Example

$$p(x) = 3x^3 - 15x^2 + 18x + 0, \qquad a = [0,18,-15,3]$$

$p_0(x) = -15x + 0$ $\qquad$ $p_1(x) = 3x + 18$

$p_{00}(x) = 0$ $\qquad$ $p_{01}(x) = -15$ $\qquad$ $p_{10}(x) = 18$ $\qquad$ $p_{11}(x) = 3$

$p_0(\omega_2^0) = p_{00}(\omega_1^0) + \omega_2^0 p_{01}(\omega_1^0) = -15$

$p_0(\omega_2^1) = p_{00}(\omega_1^0) - \omega_2^0 p_{01}(\omega_1^0) = +15$

$p_1(\omega_2^0) = p_{10}(\omega_1^0) + \omega_2^0 p_{11}(\omega_1^0) = 21$

$p_1(\omega_2^1) = p_{10}(\omega_1^0) - \omega_2^0 p_{11}(\omega_1^0) = 15$

$p(\omega_4^0) = p_0(\omega_2^0) + \omega_4^0 p_1(\omega_2^0) = -15 + 21 = 6$

$p(\omega_4^1) = p_0(\omega_2^1) + \omega_4^1 p_1(\omega_2^1) = +15 + i \cdot 15$

$p(\omega_4^2) = p_0(\omega_2^0) - \omega_4^0 p_1(\omega_2^0) = -15 - 21 = -36$

$p(\omega_4^3) = p_0(\omega_2^1) - \omega_4^1 p_1(\omega_2^1) = 15 - 15i$

Faster Polynomial Multiplication?

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

Evaluation at $\omega_{2n}^0, \omega_{2n}^1, \dots, \omega_{2n}^{2n-1}$ using **FFT** $O(n \log n)$

$2 \times 2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)\right)$ and $\left(\omega_{2n}^k, q\left(\omega_{2n}^k\right)\right)$

Point-wise multiplication $O(n)$

$2n$ point-value pairs $\left(\omega_{2n}^k, \underline{p\left(\omega_{2n}^k\right)q\left(\omega_{2n}^k\right)}\right)$

$\boxed{\textbf{Interpolation}}$

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

Interpolation

Convert point-value representation into coefficient representation

$\{x_1, \dots, x_{n-1}\} = X$

Input: $(x_0, y_0), \dots, (x_{n-1}, y_{n-1})$ with $x_i \neq x_j$ for $i \neq j$

Output:

Degree-$(n-1)$ polynomial with coefficients $a_0, \dots, a_{n-1}$ such that

$$
\left[
\begin{aligned}
p(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_{n-1} x_0^{n-1} = y_0 \\
p(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1} = y_1 \\
&\vdots \\
p(x_{n-1}) &= a_0 + a_1 x_{n-1} + a_2 x_{n-1}^2 + \cdots + a_{n-1} x_{n-1}^{n-1} = y_{n-1}
\end{aligned}
\right.
$$

→ linear system of equations for $a_0, \dots, a_{n-1}$

Interpolation

x_j

Matrix Notation:

$$\begin{pmatrix} 1 & x_0 & \cdots & x_0^{n-1} \\ 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & \cdots & x_{n-1}^{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

- System of equations solvable iff $x_i \neq x_j$ for all $i \neq j$

Special Case $x_i = \omega_n^i$:

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix}}_{W} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

Interpolation

- Linear system:

$$W \cdot \boldsymbol{a} = \boldsymbol{y} \quad \Longrightarrow \quad \boldsymbol{a} = W^{-1} \cdot \boldsymbol{y}$$

$$W_{i,j} = \omega_n^{ij}, \qquad \boldsymbol{a} = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

Claim:

$$\boldsymbol{W_{ij}^{-1} = \frac{\omega_n^{-ij}}{n}}$$

Proof: Need to show that $W^{-1}W = I_n$

DFT Matrix Inverse

$W^{-1}_{ij} = \frac{\omega_n^{-ij}}{n}$ $\qquad$ $W_{ij} = \omega_n^{ij}$

$$W^{-1}W = \begin{array}{r} \\ \text{row } i \rightarrow \\ \\ \end{array} \begin{pmatrix} & & \cdots & \\ \frac{1}{n} & \frac{\omega_n^{-i}}{n} & \cdots & \frac{\omega_n^{-(n-1)i}}{n} \\ & & \vdots & \\ & & \cdots & \end{pmatrix} \cdot \begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & \omega_n^{j} & \cdots \\ \cdots & \omega_n^{2j} & \cdots \\ & \vdots & \\ \cdots & \omega_n^{(n-1)j} & \cdots \end{pmatrix}$$

$$\left(W^{-1}W\right)_{i,j} = \frac{1}{n} \cdot \sum_{\ell=0}^{n-1} \omega_n^{-i\cdot\ell} \cdot \omega_n^{j\ell} = \frac{1}{n} \sum_{\ell=0}^{n-1} \omega_n^{\ell(j-i)}$$

