# Chapter 9
# Online Algorithms

## Algorithm Theory
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# Online Computations

- Sometimes, an algorithm has to start processing the input before the complete input is known
- For example, when storing data in a data structure, the sequence of operations on the data structure is not known

**Online Algorithm:** An algorithm that has to produce the output step-by-step when new parts of the input become available.

**Offline Algorithm:** An algorithm that has access to the whole input before computing the output.

- Some problems are inherently online
  - Especially when real-time requests have to be processed over a significant period of time

# Competitive Ratio

- Let's again consider optimization problems
  - For simplicity, assume, we have a minimization problem

**Optimal offline solution $\mathbf{OPT}(\boldsymbol{I})$:**

- Best objective value that an offline algorithm can achieve for a given input sequence $I$

$\frac{ALG}{OPT} \leq c$

**Online solution $\mathbf{ALG}(\boldsymbol{I})$:**

- Objective value achieved by an online algorithm $\mathrm{ALG}$ on $I$

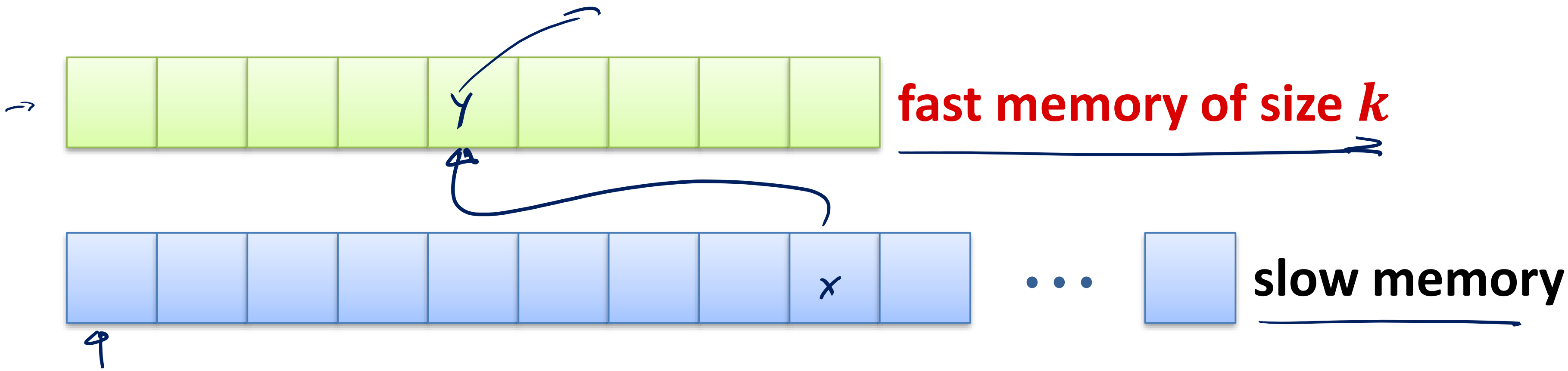
**Competitive Ratio:** An algorithm has competitive ratio $c \geq 1$ if

$$\mathbf{ALG}(\boldsymbol{I}) \leq \boldsymbol{c} \cdot \mathbf{OPT}(\boldsymbol{I}) + \boldsymbol{\alpha}.$$

- If $\alpha = 0$, we say that $\mathrm{ALG}$ is strictly $c$-competitive.

# Paging Algorithm

Assume a simple memory hierarchy:




If a memory page has to be accessed:

- Page in fast memory (hit): take page from there
- Page not in fast memory (miss): leads to a page fault
- Page fault: the page is loaded into the fast memory and some page has to be evicted from the fast memory
- Paging algorithm: decides which page to evict
- Classical online problem: we don't know the future accesses

# Paging Strategies

**Least Recently Used (LRU):**

- Replace the page that hasn't been used for the longest time

**First In First Out (FIFO):**

- Replace the page that has been in the fast memory longest

**Last In First Out (LIFO):**

- Replace the page most recently moved to fast memory

**Least Frequently Used (LFU):**

- Replace the page that has been used the least

**Longest Forward Distance (LFD):** not an online alg.

- Replace the page whose next request is latest (in the future)

→ optimal offline solution

- LFD is **not an online strategy**!

