Theoretical Computer Science - Bridging Course  
Summer Term 2016  
Sample Solution for Exercise Sheet 1

Exercise 1: Proof by Induction (5 points)

Let \( x \geq -1 \). Show that \((1 + x)^n \geq 1 + nx\) for all \( n \in \mathbb{N} \) using induction on \( n \).

Solution:

The above inequality is a special case of Bernoulli’s inequality. For \( n = 1 \) we have \( 1 + x \geq 1 + x \) which is true. In the following we assume that the inequality holds for an arbitrary but fixed \( n \) and conclude that this is also true for \( n + 1 \):

\[
(1 + x)^n \geq 1 + nx \quad \iff \quad (1 + x)(1 + x)^n \geq (1 + x)(1 + nx) \quad \text{(since } (1 + x) \geq 0) \\
\iff \quad (1 + x)^{n+1} \geq 1 + (n+1)x + nx^2 \\
\iff \quad (1 + x)^{n+1} \geq 1 + (n+1)x + nx^2 \quad \text{(since } nx^2 \geq 0) 
\]

Exercise 2: Partition of a Set (5 points)

A partition of a set \( A \) is a collection of sets \( B_i, i \in \{1, \ldots, n\} \) such that

\[
B_1 \cup \ldots \cup B_n = A \quad \text{and} \quad B_i \cap B_j = \emptyset \quad \text{for } i \neq j.
\]

Show that \( B_i := \{3k + i \mid k \in \mathbb{Z}\}, i \in \{1, 2, 3\} \) is a partition of \( \mathbb{Z} \).

**Hint:** \( \mathbb{Z} \) is the set of integers. In order to proof that two sets are equal consider an arbitrary element from one set and show that it is contained in the other set and vice versa.

Solution:

First we show \( B_1 \cup B_2 \cup B_3 = \mathbb{Z} \). Let \( z \) be an arbitrary element in \( B_1 \cup B_2 \cup B_3 \). This means \( z \in B_i \) for at least one of the \( B_i \). Since \( B_i \) is composed only of integers we conclude \( z \in \mathbb{Z} \).

Now let \( z \in \mathbb{Z} \) be an arbitrary integer and \( i := z \mod 3 \leq 2 \) the residue of the integer division of \( z \) by three. Then we have \( z = 3k + i \) for some \( k \in \mathbb{Z} \) and thus either \( z \in B_i \) for \( i = 1, 2 \) or \( z \in B_3 \) for \( i = 0 \). Therefore \( z \in B_1 \cup B_2 \cup B_3 \).

It remains to be shown that the intersection \( B_i \cap B_j \) for \( i \neq j \) is empty. For a contradiction assume there exists a \( z \in B_i \cap B_j \). Since \( z \in B_i \) and \( z \in B_j \) we have (by definition of \( B_1, B_2, B_3 \)) that \( i = z \mod 3 = j \) which is a contradiction.
Exercise 3: Counting Edges in Acyclic Graph (5 points)

A tree is an acyclic, connected, simple graph. Show that a tree with \( n \geq 1 \) nodes has \( n - 1 \) edges. A forest is a graph consisting of several unconnected trees. Show that a forest consisting of \( k \) components has \( n - k \) edges.

*Hint: A simple graph is an unweighted, undirected graph containing no self-loops or multiple edges.*

**Solution:**

First we show that an acyclic, connected graph with \( n \) nodes has exactly \( n - 1 \) edges using an induction argument on \( n \). A graph with just one node has \( n - 1 = 0 \) edges. Assume that the statement holds for graphs with an arbitrary but fixed number of nodes \( n \) and consider a graph \( G \) with \( n + 1 \) nodes. We remove one edge \( e \), which makes \( G \) disintegrate into two components \( G_1 \) and \( G_2 \) which are not connected to each other (if there were a connection between \( G_1 \) and \( G_2 \), then reattaching \( e \) to \( G \) would close a cycle).

The components \( G_1 \) and \( G_2 \) themselves are still acyclic and (internally) connected and have \( 1 \leq k, m \leq n \) nodes with \( k + m = n + 1 \). Using the induction hypothesis \((k, m \leq n)\) we have that \( G_1 \) has \( k - 1 \) edges and \( G_2 \) has \( m - 1 \) edges. Since we removed exactly one edge to obtain \( G_1 \) and \( G_2 \), \( G \) has \((k - 1) + (m - 1) + 1 = n \) edges.

Next we show that a forest \( G \) consisting of \( k \) trees \( G_1, \ldots, G_k \) has \( n - k \) edges. Let \( n_i, i \in \{1, \ldots, k\} \) the number of nodes of the \( i \)-th tree. Of course \( \sum_{i=1}^{k} n_i = n \). We already know that \( G_i \) has \( n_i - 1 \) edges. Thus \( G \) has exactly \( \sum_{i=1}^{k} n_i - 1 = n - k \) edges.

Exercise 4: Nodes with Identical Degrees (5 points)

Show that every simple graph with two or more nodes contains two nodes with the same degree.

**Solution:**

We prove this claim by contradiction. Consider a graph with \( n \geq 2 \) nodes \( u_1, u_2, \ldots, u_n \). Assume that each node has a different degree. The minimum degree a node can have is 0, in which case the node has no neighbours; and the maximum degree a node can have is \( n - 1 \), in which case the node connects to every other node in the graph. Without loss of generality, we assume that node \( u_i \) has degree \( i - 1 \), where \( 1 \leq i \leq n \) (otherwise we rename nodes). Since node \( u_n \) has degree \( n - 1 \) it must be connected to all others including \( u_1 \). However, the degree of \( u_1 \) is 0, which is a contradiction.

**Alternative approach.** You can also prove this by induction on \( n \). However, in this process, you still may have to use the trick we employed in the above proof: counting degrees carefully.