DFT Matrix Inverse

$$(W^{-1}W)_{i,j} = \sum_{\ell=0}^{n-1} \frac{\omega_n^{\ell(j-i)}}{n}$$

Need to show $(W^{-1}W)_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Case $i = j$:

$$(W^{-1}W)_{i,i} = \frac{1}{n} \sum_{\ell=0}^{n-1} \underbrace{\omega_n^{\ell \cdot 0}}_{=1} = 1 \quad \checkmark$$

DFT Matrix Inverse

$$(W^{-1}W)_{i,j} = \sum_{\ell=0}^{n-1} \frac{\omega_n^{\ell(j-i)}}{n}$$

Case $i \neq j$:

$$(W^{-1}W)_{i,j} = \frac{1}{n} \underbrace{\sum_{\ell=0}^{n-1} \left(\omega_n^{j-i}\right)^{\ell}}_{\text{geometric series}} = \frac{1}{n} \cdot \frac{\underbrace{\left(\omega_n^{j-i}\right)^{n}}_{=1} - 1}{\omega_n^{j-i} - 1} = 0 \quad \checkmark$$

$$\sum_{\ell=0}^{n-1} q^{\ell} = \frac{q^n - 1}{q - 1}$$

Inverse DFT

- $$W^{-1} = \begin{pmatrix} & & \cdots & \\ \frac{1}{n} & \frac{\omega_n^{-k}}{n} & \cdots & \frac{\omega_n^{-(n-1)k}}{n} \\ & & \vdots & \\ & & \cdots & \end{pmatrix}$$

$(W^{-1})_{ij} = \frac{\omega_n^{-ij}}{n}$

- We get $\underline{\underline{\boldsymbol{a}}} = \underline{\underline{W^{-1}}} \cdot \underline{\underline{\boldsymbol{y}}}$ and therefore

$$\underline{\underline{a_k}} = \begin{pmatrix} \frac{1}{n} & \frac{\omega_n^{-k}}{n} & \cdots & \frac{\omega_n^{-(n-1)k}}{n} \end{pmatrix} \cdot \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

$$= \frac{1}{n} \cdot \sum_{j=0}^{n-1} \omega_n^{-kj} \cdot y_j$$

DFT and Inverse DFT

Inverse DFT:

$= (\omega_n^{-k})^j$

$z = \omega_n^{-k}$

$$a_k = \frac{1}{n} \cdot \sum_{j=0}^{n-1} \omega_n^{-kj} \cdot y_j$$

- Define polynomial $q(x) = y_0 + y_1 x + \cdots + y_{n-1} x^{n-1}$:

$$a_k = \frac{1}{n} \cdot q(\omega_n^{-k})$$

$\omega_n^{-k} = \omega_n^{n-k}$

DFT:

- Polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$:

$$y_k = p(\omega_n^k)$$

$y_0, \ldots, y_{n-1}$

DFT and Inverse DFT

$$q(x) = y_0 + y_1 x + \cdots + y_{n-1} x^{n-1}, \qquad a_k = \frac{1}{n} \cdot q\left(\omega_n^{-k}\right):$$

- Therefore:

$$(a_0, a_1, \dots, a_{n-1})$$
$$= \frac{1}{n} \cdot \left(q(\omega_n^{-0}), q(\omega_n^{-1}), q(\omega_n^{-2}), \dots, q\left(\omega_n^{-(n-1)}\right)\right)$$
$$= \frac{1}{n} \cdot \left(q(\omega_n^{0}), q(\omega_n^{n-1}), q(\omega_n^{n-2}), \dots, q(\omega_n^{1})\right)$$

- Recall:

$$\mathrm{DFT}_n(\boldsymbol{y}) = \left(q(\omega_n^0), q(\omega_n^1), q(\omega_n^2), \dots, q(\omega_n^{n-1})\right)$$
$$= n \cdot (a_0, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$$

DFT and Inverse DFT

- We have $\mathrm{DFT}_n(\boldsymbol{y}) = n \cdot (a_0, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$:

$$a_i = \begin{cases} \frac{1}{n} \cdot (\mathrm{DFT}_n(\boldsymbol{y}))_0 & \text{if } i = 0 \\ \frac{1}{n} \cdot (\mathrm{DFT}_n(\boldsymbol{y}))_{n-i} & \text{if } i \neq 0 \end{cases}$$

- DFT and inverse DFT can both be computed using FFT algorithm in $O(n \log n)$ time.