# LFD is Optimal

1, ..., n

**Theorem:** LFD (longest forward distance) is an optimal offline alg.

**Proof:**

$\sigma$: sequence of requests

- For contradiction, assume that LFD is not optimal
- Then there exists a finite input sequence $\sigma$ on which LFD is not optimal (assume that the length of $\sigma$ is $|\sigma| = n$)
- Let OPT be an optimal solution for $\sigma$ such that $0 \leq i \leq n-1$
  - OPT processes requests $1, \dots, i$ in exactly the same way as LFD
  - OPT processes request $i + 1$ differently than LFD
  - Any other optimal strategy processes one of the first $i + 1$ requests differently than LFD
- Hence, OPT is the optimal solution that behaves in the same way as LFD for as long as possible → we have $i < n$
- Goal: Construct OPT′ that is identical with LFD for req. $1, \dots, i + 1$

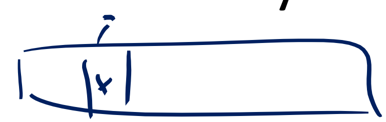
# LFD is Optimal

**Theorem:** LFD (longest forward distance) is an optimal offline alg.

**Proof:**

**Case 1:** Request $i+1$ does **not** lead to a page fault

- LFD does not change the content of the fast memory



- OPT behaves differently than LFD
  → OPT replaces some page in the fast memory
  - As up to request $i+1$, both algorithms behave in the same way, they also have the same fast memory content
  - OPT therefore does not require the new page for request $i+1$
  - Hence, OPT can also load that page later (without extra cost) → OPT′

# LFD is Optimal

LFD: $p$ OPT: $p'$

**Theorem:** LFD (longest forward distance) is an optimal offline alg.

**Proof:**

**Case 2:** Request $i + 1$ does lead to a **page fault**

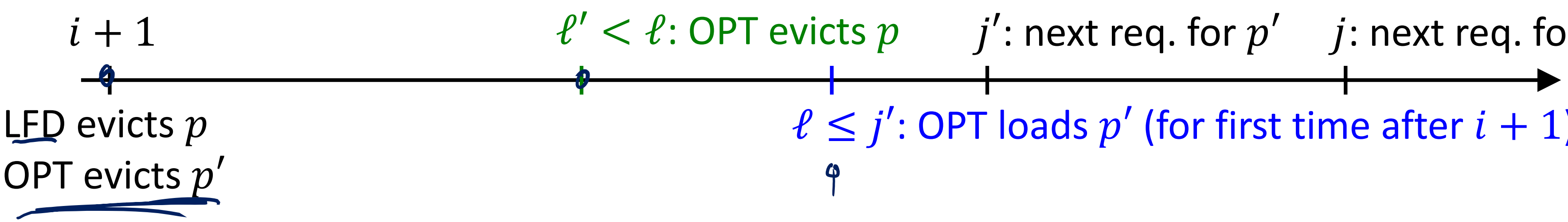
- LFD and OPT move the same page into the fast memory, but they evict different pages
  - If OPT loads more than one page, all pages that are not required for request $i + 1$ can also be loaded later
- Say, LFD evicts page $p$ and OPT evicts page $p'$
- By the definition of LFD, $p'$ is required again before page $p$

# LFD is Optimal

**Theorem:** LFD (longest forward distance) is an optimal offline alg.

**Proof:**

**Case 2:** Request $i + 1$ does lead to a **page fault**




a) OPT keeps $p$ in fast memory until request $\ell$

- Evict $p$ at request $i + 1$, keep $p'$ instead and load $p$ (instead of $p'$) back into the fast memory at request $\ell$

b) OPT evicts $p$ at request $\ell' < \ell$

- Evict $p$ at request $i + 1$ and $p'$ at request $\ell'$ (switch evictions of $p$ and $p'$)

# Phase Partition

We partition a given request sequence $\sigma$ into phases as follows:

