# Chapter 6
# Graph Algorithms

## Algorithm Theory
## WS 2017/18

**Fabian Kuhn**

# Ford-Fulkerson Algorithm

- Improve flow using an augmenting path as long as possible:

1. Initially, $f(e) = 0$ for all edges $e \in E$, $G_f = G$
2. **while** there is an augmenting $s$-$t$-path $P$ in $G_f$ **do**
3. Let $P$ be an augmenting $s$-$t$-path in $G_f$;
4. $f' \coloneqq \text{augment}(f, P)$;
5. update $f$ to be $f'$;
6. update the residual graph $G_f$
7. **end**;

# Conclusions Ford Fulkerson Algorithm

**Theorem: (Max-Flow Min-Cut Theorem)**

In every flow network, the maximum value of an $s$-$t$ flow is equal to the minimum capacity of an $s$-$t$ cut.

**Theorem: (Integer-Valued Flows)**

If all capacities in the flow network are integers, then there is a maximum flow $f$ for which the flow $f(e)$ of every edge $e$ is an integer.

# Strongly Polynomial Algorithm

- Time of regular Ford-Fulkerson algorithm with integer capacities:

$$O(mC)$$

- Time of algorithm with scaling parameter:

$$O(m^2 \log C)$$

- $O(\log C)$ is polynomial in the size of the input, but not in $n$

- Can we get an algorithm that runs in time polynomial in $n$?

- Always picking a shortest augmenting path leads to running time

$$O(m^2 n)$$

  – also works for arbitrary real-valued weights

# Other Algorithms

- There are many other algorithms to solve the maximum flow problem, for example:

- **Preflow-push algorithm:**
  - Maintains a preflow (∀ nodes: inflow ≥ outflow)
  - Alg. guarantees: As soon as we have a flow, it is optimal
  - Detailed discussion in 2012/13 lecture
  - Running time of basic algorithm: $O(m \cdot n^2)$
  - Doing steps in the "right" order: $O(n^3)$

- **Current best known complexity:** $\boldsymbol{O(m \cdot n)}$
  - For graphs with $m \geq n^{1+\epsilon}$ (for every constant $\epsilon > 0$) [King,Rao,Tarjan 1992/1994]
  - For sparse graphs with $m \leq n^{16/15-\delta}$ [Orlin, 2013]

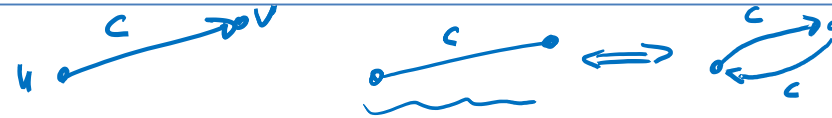
# Maximum Flow Applications

- Maximum flow has many applications

- Reducing a problem to a max flow problem can even be seen as an important algorithmic technique

- Examples:
  - related network flow problems
  - computation of small cuts
  - computation of matchings
  - computing disjoint paths
  - scheduling problems
  - assignment problems with some side constraints
  - …

# Undirected Edges and Vertex Capacities



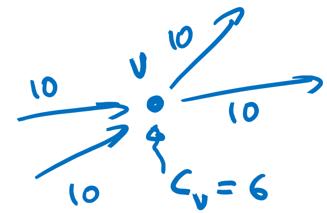
**Undirected Edges:**

- Undirected edge $\{u, v\}$: add edges $(u, v)$ and $(v, u)$ to network

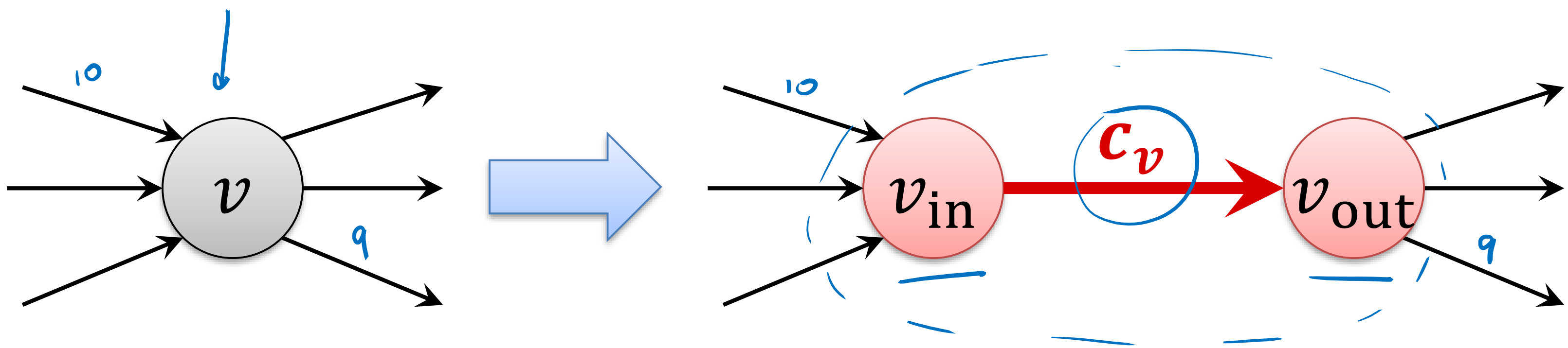
**Vertex Capacities:**

- Not only edges, but also (or only) nodes have capacities
- Capacity $c_v$ of node $v \notin \{s, t\}$:

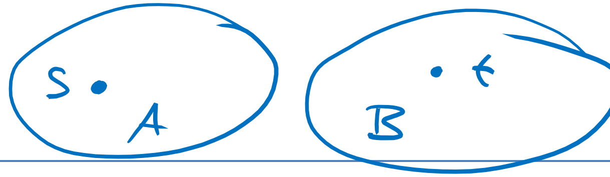
$$f^{\text{in}}(v) = f^{\text{out}}(v) \leq c_v$$



- Replace node $v$ by edge $e_v = \{v_{\text{in}}, v_{\text{out}}\}$:



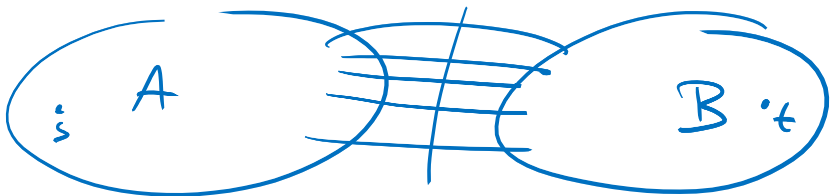
# Minimum $s$-$t$ Cut




**Given:** undirected graph $G = (V, E)$, nodes $s, t \in V$

**$s$-$t$ cut:** Partition $(A, B)$ of $V$ such that $s \in A$, $t \in B$

**Size of cut $(A, B)$:** number of edges crossing the cut




running time:

$O(m \cdot n)$

size of cut = # edges crossing the cut

**Objective:** find $s$-$t$ cut of minimum size

create flow network

1) make edges directed

2) edge cap. = 1

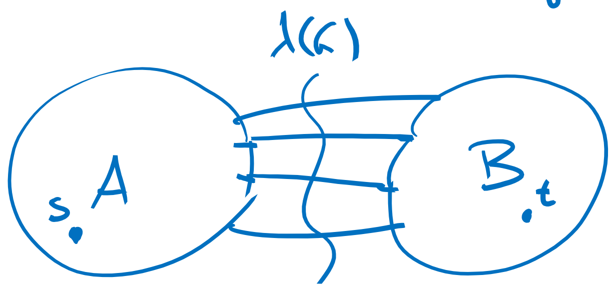
Size of cut in G = cap. cut in flow network

# Edge Connectivity

**Definition:** A graph $G = (V, E)$ is $k$-edge connected for an integer $k \geq 1$ if the graph $G_X = (V, E \setminus X)$ is connected for every edge set

$$X \subseteq E, |X| \leq k - 1.$$

need to remove ≥ k edges to make G disconnected




edge connectivity λ(G):
max. k s.t. G is
k-edge connected

**Goal:** Compute edge connectivity $\lambda(G)$ of $G$
(and edge set $X$ of size $\lambda(G)$ that divides $G$ into $\geq 2$ parts)

- minimum set $X$ is a minimum $s$-$t$ cut for some $s, t \in V$
  - Actually for all $s, t$ in different components of $G_X = (V, E \setminus X)$

need to call min s-t-cut alg. n-1 times

- Possible algorithm: fix $s$ and find min $s$-$t$ cut for all $t \neq s$

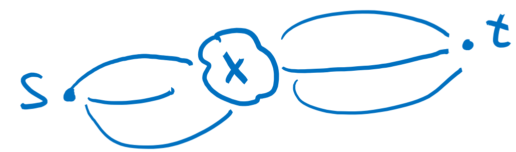
running time: $O(mn^2)$

# Minimum $s$-$t$ Vertex-Cut

**Given:** undirected graph $G = (V, E)$, nodes $s, t \in V$

**$s$-$t$ vertex cut:** Set $X \subset V$ such that $s, t \notin X$ and $s$ and $t$ are in different components of the sub-graph $G[V \setminus X]$ induced by $V \setminus X$

**Size of vertex cut:** $|X|$




**Objective:** find $s$-$t$ vertex-cut of minimum size

- Replace undirected edge $\{u, v\}$ by $(u, v)$ and $(v, u)$
- Compute max $s$-$t$ flow for edge capacities $\infty$ and node capacities

$$c_v = 1 \text{ for } v \neq s, t$$

time: $O(m \cdot n)$

- Replace each node $v$ by $v_{\text{in}}$ and $v_{\text{out}}$:


- Min edge cut corresponds to min vertex cut in $G$

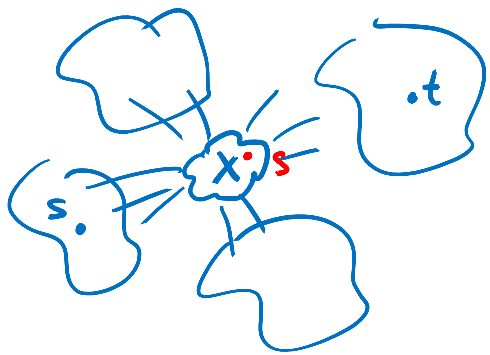
# Vertex Connectivity

$\tilde{O}(n^2)$

**Definition:** A graph $G = (V, E)$ is $k$-vertex connected for an integer $k \geq 1$ if the sub-graph $G[V \setminus X]$ induced by $V \setminus X$ is connected for every edge set

$$X \subseteq V, |X| \leq k - 1.$$

need to remove at least $k$ nodes to disconnect $G$




vertex conn. $\kappa(G)$

max. $k$ s.t.
$G$ is $k$-vertex connected

**Goal:** Compute vertex connectivity $\kappa(G)$ of $G$

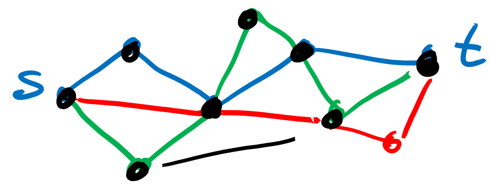
(and node set $X$ of size $\kappa(G)$ that divides $G$ into $\geq 2$ parts)

- Compute minimum $s$-$t$ vertex cut for ~~fixed~~ all $s$ and all $t \neq s$

# Edge-Disjoint Paths

**Given:** Graph $G = (V, E)$ with nodes $s, t \in V$

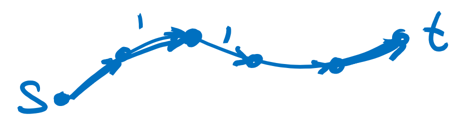
**Goal:** Find as many edge-disjoint $s$-$t$ paths as possible




**Solution:**

- Find max $s$-$t$ flow in $G$ with edge capacities $c_e = 1$ for all $e \in E$

Flow $f$ induces $|f|$ edge-disjoint paths:

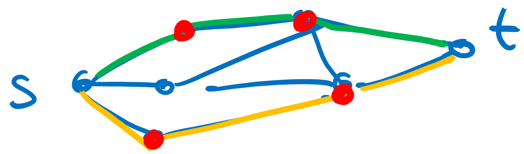



- Integral capacities → can compute integral max flow $f$
- Get $|f|$ edge-disjoint paths by greedily picking them
- Correctness follows from flow conservation $f^{\text{in}}(v) = f^{\text{out}}(v)$

# Vertex-Disjoint Paths

**Given:** Graph $G = (V, E)$ with nodes $s, t \in V$

**Goal:** Find as many internally vertex-disjoint $s$-$t$ paths as possible




**Solution:**

- Find max $s$-$t$ flow in $G$ with node capacities $c_v = 1$ for all $v \in V$

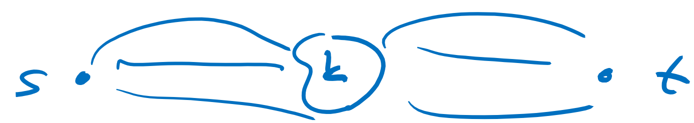
Flow $f$ induces $|f|$ vertex-disjoint paths:

- Integral capacities → can compute integral max flow $f$
- Get $|f|$ vertex-disjoint paths by greedily picking them
- Correctness follows from flow conservation $f^{\text{in}}(v) = f^{\text{out}}(v)$

# Menger's Theorem

**Theorem: (edge version)**
For every graph $G = (V, E)$ with nodes $s, t \in V$, the size of the minimum $s$-$t$ (edge) cut equals the maximum number of pairwise edge-disjoint paths from $s$ to $t$.


