Chapter 5
Data Structures

Algorithm Theory
WS 2017/18

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Priority Queue / Heap

- Stores (*key,data*) pairs (like dictionary)
- But, different set of operations:

- **Initialize-Heap**: creates new empty heap
- **Is-Empty**: returns true if heap is empty
- **Insert**(*key,data*): inserts (*key,data*)-pair, returns pointer to entry
- **Get-Min**: returns (*key,data*)-pair with minimum *key*
- **Delete-Min**: deletes minimum (*key,data*)-pair

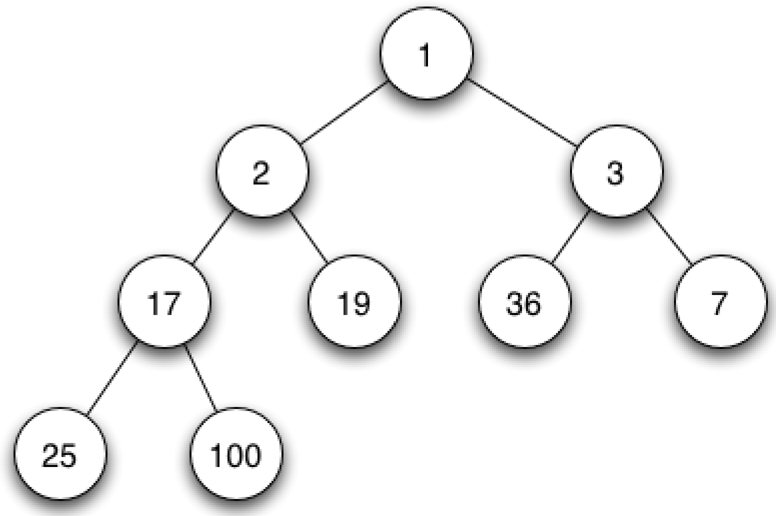
consistent

- **Decrease-Key**(*entry,newkey*): decreases *key* of *entry* to *newkey*
- **Merge**: merges two heaps into one

Priority Queue Implementation

Implementation as min-heap:

→ complete binary tree,
e.g., stored in an array



- **Initialize-Heap**: $\boldsymbol{O(1)}$
- **Is-Empty**: $\boldsymbol{O(1)}$
- n **Insert**: $\boldsymbol{O(\log n)}$
- n **Get-Min**: $\boldsymbol{O(1)}$
- n **Delete-Min**: $\boldsymbol{O(\log n)}$
- m **Decrease-Key**: $\boldsymbol{O(\log n)}$
- **Merge** (heaps of size m and n, $m \leq n$): $\boldsymbol{O(m \log n)}$

Dijkstra: $G = (V, E)$

$n = |V|, \; m = |E|$

$O(m \log n)$

Can We Do Better?

- Cost of **Dijkstra** with **complete binary min-heap** implementation:

$$O(|E|\log|V|)$$

- **Binary heap:**
  insert, delete-min, and decrease-key cost $O(\log n)$
  merging two heaps is expensive

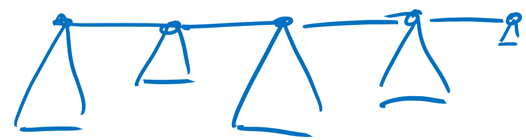
- One of the operations insert or delete-min must cost $\Omega(\log n)$:
  - Heap-Sort:
    Insert n elements into heap, then take out the minimum n times
  - (Comparison-based) sorting costs at least $\Omega(n \log n)$.

- But maybe we can improve merge, decrease-key, and one of the other two operations?

Fibonacci Heaps

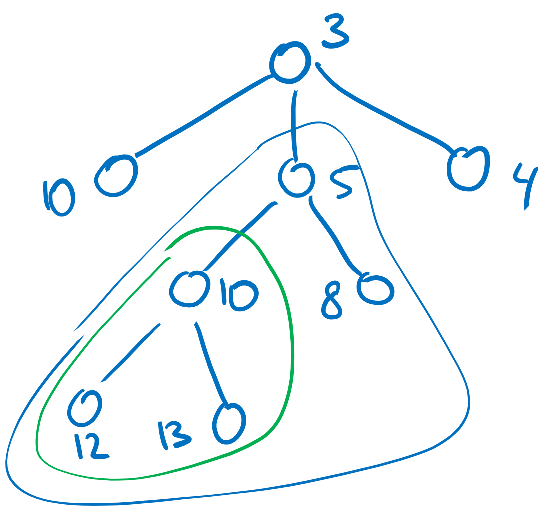
Structure:

A Fibonacci heap H consists of a collection of trees satisfying the **min-heap** property.



Min-Heap Property:

Key of a node $v \leq$ keys of all nodes in any sub-tree of v



Fibonacci Heaps

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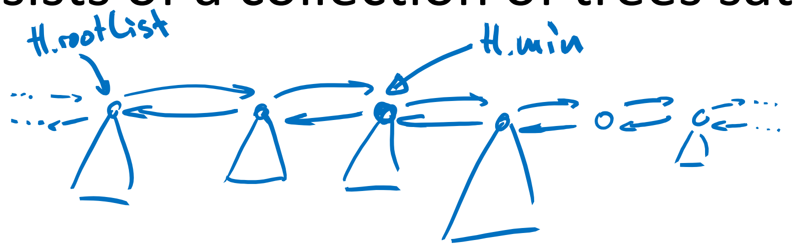


Variables:

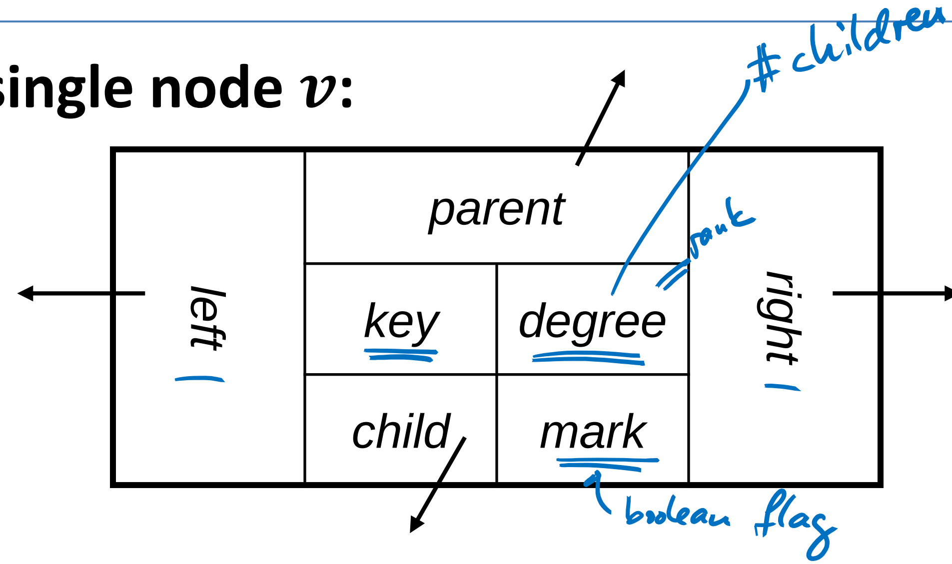
- $H.min$: root of the tree containing the (a) minimum key
- $H.rootlist$: circular, doubly linked, unordered list containing the roots of all trees
- $H.size$: number of nodes currently in H

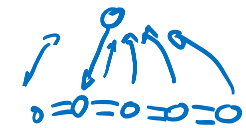
Lazy Merging:

- To reduce the number of trees, sometimes, trees need to be merged
- Lazy merging: Do not merge as long as possible...

