only by an $O(\log n)$-factor slower than the basic contraction algorithm.
- The success probability is exponentially better!

Number of Minimum Cuts

- Given a graph G, how many minimum cuts can there be?
- Or alternatively: If G has edge connectivity k, how many ways are there to remove k edges to disconnect G?
- Note that the total number of cuts is large.

Number of Minimum Cuts

Example: Ring with n nodes



- Minimum cut size: 2
- Every two edges induce a min cut
- Number of edge pairs:

$$\binom{n}{2}$$

- Are there graphs with more min cuts?

Number of Min Cuts

Theorem: The number of minimum cuts of a graph is at most $\binom{n}{2}$.

Proof:

- Assume there are s min cuts
- For $i \in \{1, \dots, s\}$, define event $\mathcal{C}_i$:

  $\mathcal{C}_i \coloneqq \{\text{basic contraction algorithm returns min cut } i\}$
- We know that for $i \in \{1, \dots, s\}$: $\mathbb{P}(\mathcal{C}_i) \geq 1/\binom{n}{2}$
- Events $\mathcal{C}_1, \dots, \mathcal{C}_s$ are disjoint:

$$\mathbb{P}\left(\bigcup_{i=1}^{s} \mathcal{C}_i\right) = \sum_{i=1}^{s} \mathbb{P}(\mathcal{C}_i) \geq \frac{s}{\binom{n}{2}}$$