**Theorem: (node version)**
For every graph $G = (V, E)$ with nodes $s, t \in V$, the size of the minimum $s$-$t$ vertex cut equals the maximum number of pairwise internally vertex-disjoint paths from $s$ to $t$




- Both versions can be seen as a special case of the max flow min cut theorem

# Baseball Elimination

| Team $i$ | Wins $w_i$ | Losses $\ell_i$ | To Play $r_i$ | Against = $r_{ij}$ | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | NY | Balt. | T. Bay | Tor. | Bost. |
| New York | 81 | 69 | 12 | - | 2 | 5 | 2 | 3 |
| Baltimore | 79 | 77 | 6 | 2 | - | 2 | 1 | 1 |
| Tampa Bay | 79 | 74 | 9 | 5 | 2 | - | 1 | 1 |
| Toronto | 76 | 80 | 6 | 2 | 1 | 1 | - | 2 |
| Boston | 71 | 84 | 7 | 3 | 1 | 1 | 2 | - |

- Only wins/losses possible (no ties), winner: team with most wins
- Which teams can still win (as least as many wins as top team)?
- Boston is eliminated (cannot win):
  - Boston can get at most 78 wins, New York already has 81 wins
- If for some $i, j$: $w_i + r_i < w_j$ → team $i$ is eliminated
- Sufficient condition, but not a necessary one!

# Baseball Elimination

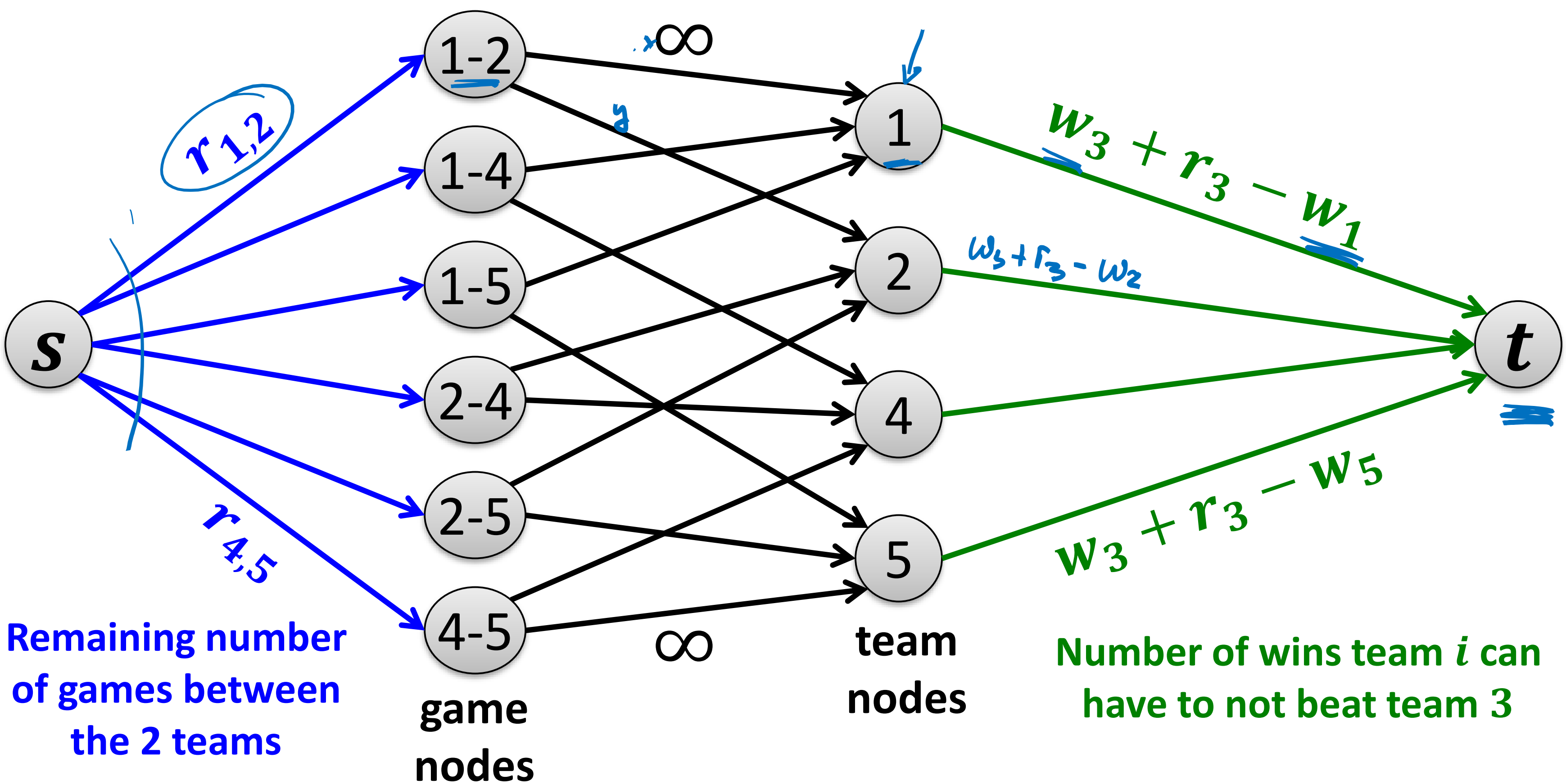
| Team $i$ | Wins $w_i$ | Losses $\ell_i$ | To Play $r_i$ | Against = $r_{ij}$ | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | NY | Balt. | T. Bay | Tor. | Bost. |
| New York | 81 | 69 | 12 | - | 2 | 5 | 2 | 3 |
| Baltimore | 79 | 77 | 6 | 2 | - | 2 | 1 | 1 |
| Tampa Bay | 79 | 74 | 9 | 5 | 2 | - | 1 | 1 |
| Toronto | 76 | 80 | 6 | 2 | 1 | 1 | - | 2 |
| Boston | 71 | 84 | 7 | 3 | 1 | 1 | 2 | - |

- Can Toronto still finish first?
- Toronto can get $82 > 81$ wins, but:
  NY and Tampa have to play 5 more times against each other
  → if NY wins two, it gets 83 wins, otherwise, Tampa has 83 wins
- Hence: Toronto cannot finish first
- How about the others? How can we solve this in general?

# Max Flow Formulation

$w_i$ : # wins so far (team $i$)
$r_i$ : # remaining games

- Can team 3 finish with most wins?