Trees in Fibonacci Heaps

Structure of a single node v:





- $v.child$: points to circular, doubly linked and unordered list of the children of v
- $v.left, v.right$: pointers to siblings (in doubly linked list)
- $v.mark$: will be used later…

Advantages of circular, doubly linked lists:

- Deleting an element takes constant time
- Concatenating two lists takes constant time

Example

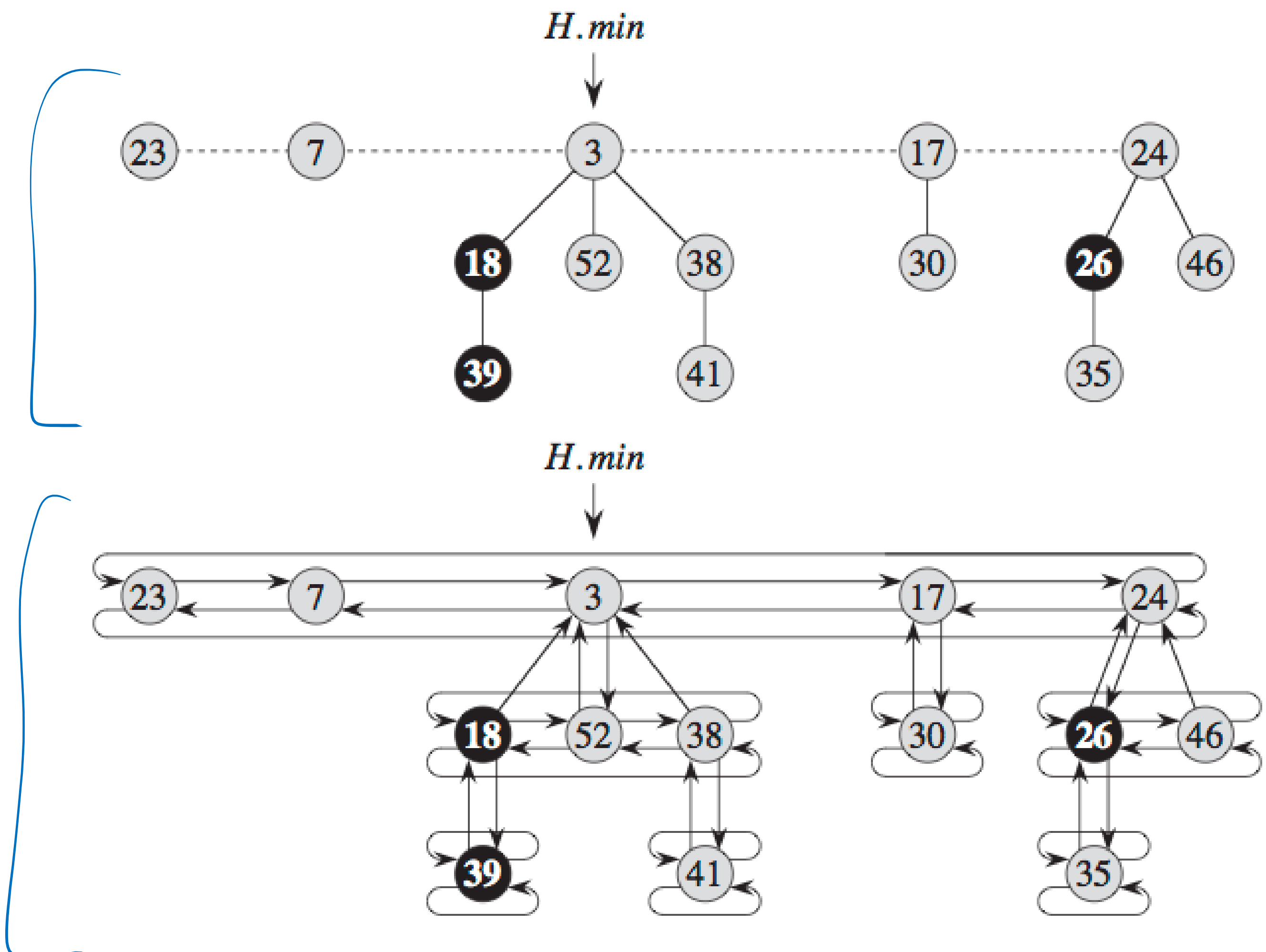


Figure: Cormen et al., Introduction to Algorithms

Simple (Lazy) Operations

Initialize-Heap H:

- $H.rootlist := H.min := null$

Merge heaps H and H':

- concatenate root lists
- update $H.min$

Insert element e into H:

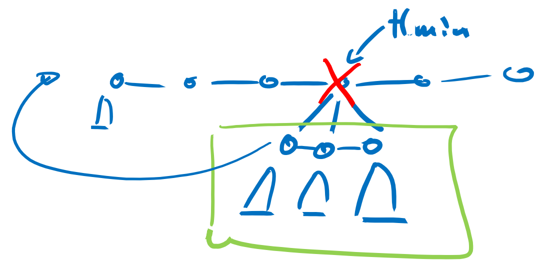
- create new one-node tree containing e → H'
  - mark of root node is set to $\mathbf{false}$
- merge heaps H and H'

Get minimum element of H:

- return $H.min$

Operation Delete-Min

Delete the node with minimum key from H and return its element:



1. $m := H.min$;
2. **if** $H.size > 0$ **then**
3. remove $H.min$ from $H.rootlist$;
4. add $H.min.child$ (list) to $H.rootlist$
5. $\boxed{\boldsymbol{H.Consolidate}();}$

   // Repeatedly merge nodes with equal degree in the root list
   // until degrees of nodes in the root list are distinct.
   // Determine the element with minimum key

6. **return** m

cost: $O(|H.rootlist| + D(n))$

Rank and Maximum Degree

Ranks of nodes, trees, heap:

Node v:

- $rank(v)$: degree of v (number of children of v)

Tree T:

- $rank(T)$: rank (degree) of root node of T

Heap H:

- $rank(H)$: maximum degree (#children) of any node in H

Assumption (n: number of nodes in H):

$$rank(H) \leq D(n) = O(\log n)$$

- for a known function $D(n)$

Merging Two Trees

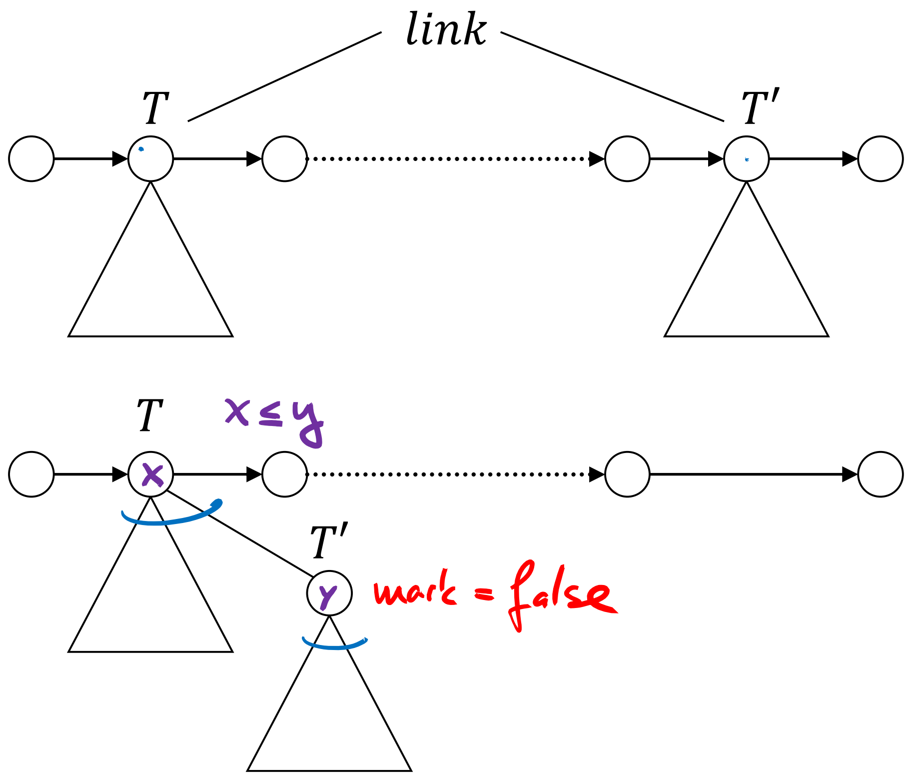
min-heap prop.