- **Phase $\mathbf{0}$**: empty sequence
- **Phase $\boldsymbol{i}$** : maximal sequence that immediately follows phase $i-1$ and contains at most $k$ distinct page requests

**Example sequence ($\boldsymbol{k=4}$):**

$$\underbrace{2,5,12,5,4,2}_{\text{phase 1}} \mid \underbrace{10,8,3,6}_{\text{ph. 2}} \mid \underbrace{2,2,6,6,8,3,2,6}_{\text{ph. 3}} \mid \underbrace{9,10,6,3,10}_{\text{ph. 4}} \mid \underbrace{2,1,3,5}_{\text{ph. 5}}$$

$$2,\underbrace{5,12,5,4,2,10}_{\text{phase 1 interval}},\underbrace{8,3,6,2}_{\text{ph. 2 interval}},\underbrace{2,6,6,8,3,2,6,9}_{},\underbrace{10,6,3,10,2}_{},1,3,5$$

**Phase $\boldsymbol{i}$ Interval:** interval starting with the second request of phase $i$ and ending with the first request of phase $i+1$

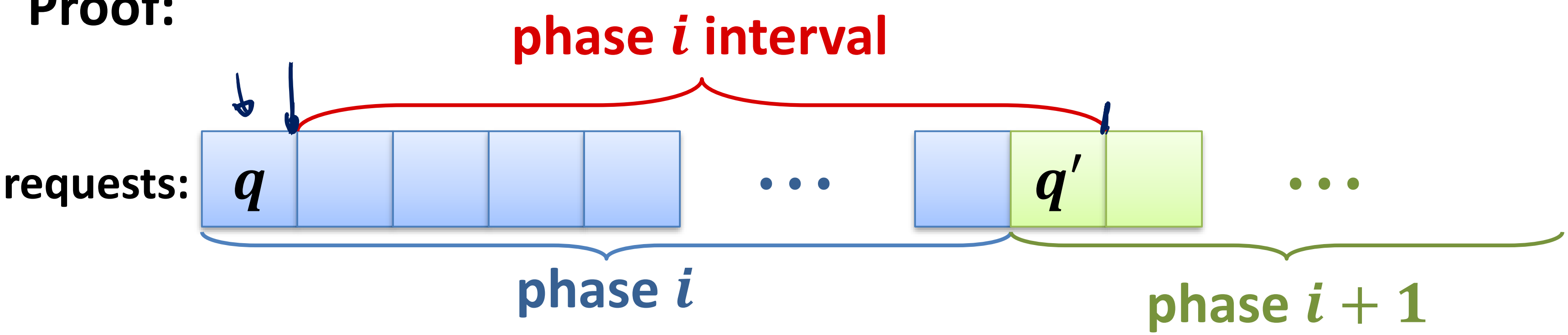
- If the last phase is phase $p$, phase $i$ interval is defined for $i = 1, \dots, p-1$

# Optimal Algorithm

$p-1$

**Lemma:** Algorithm LFD has at least one page fault in each phase $i$ interval (for $i = 1, \dots, p-1$, where $p$ is the number of phases).

**Proof:**




- $q$ is in fast memory after first request of phase $i$
- Number of distinct requests in phase $i$: $k$
- By maximality of phase $i$: $q'$ does not occur in phase $i$
- Number of distinct requests $\neq q$ in phase interval $i$: $k$

→ at least one page fault

# LRU and FIFO Algorithms

**Lemma:** Algorithm LFD has at least one page fault in each phase $i$ interval (for $i = 1, \dots, p-1$, where $p$ is the number of phases).

**Corollary:** The number of page faults of an optimal offline algorithm is at least $p - 1$, where $p$ is the number of phases

**Theorem:** The LRU and the FIFO algorithms both have a competitive ratio of at most $k$.

**Proof:**

- We will show that both have at most $k$ page faults per phase
- We then have (for every input $I$):

$$\mathrm{LRU}(I), \mathrm{FIFO}(I) \leq k \cdot p \leq k \cdot \mathrm{OPT}(I) + k$$