- Team 3 can finish first iff all source-game edges are saturated

# Reason for Elimination

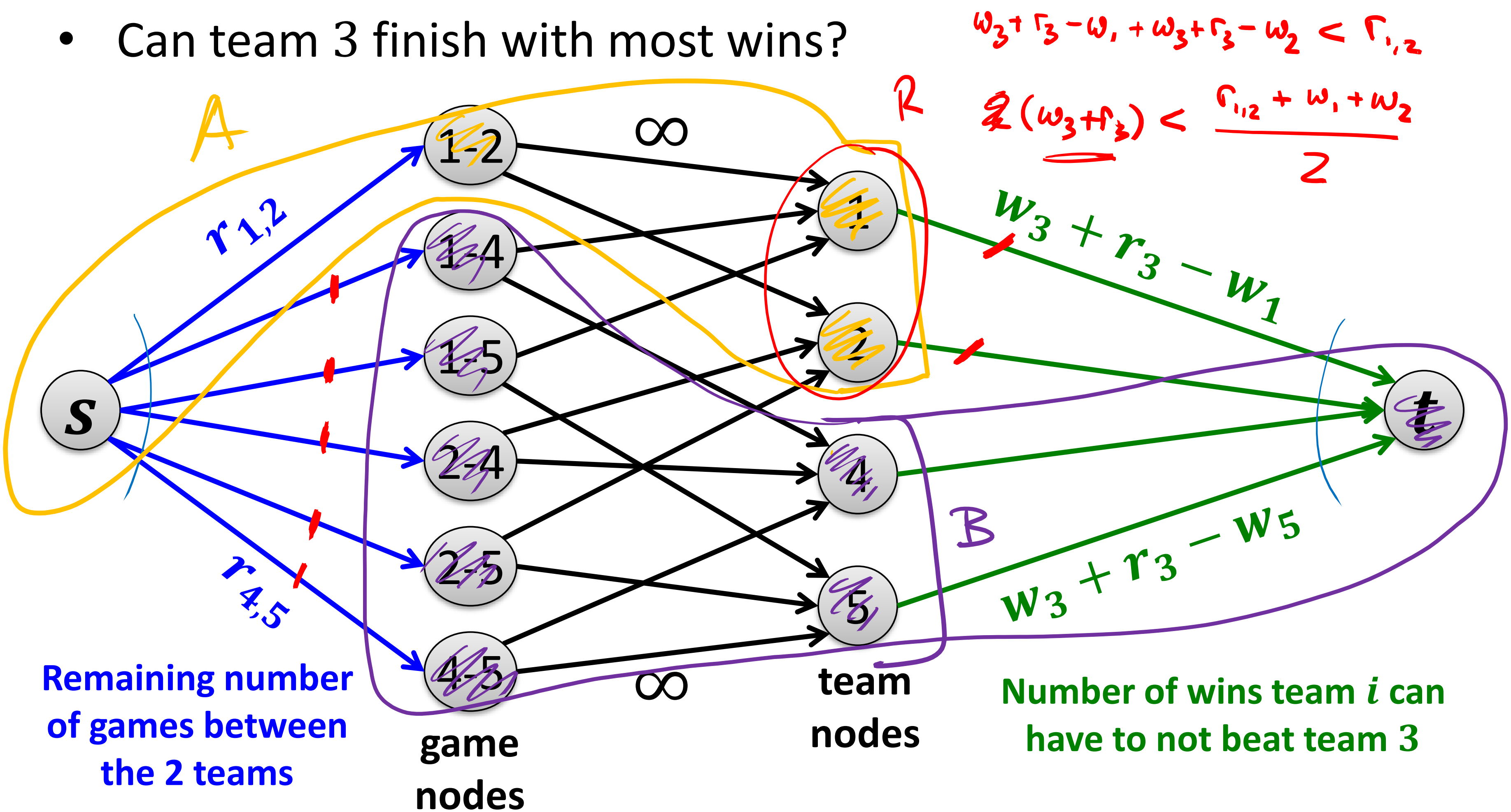
AL East: Aug 30, 1996

| Team $i$ | Wins $w_i$ | Losses $\ell_i$ | To Play $r_i$ | Against = $r_{ij}$ | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | NY | Balt. | Bost. | Tor. | Detr. |
| New York | 75 | 59 | 28 | - | 3 | 8 | 7 | 3 |
| Baltimore | 71 | 63 | 28 | 3 | - | 2 | 7 | 4 |
| Boston | 69 | 66 | 27 | 8 | 2 | - | 0 | 0 |
| Toronto | 63 | 72 | 27 | 7 | 7 | 0 | - | 0 |
| **Detroit** | 49 | 86 | 27 | 3 | 4 | 0 | 0 | - |

- Detroit could finish with $49 + 27 = 76$ wins
- Consider $R = \{\text{NY}, \text{Bal}, \text{Bos}, \text{Tor}\}$
  - Have together already won $w(R) = 278$ games
  - Must together win at least $r(R) = 27$ more games
- On average, teams in $R$ win $\frac{278+27}{4} = 76.25$ games

# Reason for Elimination

- Can team 3 finish with most wins?




- Team 3 cannot finish first ⟺ min cut of size < "all blue edges"

# Reason for Elimination

**Certificate of elimination:**

$$R \subseteq X, \qquad w(R) := \underbrace{\sum_{i \in R} w_i}_{\text{\#wins of nodes in } R}, \qquad r(R) := \underbrace{\sum_{i,j \in R} r_{i,j}}_{\text{\#remaining games among nodes in } R}$$

Team $x \in X$ is eliminated by $R$ if

$$\frac{w(R) + r(R)}{|R|} > w_x + r_x.$$

# Reason for Elimination

**Theorem:** Team $x$ is eliminated if and only if there exists a subset $R \subseteq X$ of the teams $X$ such that $x$ is eliminated by $R$.

**Proof Idea:**

- Minimum cut gives a certificate…
- If $x$ is eliminated,  max flow solution does not saturate all outgoing edges of the source.
- Team nodes of unsaturated source-game edges are saturated
- Source side of min cut contains all teams of saturated team-dest. edges of unsaturated source-game edges
- Set of team nodes in source-side of min cut give a certificate $R$

# Circulations with Demands

**Given:** Directed network with positive edge capacities

**Sources & Sinks:** Instead of one source and one destination, several sources that generate flow and several sinks that absorb flow.

**Supply & Demand:** sources have supply values, sinks demand values

**Goal:** Compute a flow such that source supplies and sink demands are exactly satisfied

- The circulation problem is a feasibility rather than a maximization problem

# Circulations with Demands: Formally

**Given:** Directed network $G = (V, E)$ with

- Edge capacities $c_e > 0$ for all $e \in E$
- Node demands $d_v \in \mathbb{R}$ for all $v \in V$
  - $d_v > 0$: node needs flow and therefore is a sink
  - $d_v < 0$: node has a supply of $-d_v$ and is therefore a source
  - $d_v = 0$: node is neither a source nor a sink