Given: Heap-ordered trees T, T' with $rank(T) = rank(T')$

- Assume: min-key of T < min-key of T'

Operation $link(T, T')$:

- Removes tree T' from root list and adds T' to child list of T
- $rank(T) \coloneqq rank(T) + 1$
- $(T'.mark = \textbf{false})$



Consolidation of Root List

Array A pointing to find roots with the same rank:

			T'	T''	…	
0	1	2				$D(n)$

before op

Consolidate:

Time:
$O(|H.rootlist| + D(n))$

1. **for** $i := 0$ **to** $D(n)$ **do** $A[i] :=$ null;
2. **while** $H.rootlist \neq$ null **do**
3. $\quad T :=$ "delete and return first element of $H.rootlist$"
4. $\quad$ **while** $A[rank(T)] \neq$ null **do**
5. $\quad\quad T' := A[rank(T)];$
6. $\quad\quad A[rank(T)] := null;$
7. $\quad\quad T := link(T, T')$
8. $\quad A[rank(T)] := T$
9. Create new $H.rootlist$ and $H.min$

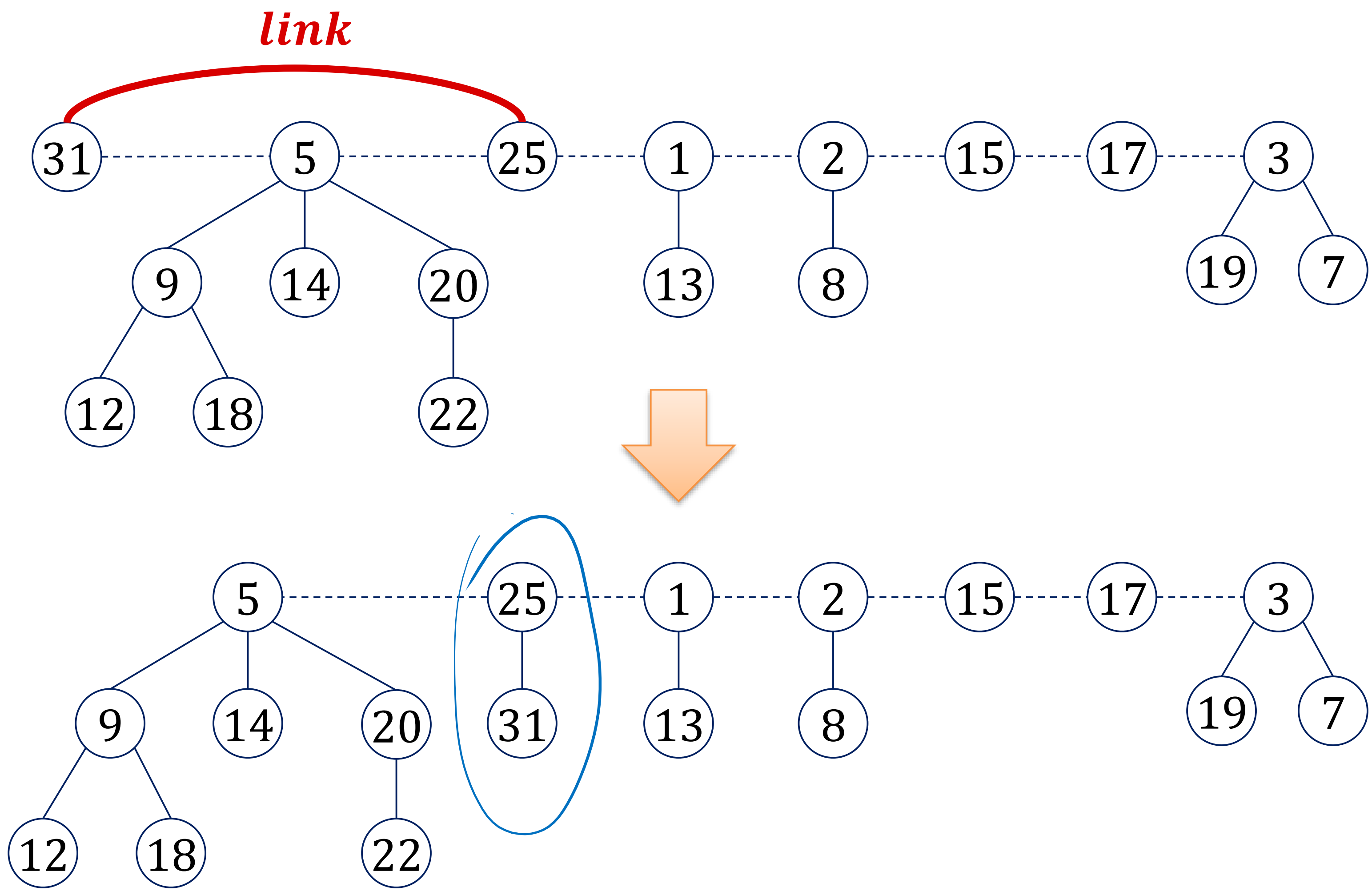
T, T'

time:

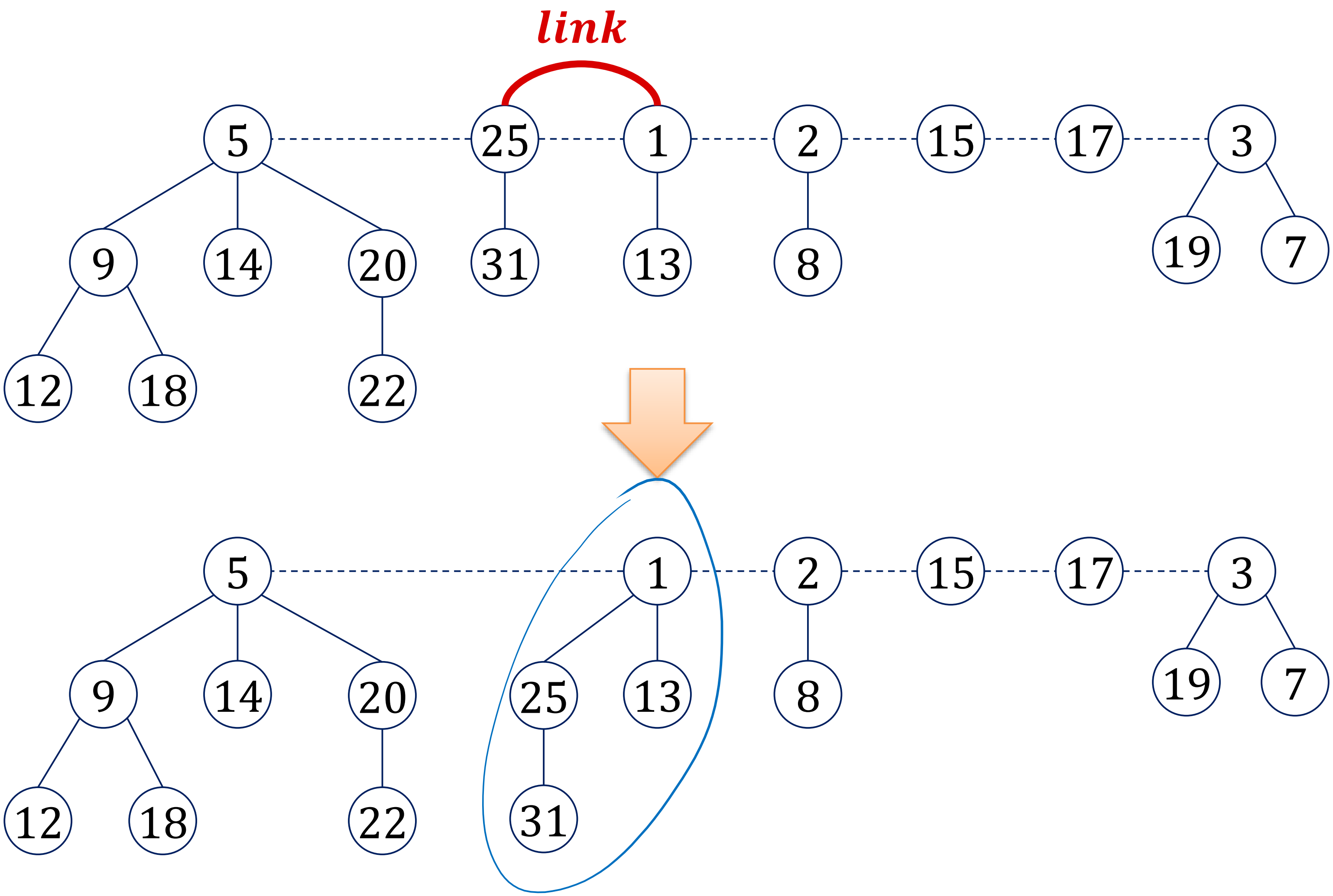
$O(|H.rootlist| + D(n) + \#\text{link op.})$