- 2 polynomials of degr. $< n$ can be multiplied in time $O(n \log n)$.

Faster Polynomial Multiplication?

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):

p, q of degree $n-1$, n coefficients

Evaluation at $\omega_{2n}^0, \omega_{2n}^1, \dots, \omega_{2n}^{2n-1}$ using **FFT**

$2 \times 2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)\right)$ and $\left(\omega_{2n}^k, q\left(\omega_{2n}^k\right)\right)$

Point-wise multiplication

$2n$ point-value pairs $\left(\omega_{2n}^k, p\left(\omega_{2n}^k\right)q\left(\omega_{2n}^k\right)\right)$

Interpolation using **FFT**

$p(x)q(x)$ of degree $2n-2$, $2n-1$ coefficients

$O(n \log n \log\log n)$

Convolution

- More generally, the polynomial multiplication algorithm computes the convolution of two vectors:

$$\boldsymbol{a} = (a_0, a_1, \dots, a_{m-1})$$
$$\boldsymbol{b} = (b_0, b_1, \dots, b_{n-1})$$

$$\boldsymbol{a} * \boldsymbol{b} = (c_0, c_1, \dots, c_{m+n-2}),$$
$$\text{where } c_k = \sum_{\substack{(i,j):i+j=k \\ i<m, j<n}} a_i b_j$$

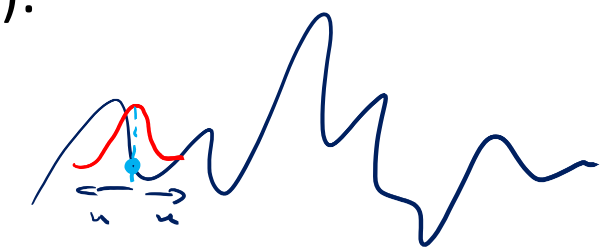
- c_k is exactly the coefficient of x^k in the product polynomial of the polynomials defined by the coefficient vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

More Applications of Convolutions

Signal Processing Example:

- Assume $\boldsymbol{a} = (a_0, \dots, a_{n-1})$ represents a sequence of measurements over time
- Measurements might be noisy and have to be smoothed out
- Replace a_i by weighted average of nearby last m and next m measurements (e.g., Gaussian smoothing):

$$a_i' = \frac{1}{Z} \cdot \sum_{j=i-m}^{i+m} a_j e^{-(i-j)^2}$$



- New vector $\boldsymbol{a}'$ is the convolution of $\boldsymbol{a}$ and the weight vector

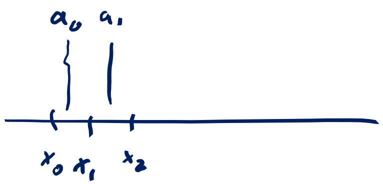
$$\frac{1}{Z} \cdot \left(e^{-m^2}, e^{-(m-1)^2}, \dots, e^{-1}, 1, e^{-1}, \dots, e^{-(m-1)^2}, e^{-m^2}\right)$$

- Might need to take care of boundary points…

More Applications of Convolutions

Combining Histograms:

- Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ represent two histograms



- E.g., annual income of all men & annual income of all women

- Goal: Get new histogram $\boldsymbol{c}$ representing combined income of all possible pairs of men and women:

$$\boldsymbol{c} = \boldsymbol{a} * \boldsymbol{b}$$

Also, the DFT (and thus the FFT alg.) has many other applications!

DFT in Signal Processing

$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$

Assume that $y(0), y(1), y(2), \dots, y(T-1)$ are measurements of a time-dependent signal.

Inverse DFT_N of $\big(y(0), \dots, y(T-1)\big)$ is a vector $(c_0, \dots, c_{N-1})$ s.t.

$N = T$

$$y(t) = \sum_{k=0}^{N-1} c_k \cdot e^{\frac{2\pi i \cdot k}{N} \cdot t} \quad (\omega_N^t)^k$$

$$= \sum_{k=0}^{T-1} c_k \cdot \left(\cos\left(\frac{2\pi \cdot k}{N} \cdot t\right) + i \sin\left(\frac{2\pi \cdot k}{N} \cdot t\right)\right)$$

- Converts signal from time domain to frequency domain
- Signal can then be edited in the frequency domain
  - e.g., setting some $c_k = 0$ filters out some frequencies