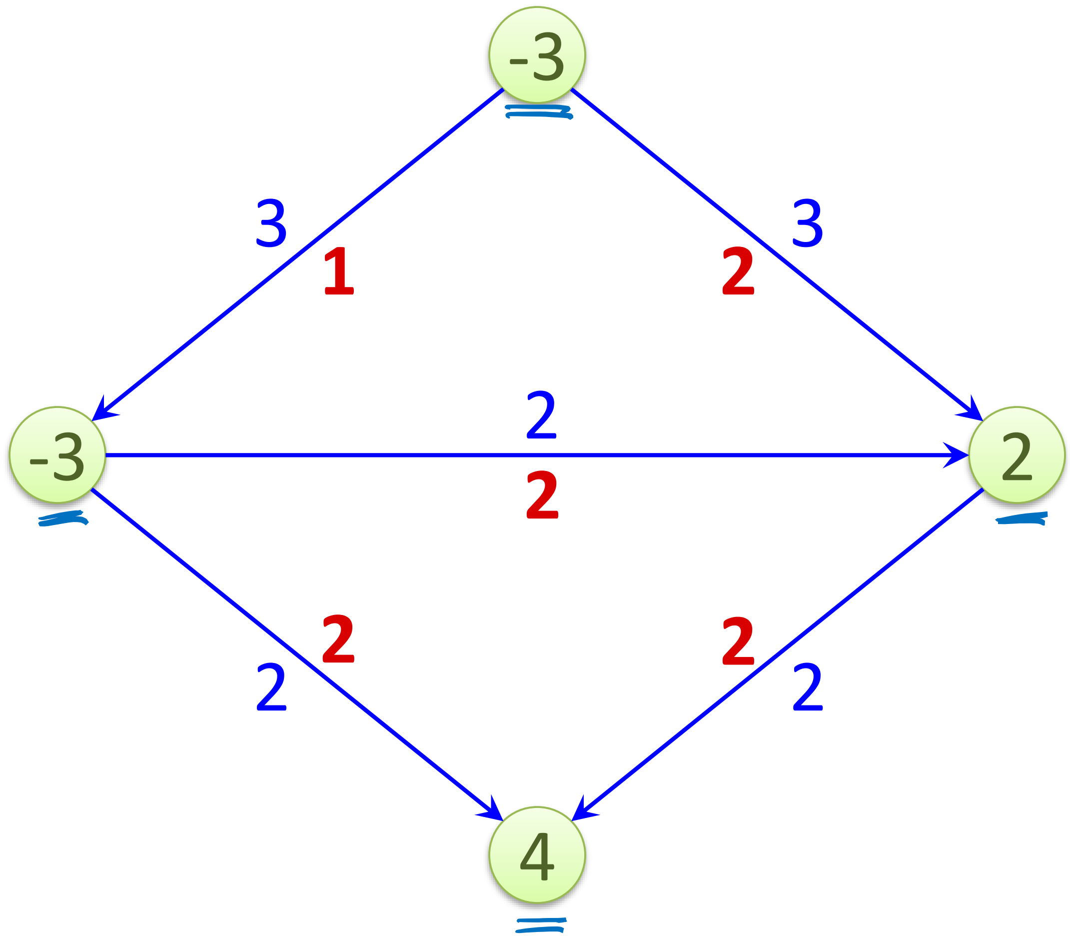
$d_v = 3$ $d_v = -3$

**Flow:** Function $f : E \to \mathbb{R}_{\geq 0}$ satisfying

- *Capacity Conditions*: $\forall e \in E$: $0 \leq f(e) \leq c_e$
- *Demand Conditions*: $\forall v \in V$: $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

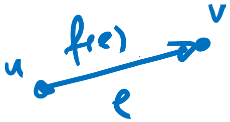
**Objective:** Does a flow $f$ satisfying all conditions exist?
If yes, find such a flow $f$.

# Example



# Condition on Demands

**Claim:** If there exists a feasible circulation with demands $d_v$ for $v \in V$, then




$$\sum_{v \in V} d_v = 0.$$

$d(v) = f^{in}(v) - f^{out}(v)$

**Proof:**

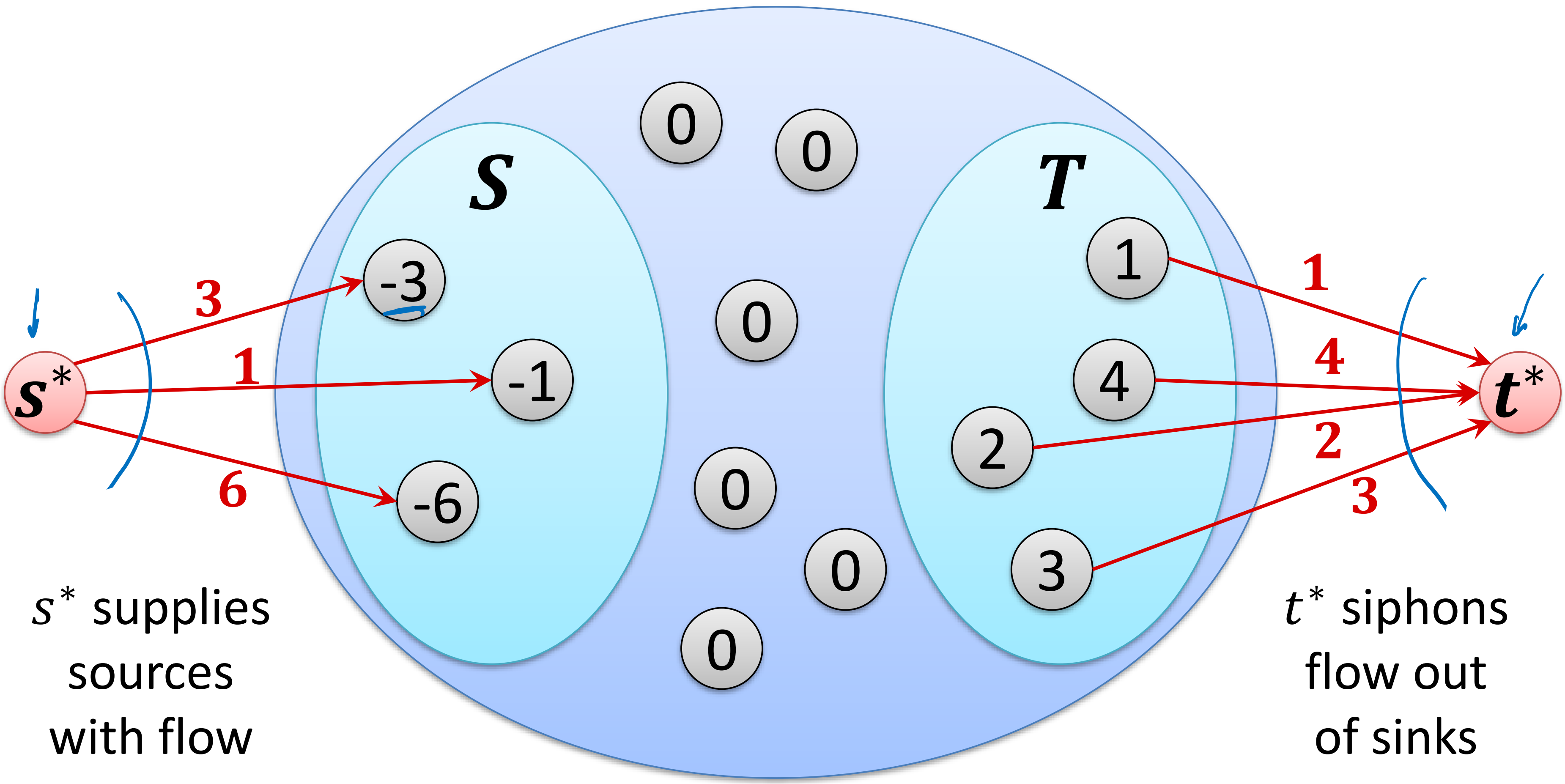
- $\sum_v d_v = \sum_v \left(f^{\text{in}}(v) - f^{\text{out}}(v)\right) = \sum_v f^{in}(v) - \sum_v f^{out}(v) = 0$
- $f(e)$ of each edge $e$ appears twice in the above sum with different signs → overall sum is 0