$\leq |H.root\ list| - 1$

Consolidate Example



Consolidate Example



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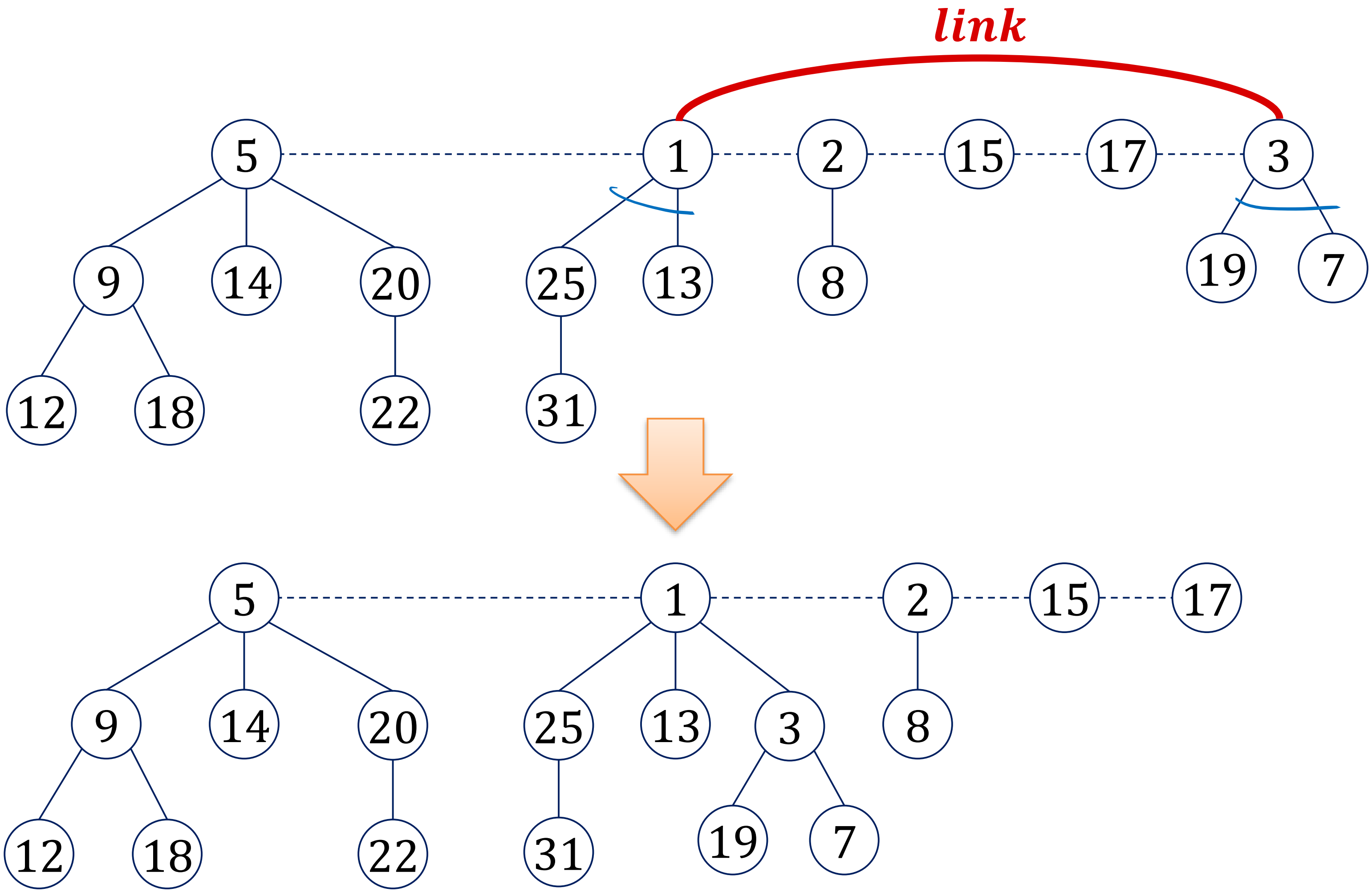
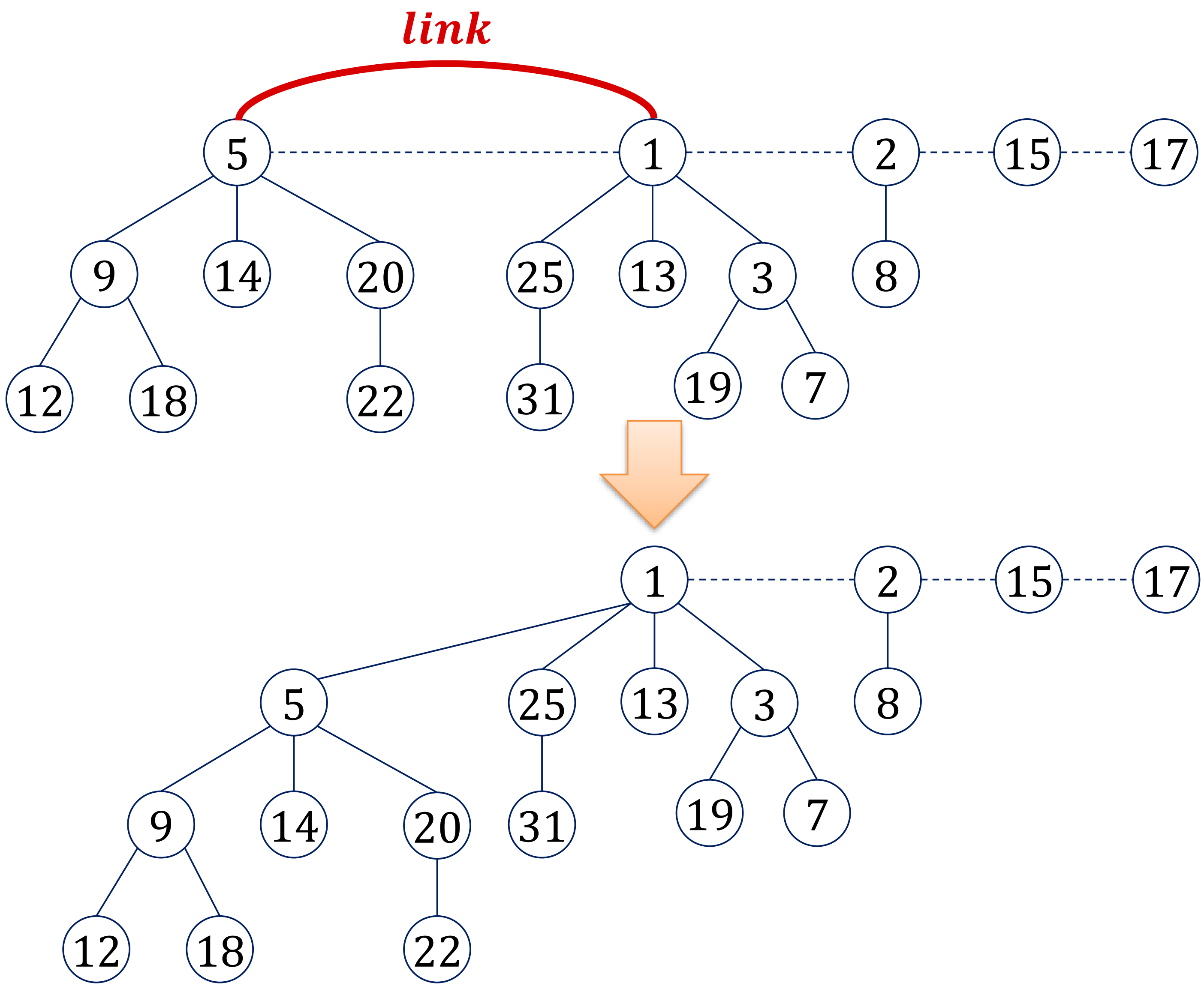