**Total supply = total demand:**

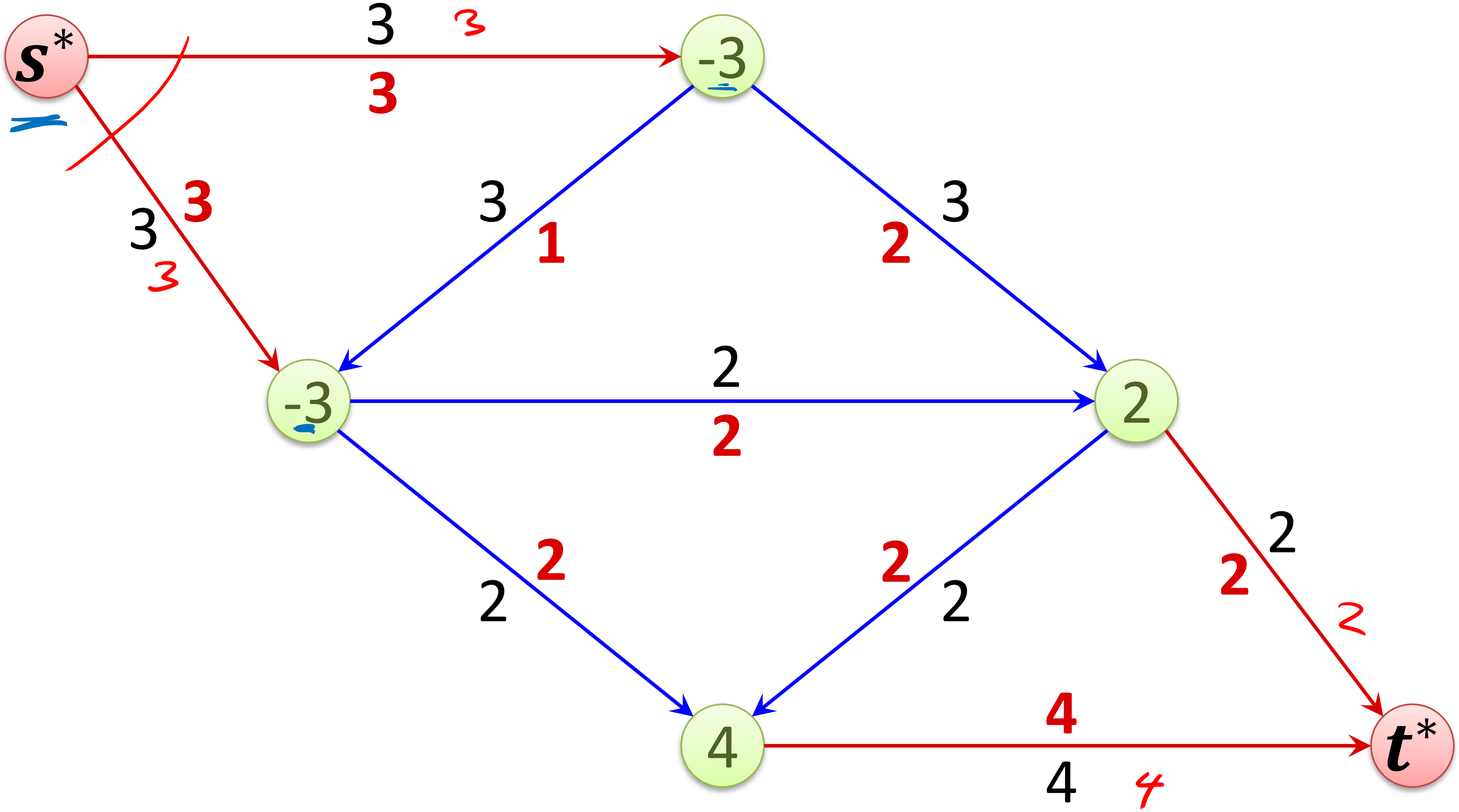
$$\text{Define } \boldsymbol{D} := \sum_{v: d_v > 0} \boldsymbol{d_v} = \sum_{v: d_v < 0} -\boldsymbol{d_v}$$

# Reduction to Maximum Flow

- Add "super-source" $s^*$ and "super-sink" $t^*$ to network



# Example



# Formally…

**Reduction:** Get graph $G'$ from graph as follows

- Node set of $G'$ is $V \cup \{s^*, t^*\}$
- Edge set is $E$ and edges
  - $(s^*, v)$ for all $v$ with $d_v < 0$, capacity of edge is $-d_v$ $=|d_v|$
  - $(v, t^*)$ for all $v$ with $d_v > 0$, capacity of edge is $d_v$

**Observations:**

- Capacity of min $s^*$-$t^*$ cut is at most $D$ (e.g., the cut $(s^*, V \cup \{t^*\})$)
- A feasible circulation on $G$ can be turned into a feasible flow of value $D$ of $G'$ by saturating all $(s^*, v)$ and $(v, t^*)$ edges.
- Any flow of $G'$ of value $D$ induces a feasible circulation on $G$
  - $(s^*, v)$ and $(v, t^*)$ edges are saturated
  - By removing these edges, we get exactly the demand constraints

# Circulation with Demands

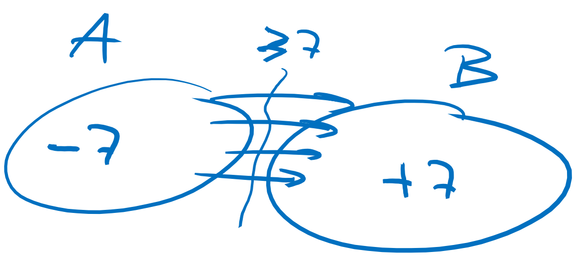
**Theorem:** There is a feasible circulation with demands $d_v, v \in V$ on graph $G$ if and only if there is a flow of value $D$ on $G'$.

- If all capacities and demands are integers, there is an integer circulation

The max flow min cut theorem also implies the following:

**Theorem:** The graph $G$ has a feasible circulation with demands $d_v, v \in V$ if and only if for all cuts $(A, B)$,

$$\sum_{v \in B} d_v \leq c(A, B).$$



# Circulation: Demands and Lower Bounds

**Given:** Directed network $G = (V, E)$ with

- Edge capacities $c_e > 0$ and **lower bounds $0 \leq \ell_e \leq c_e$ for $e \in E$**
- Node demands $d_v \in \mathbb{R}$ for all $v \in V$
  - $d_v > 0$: node needs flow and therefore is a sink
  - $d_v < 0$: node has a supply of $-d_v$ and is therefore a source
  - $d_v = 0$: node is neither a source nor a sink

**Flow:** Function $f: E \rightarrow \mathbb{R}_{\geq 0}$ satisfying

- *Capacity Conditions*: $\forall e \in E$: $\boldsymbol{\ell_e \leq f(e) \leq c_e}$
- *Demand Conditions*: $\forall v \in V$: $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

**Objective:** Does a flow $f$ satisfying all conditions exist?
If yes, find such a flow $f$.

# Solution Idea

$f(e) = f_0(e) + f'(e)$

- Define initial circulation $f_0(e) = \ell_e$
  Satisfies capacity constraints: $\forall e \in E: \ell_e \leq f_0(e) \leq c_e$

- Define

$$L_v := f_0^{\text{in}}(v) - f_0^{\text{out}}(v) = \sum_{e \text{ into } v} \ell_e - \sum_{e \text{ out of } v} \ell_e$$

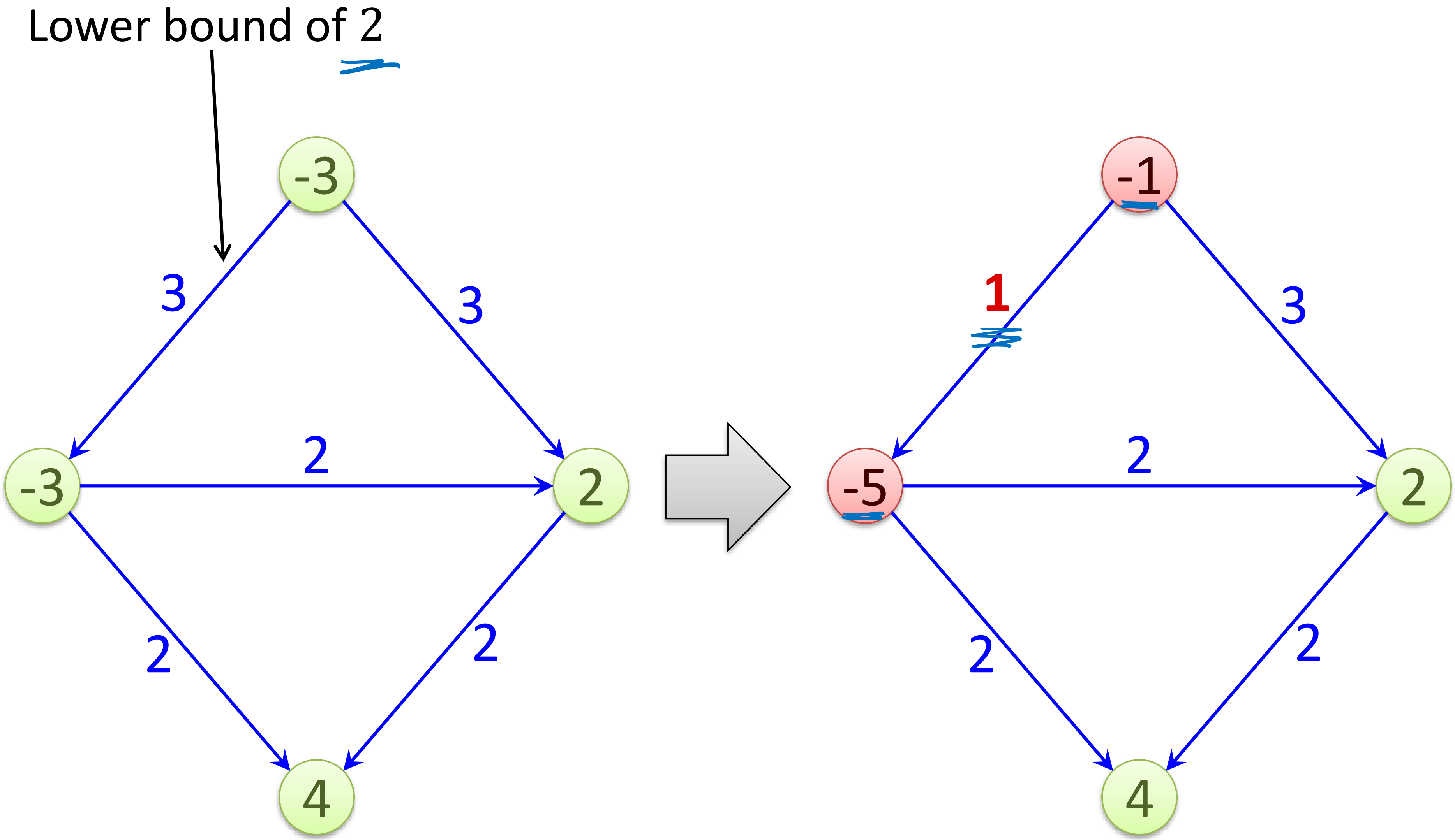
- If $L_v = d_v$, demand condition is satisfied at $v$ by $f_0$, otherwise, we need to superimpose another circulation $f_1$ such that

$$d'_v := f_1^{\text{in}}(v) - f_1^{\text{out}}(v) = d_v - L_v$$

- Remaining capacity of edge $e$: $c'_e := c_e - \ell_e$

- We get a circulation problem with new demands $d'_v$, new capacities $c'_e$, and no lower bounds

# Eliminating a Lower Bound: Example

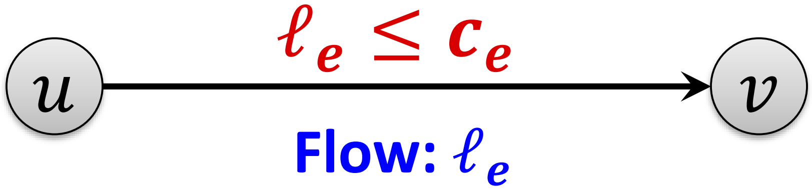


# Reduce to Problem Without Lower Bounds

**Graph $G = (V, E)$:**

- Capacity: For each edge $e \in E$: $\ell_e \leq f(e) \leq c_e$
- Demand: For each node $v \in V$: $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

**Model lower bounds with supplies & demands:**




**Create Network $G'$ (without lower bounds):**

- For each edge $e \in E$: $c'_e = c_e - \ell_e$
- For each node $v \in V$: $d'_v = d_v - L_v$

$$L_v = \sum_{e \text{ into } v} \ell_e - \sum_{e \text{ out of } v} \ell_e$$

# Circulation: Demands and Lower Bounds

**Theorem:** There is a feasible circulation in $G$ (with lower bounds) if and only if there is feasible circulation in $G'$ (without lower bounds).