Consolidate Example



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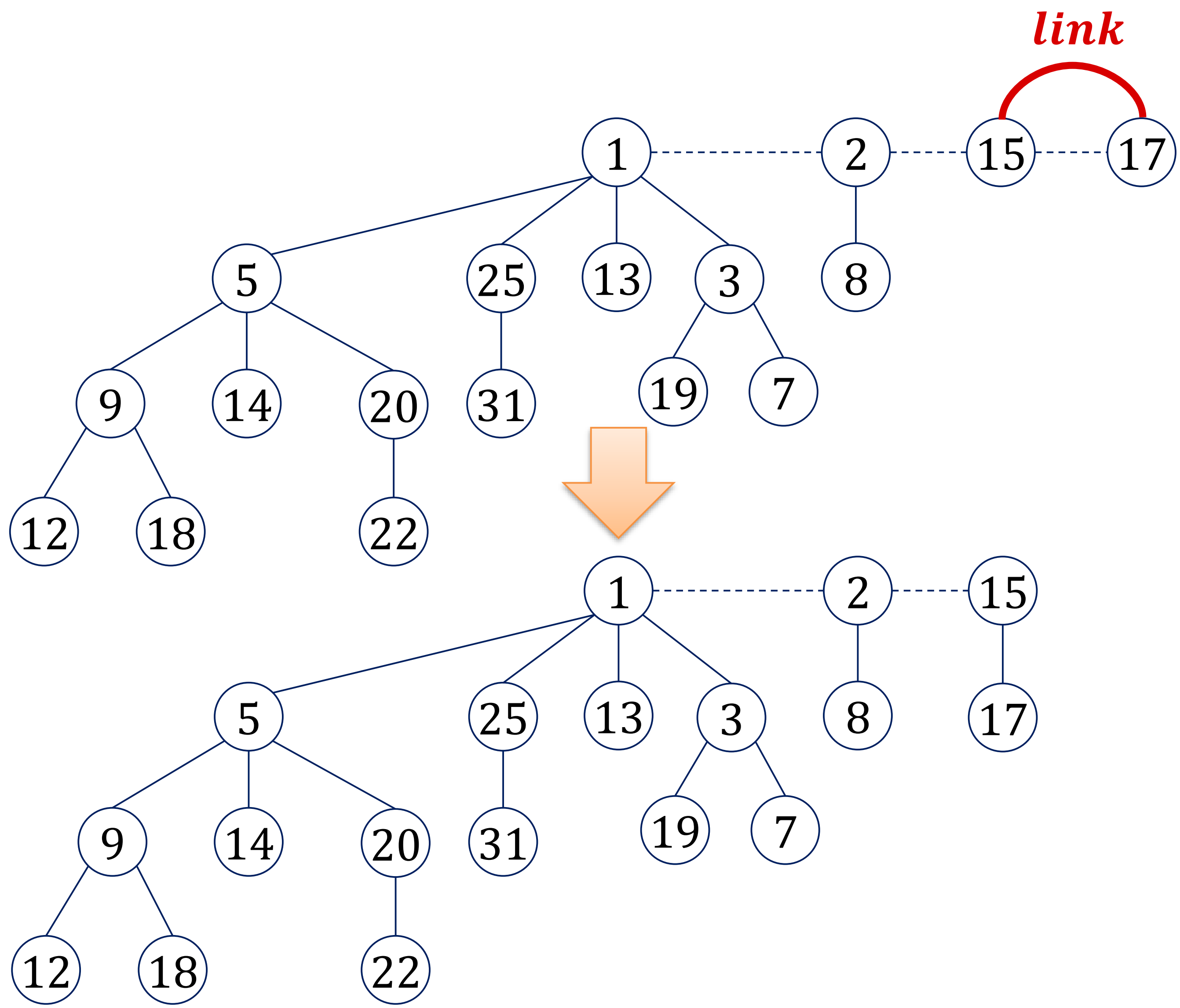
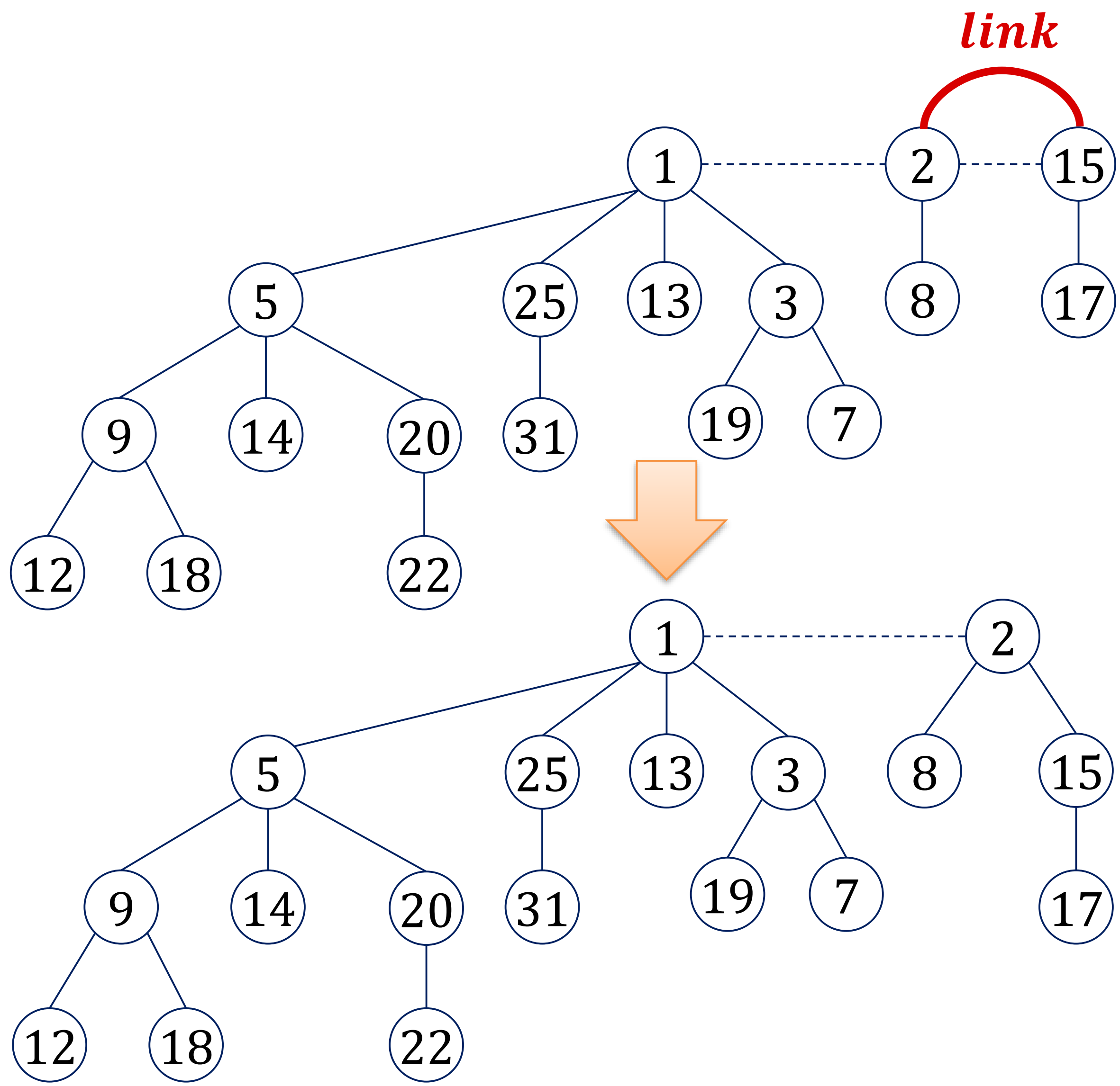


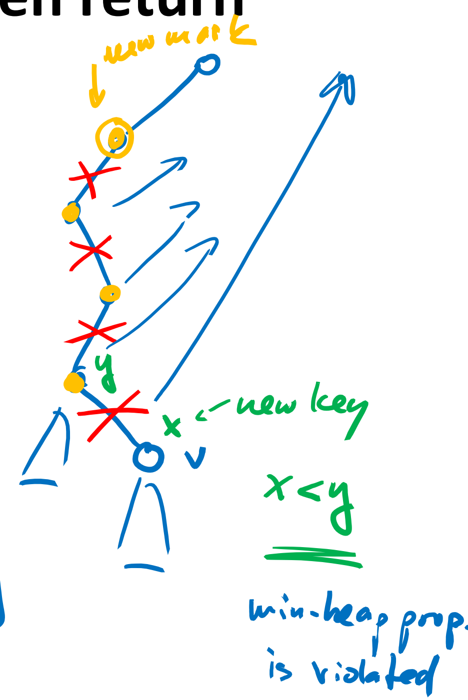
Consolidate Example



Operation Decrease-Key

Decrease-Key$(\boldsymbol{v}, \boldsymbol{x})$**:** (decrease key of node v to new value x)

1. **if** $x \geq v.key$ **then return**;
2. $v.key := x$; update $H.min$;
3. **if** $v \in H.rootlist \ \vee \ x \geq v.parent.key$ **then return**
4. **repeat**
5. $\quad parent := v.parent;$
6. $\quad \boldsymbol{H.cut(v)};$
7. $\quad v := parent;$
8. **until** $\neg(\boldsymbol{v.mark}) \ \vee \ v \in H.rootlist;$
9. **if** $v \notin H.rootlist$ **then** $\boldsymbol{v.mark := \mathbf{true}}$;

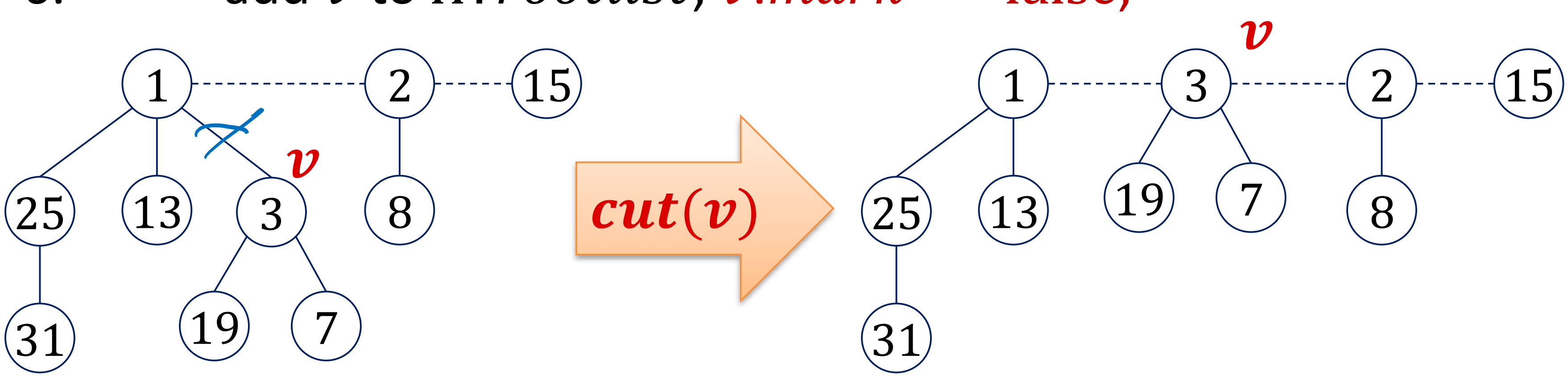


Operation Cut(v)

Operation $H.cut(v)$:

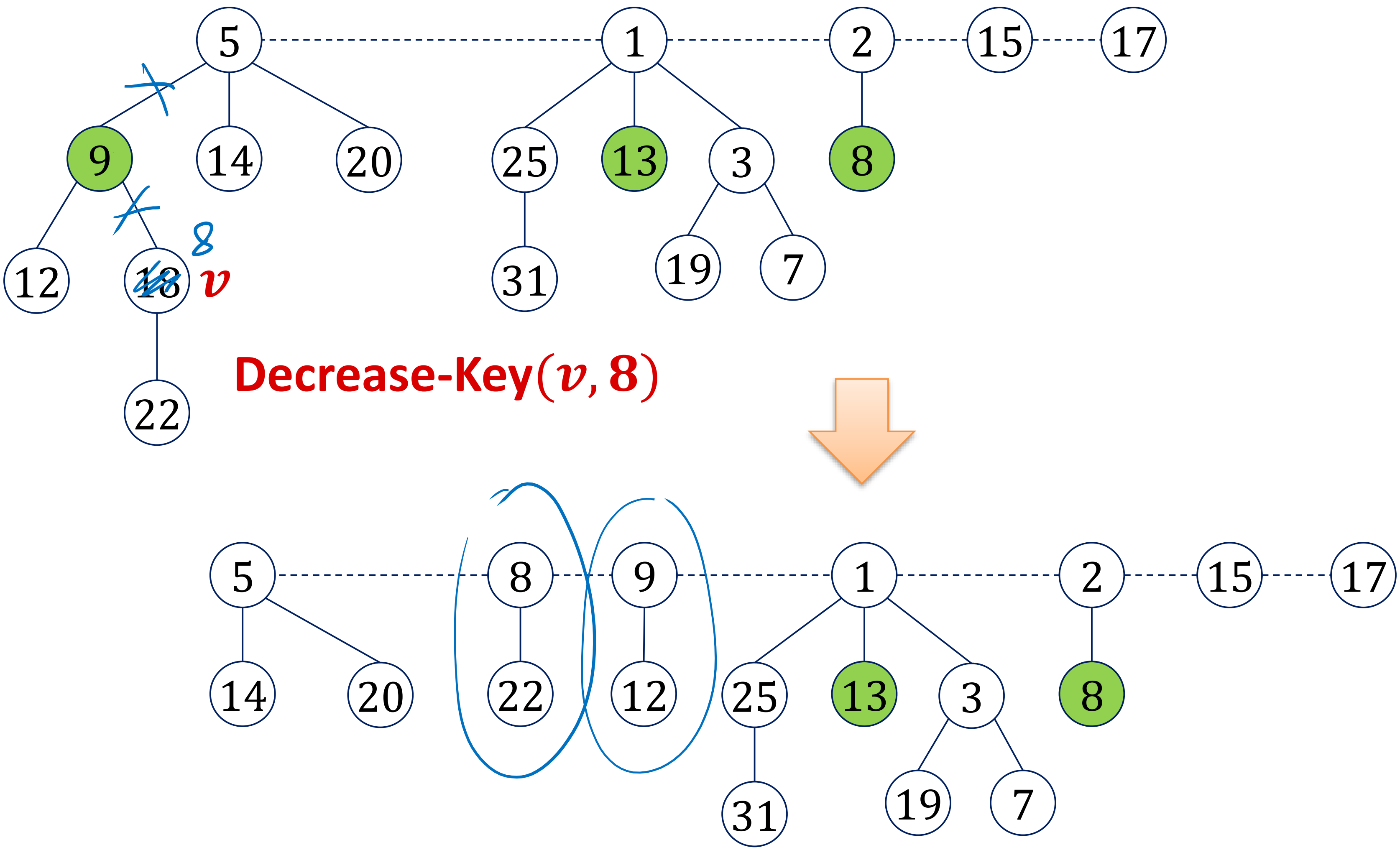
- Cuts v's sub-tree from its parent and adds v to rootlist

1. **if** $v \notin H.rootlist$ **then**
2. // cut the link between v and its parent
3. $rank(v.parent) \coloneqq rank(v.parent) - 1;$
4. remove v from $v.parent.child$ (list)
5. $v.parent \coloneqq \text{null};$
6. add v to $H.rootlist$; $v.mark \coloneqq \text{false};$



Decrease-Key Example

- Green nodes are marked



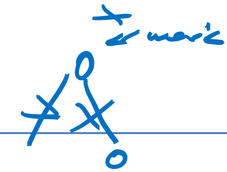
Fibonacci Heaps Marks

- Nodes in the root list (the tree roots) are always unmarked
  → If a node is added to the root list (insert, decrease-key), the mark of the node is set to false.

- Nodes not in the root list can only get marked when a subtree is cut in a decrease-key operation

- A node v is marked if and only if v is not in the root list and v has lost a child since v was attached to its current parent
  - a node can only change its parent by being moved to the root list

Fibonacci Heap Marks



History of a node v:

v is being linked to a node ⇒ $\boldsymbol{v.mark = \text{false}}$

a child of v is cut ⇒ $\boldsymbol{v.mark := \text{true}}$

a second child of v is cut ⇒ $\boldsymbol{H.cut(v);}$
$\boldsymbol{v.mark := false}$

- Hence, the boolean value $v.mark$ indicates whether node v has lost a child since the last time v was made the child of another node.
- Nodes v in the root list always have $v.mark = \text{false}$

Cost of Delete-Min & Decrease-Key

Delete-Min:

1. Delete min. root r and add $r.child$ to $H.rootlist$

   time: $O(1)$

2. Consolidate $H.rootlist$

   time: $O(\text{length of } H.rootlist + D(n))$

- Step 2 can potentially be linear in n (size of H)

Decrease-Key (at node v):

1. If new key $<$ parent key, cut sub-tree of node v

   time: $O(1)$

2. Cascading cuts up the tree as long as nodes are marked

   time: $O(\text{number of consecutive marked nodes})$

- Step 2 can potentially be linear in n

Exercises: Both operations can take $\Theta(n)$ time in the worst case!