- Given circulation $f'$ in $G'$, $f(e) = f'(e) + \ell_e$ is circulation in $G$
  - The capacity constraints are satisfied because $f'(e) \leq c_e - \ell_e$
  - Demand conditions:

$$f^{\text{in}}(v) - f^{\text{out}}(v) = \sum_{e \text{ into } v} \left(\ell_e + f'(e)\right) - \sum_{e \text{ out of } v} \left(\ell_e + f'(e)\right)$$

$$= L_v + (d_v - L_v) = d_v$$

- Given circulation $f$ in $G$, $f'(e) = f(e) - \ell_e$ is circulation in $G'$
  - The capacity constraints are satisfied because $\ell_e \leq f(e) \leq c_e$
  - Demand conditions:

$$f'^{\text{in}}(v) - f'^{\text{out}}(v) = \sum_{e \text{ into } v} \left(f(e) - \ell_e\right) - \sum_{e \text{ out of } v} \left(f(e) - \ell_e\right)$$

$$= d_v - L_v$$

# Integrality

**Theorem:** Consider a circulation problem with integral capacities, flow lower bounds, and node demands. If the problem is feasible, then it also has an integral solution.

**Proof:**

- Graph $G'$ has only integral capacities and demands
- Thus, the flow network used in the reduction to solve circulation with demands and no lower bounds has only integral capacities
- The theorem now follows because a max flow problem with integral capacities also has an optimal integral solution

- It also follows that with the max flow algorithms we studied, we get an integral feasible circulation solution.

# Matrix Rounding

- **Given:** $p \times q$ matrix $D = \{d_{i,j}\}$ of real numbers
- **row *i* sum:** $a_i = \sum_j d_{i,j}$, **column *j* sum:** $b_j = \sum_i d_{i,j}$
- **Goal:** Round each $d_{i,j}$, as well as $a_i$ and $b_j$ up or down to the next integer so that the sum of rounded elements in each row (column) equals the rounded row (column) sum
- **Original application:** publishing census data

**Example:**

| | | | |
|---|---|---|---|
| 3.14 | 6.80 | 7.30 | 17.24 |
| 9.60 | 2.40 | 0.70 | 12.70 |
| 3.60 | 1.20 | 6.50 | 11.30 |
| 16.34 | 10.40 | 14.50 | |

**original data**

→

| | | | |
|---|---|---|---|
| 3 | 7 | 7 | 17 |
| 10 | 2 | 1 | 13 |
| 3 | 1 | 7 | 11 |
| 16 | 10 | 15 | |

**possible rounding**

# Matrix Rounding

**Theorem:** For any matrix, there exists a feasible rounding.

**Remark:** Just rounding to the nearest integer doesn't work

| | | | |
|---|---|---|---|
| 0.35 | 0.35 | 0.35 | 1.05 |
| 0.55 | 0.55 | 0.55 | 1.65 |
| 0.90 | 0.90 | 0.90 | |

**original data**

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 3 |
| 1 | 1 | 1 | |

**rounding to nearest integer**

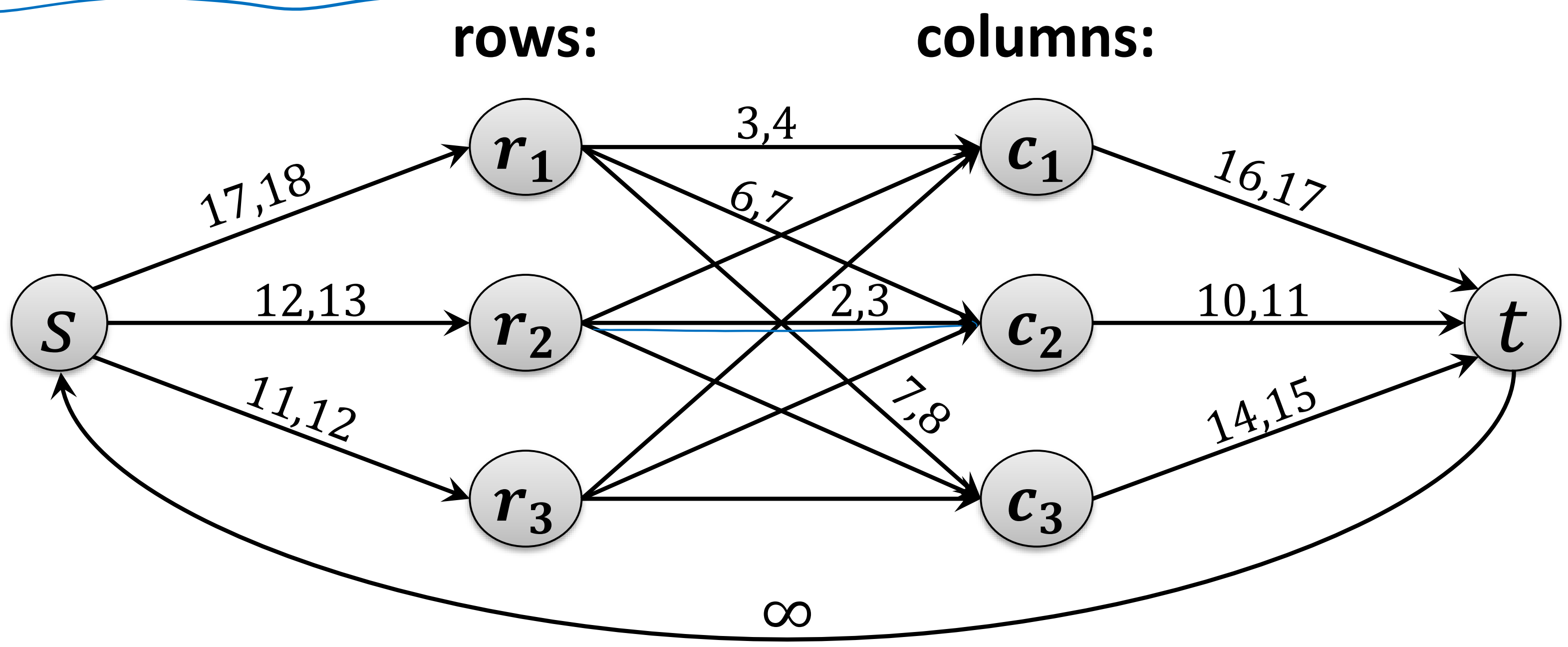
| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 2 |
| 1 | 1 | 1 | |

**feasible rounding**

# Reduction to Circulation

| | | | |
|---|---|---|---|
| 3.14 | 6.80 | 7.30 | 17.24 |
| 9.60 | 2.40 | 0.70 | 12.70 |
| 3.60 | 1.20 | 6.50 | 11.30 |
| 16.34 | 10.40 | 14.50 | |

Matrix elements and row/column sums give a feasible circulation that satisfies all lower bound, capacity, and demand constraints




all demands $d_v = 0$

# Matrix Rounding

**Theorem:** For any matrix, there exists a feasible rounding.

**Proof:**

- The matrix entries $d_{i,j}$ and the row and column sums $a_i$ and $b_j$ give a feasible circulation for the constructed network
- Every feasible circulation gives matrix entries with corresponding row and column sums (follows from demand constraints)
- Because all demands, capacities, and flow lower bounds are integral, there is an integral solution to the circulation problem

→ **gives a feasible rounding!**