Cost of Delete-Min & Decrease-Key

- Cost of delete-min and decrease-key can be $\Theta(n)$…
  - Seems a large price to pay to get insert and merge in $O(1)$ time

- Maybe, the operations are efficient most of the time?
  - It seems to require a lot of operations to get a long rootlist and thus, an expensive consolidate operation
  - In each decrease-key operation, at most one node gets marked:
    We need a lot of decrease-key operations to get an expensive decrease-key operation

- Can we show that the average cost per operation is small?

- We can → requires **amortized analysis**

Fibonacci Heaps Complexity

- Worst-case cost of a single delete-min or decrease-key operation is $\Omega(n)$

- Can we prove a small worst-case amortized cost for delete-min and decrease-key operations?

Recall:

- Data structure that allows operations $O_1, \ldots, O_k$

- We say that operation O_p has amortized cost a_p if for every execution the total time is

$$T \leq \sum_{p=1}^{k} n_p \cdot a_p ,$$

where n_p is the number of operations of type O_p

Amortized Cost of Fibonacci Heaps

- **Initialize-heap**, **is-empty**, **get-min**, **insert**, and **merge** have **worst-case cost $O(1)$** and amortized cost $O(1)$

- **Delete-min** has **amortized cost $O(\log n)$**
- **Decrease-key** has **amortized cost $O(1)$**

- Starting with an empty heap, any sequence of n operations with at most n_d delete-min operations has total cost (time)

$$T = O(n + n_d \log n).$$

- We will now need the marks…

$O(m \log n) \rightarrow O(m + n \log n)$

- Cost for Dijkstra: $O(|E| + |V| \log |V|)$

Fibonacci Heaps: Marks

Cycle of a node:

1. Node v is removed from root list and linked to a node

   $\boldsymbol{v.mark = \text{false}}$

2. Child node u *of* v is cut and added to root list

   $\boldsymbol{v.mark := \text{true}}$

3. Second child of v is cut

   node $\boldsymbol{v}$ is cut as well and moved to root list

   $\boldsymbol{v.mark := \text{false}}$

The boolean value $v.mark$ indicates whether node v has lost a child since the last time v was made the child of another node.

Potential Function

$R = |H.rootList|$

System state characterized by two parameters:

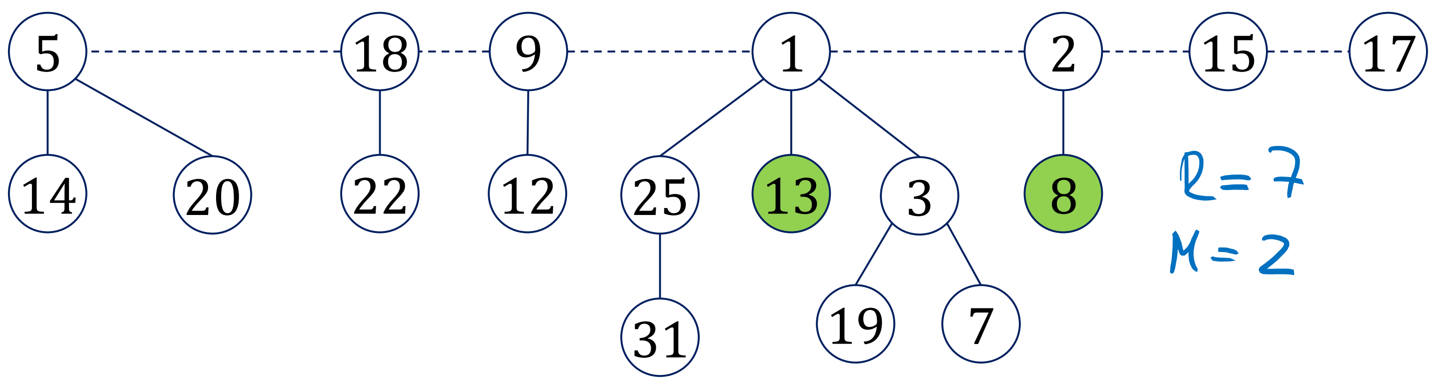
$\Phi = R + M$

- $\boldsymbol{R}$: number of trees (length of $H.rootlist$)
- $\boldsymbol{M}$: number of marked nodes (not in the root list)

Potential function:

$$\boldsymbol{\Phi := R + 2M}$$

Example:



$R = 7$
$M = 2$

- $R = 7, M = 2 \rightarrow \Phi = 11$

Actual Time of Operations

- Operations: ***initialize-heap***, ***is-empty***, ***insert***, ***get-min***, ***merge***

  actual time: $O(1)$

  – Normalize unit time such that

  $$t_{init}, t_{is-empty}, t_{insert}, t_{get-min}, t_{merge} \leq 1$$

- Operation ***delete-min***:
  – Actual time: $O(\text{length of } H.rootlist + D(n))$
  – Normalize unit time such that

  $$t_{del-min} \leq D(n) + \text{ length of } H.rootlist \quad = D(n) + R$$

- Operation **descrease-key**:
  – Actual time: $O(\text{length of path to next unmarked ancestor})$
  – Normalize unit time such that

  $$t_{decr-key} \leq \text{length of path to next unmarked ancestor}$$

Amortized Times

Assume operation i is of type:

- **initialize-heap:**
  - actual time: $t_i \leq 1$, potential: $\Phi_{i-1} = \Phi_i = 0$
  - amortized time: $a_i = t_i + \Phi_i - \Phi_{i-1} \leq 1$

- **is-empty, get-min:**
  - actual time: $t_i \leq 1$, potential: $\Phi_i = \Phi_{i-1}$ (heap doesn't change)
  - amortized time: $a_i = t_i + \Phi_i - \Phi_{i-1} \leq 1$

- **merge:**
  - Actual time: $t_i \leq 1$
  - combined potential of both heaps: $\Phi_i = \Phi_{i-1}$
  - amortized time: $a_i = t_i + \Phi_i - \Phi_{i-1} \leq 1$

Amortized Time of Insert

$\Phi = R + 2M$

Assume that operation i is an *insert* operation:

- **Actual time:** $t_i \leq 1$

- **Potential function:**
  - M remains unchanged (no nodes are marked or unmarked, no marked nodes are moved to the root list)
  - R grows by 1 (one element is added to the root list)

$$M_i = M_{i-1}, \qquad R_i = R_{i-1} + 1$$
$$\Phi_i = \Phi_{i-1} + 1$$

- **Amortized time:**

$$\boldsymbol{a_i = \underbrace{t_i}_{\leq 1} + \underbrace{\Phi_i - \Phi_{i-1}}_{=1} \leq 2}$$

Amortized Time of Delete-Min

Assume that operation i is a *delete-min* operation:

Actual time: $t_i \leq D(n) + |H.rootlist|$

$R_i - R_{i-1} \leq D(n)+1 - |H.rootlist|$

Potential function $\boldsymbol{\Phi = R + 2M}$:

- R: changes from $|H.rootlist|$ to at most $D(n) + 1$
- M: (# of marked nodes that are not in the root list)
  - Number of marks does not ~~increase~~ change

$$M_i = M_{i-1}, \qquad R_i \leq R_{i-1} + D(n) + 1 - |H.rootlist|$$
$$\Phi_i \leq \Phi_{i-1} + D(n) + 1 - |H.rootlist|$$

Amortized Time:

$$\boldsymbol{a_i = t_i + \Phi_i - \Phi_{i-1} \leq 2D(n) + 1}$$

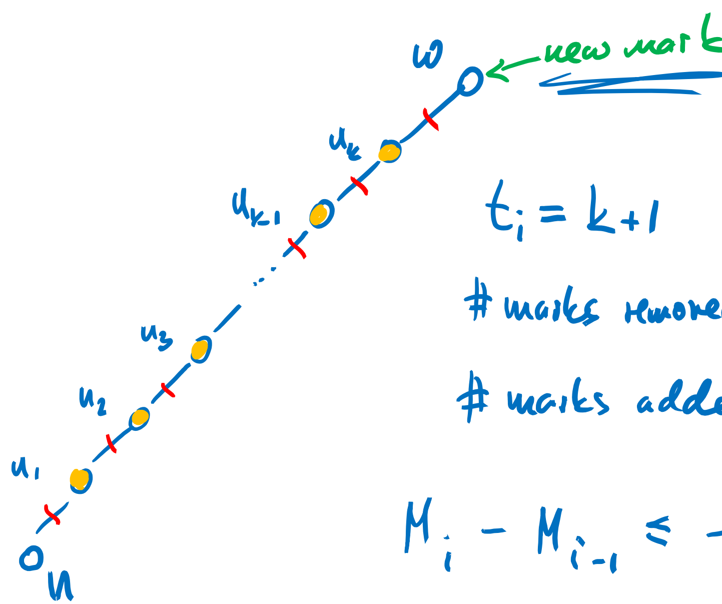
Amortized Time of Decrease-Key

Assume that operation i is a *decrease-key* operation at node u:

Actual time: $t_i \leq$ length of path to next unmarked ancestor v

Potential function $\boldsymbol{\Phi = R + 2M}$:

- Assume, node u and nodes $u_1, \dots, u_k$ are moved to root list
  - $u_1, \dots, u_k$ are marked and moved to root list, $v.\text{mark}$ is set to true



$t_i = k+1$

$\#$ marks removed $\geq k$

$\#$ marks added ≤ 1

$M_i - M_{i-1} \leq -(k-1)$

Amortized Time of Decrease-Key

Assume that operation i is a *decrease-key* operation at node u:

Actual time: $t_i \leq$ length of path to next unmarked ancestor v

Potential function $\boldsymbol{\Phi = R + 2M}$:

- Assume, node u and nodes $u_1, \dots, u_k$ are moved to root list
  - $u_1, \dots, u_k$ are marked and moved to root list, $v.\text{mark}$ is set to true
- $\geq k$ marked nodes go to root list, ≤ 1 node gets newly marked
- R grows by $\leq k + 1$, M grows by 1 and is decreased by $\geq k$

$$R_i \leq R_{i-1} + k + 1, \qquad M_i \leq M_{i-1} + 1 - k$$
$$\Phi_i \leq \Phi_{i-1} + (k+1) - 2(k-1) = \Phi_{i-1} + 3 - k$$

Amortized time:

$$\boldsymbol{a_i = t_i + \Phi_i - \Phi_{i-1} \leq \underbrace{k + 1}_{t_i} + \underbrace{3 - k}_{\Delta\phi} = 4}$$

Complexities Fibonacci Heap

- **Initialize-Heap**: $\boldsymbol{O(1)}$
- **Is-Empty**: $\boldsymbol{O(1)}$
- **Insert**: $\boldsymbol{O(1)}$
- **Get-Min**: $\boldsymbol{O(1)}$
- **Delete-Min**: $\boldsymbol{O(D(n))}$
- **Decrease-Key**: $\boldsymbol{O(1)}$

amortized

- **Merge** (heaps of size m and n, $m \leq n$): $\boldsymbol{O(1)}$

- **How large can $D(n)$ get?**

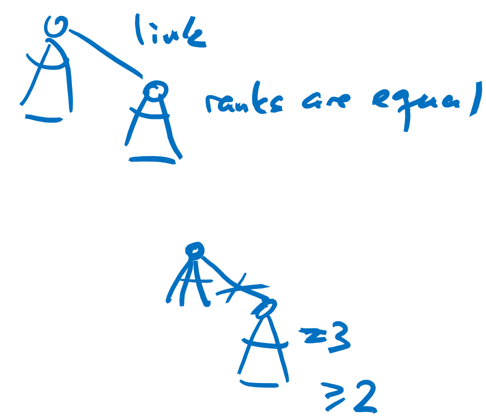
Rank of Children

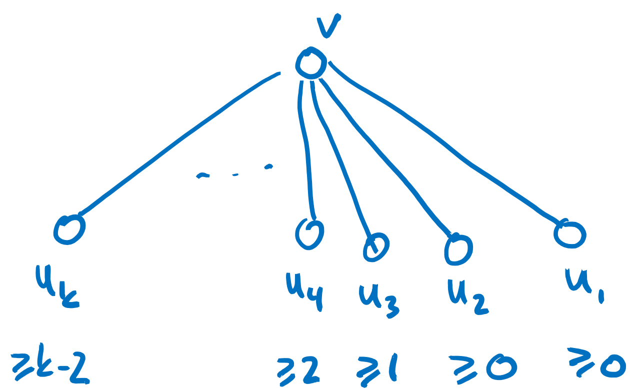
Lemma:

Consider a node v of rank k and let $u_1, \ldots, u_k$ be the children of v in the order in which they were linked to v. Then,

$$\boldsymbol{rank(u_i) \geq i - 2}.$$

Proof:





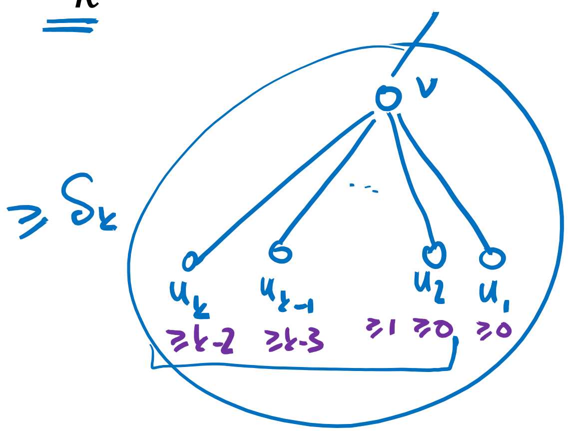
Fibonacci Numbers:

$$F_0 = 0, \qquad F_1 = 1, \qquad \forall k \geq 2: F_k = F_{k-1} + F_{k-2}$$

Lemma:
In a Fibonacci heap, the size of the sub-tree of a node v with rank k is at least F_{k+2}.

Proof:

- S_k: minimum size of the sub-tree of a node of rank k



$S_0 = 1, \quad S_1 = 2$

$k \geq 2$: $\quad S_k \geq 2 + \sum_{i=0}^{k-2} S_i$

by prev. Lemma

Size of Trees

$F_{k+2} = F_k + F_{k+1}$

$$S_0 = 1, \qquad S_1 = 2, \qquad \forall k \geq 2: S_k \geq 2 + \sum_{i=0}^{k-2} S_i$$

- Claim about Fibonacci numbers:

$$\forall k \geq 0: F_{k+2} = 1 + \sum_{i=0}^{k} F_i$$

Proof: (induction on k)

k = 0: $F_2 = 1 + F_0 = F_1 + F_0$ ✓

k > 0: $F_{k+2} = F_k + \underbrace{F_{k+1}}_{} = F_k + 1 + \sum_{i=0}^{k-1} F_i = 1 + \sum_{i=0}^{k} F_i$

I.H.:
$F_{k+1} = 1 + \sum_{i=0}^{k-1} F_i$

$\square$

Size of Trees

$F_0 = 0,\ F_1 = 1$

$$S_0 = 1, S_1 = 2, \forall k \geq 2: S_k \geq 2 + \sum_{i=0}^{k-2} S_i, \qquad F_{k+2} = 1 + \sum_{i=0}^{k} F_i$$

- Claim of lemma: $S_k \geq F_{k+2}$

Ind. on k

base: $S_0 \geq F_2$ $(S_0 = 1,\ F_2 = 1)$ ✓ $S_1 \geq F_3$ $(S_1 = 2,\ F_3 = 2)$ ✓

step: $k \geq 2$

$$S_k \geq 2 + \sum_{i=0}^{k-2} S_i \overset{(I.H.)}{\geq} 2 + \sum_{i=0}^{k-2} F_{i+2}$$

$$= 2 + \sum_{j=2}^{k} F_j$$

$$= 1 + \sum_{j=0}^{k} F_j = F_{k+2}$$

$\square$

Size of Trees

Lemma:
In a Fibonacci heap, the size of the sub-tree of a node v with rank k is at least F_{k+2}.

Theorem:
The maximum rank of a node in a Fibonacci heap of size n is at most

$$\boldsymbol{D(n) = O(\log n)}\,.$$

Proof:

- The Fibonacci numbers grow exponentially:

0

$$F_k = \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right)$$

- For $D(n) \geq k$, we need $n \geq F_{k+2}$ nodes.

Summary: Binomial and Fibonacci Heaps

	Binary Heap	Fibonacci Heap
initialize	$O(1)$	$O(1)$
insert	$O(\log n)$	$O(1)$
get-min	$O(1)$	$O(1)$
delete-min	$O(\log n)$	$O(\log n)$ *
decrease-key	$O(\log n)$	$O(1)$ *
merge	$O(m \cdot \log n)$	$O(1)$
is-empty	$O(1)$	$O(1)$

* **amortized time**