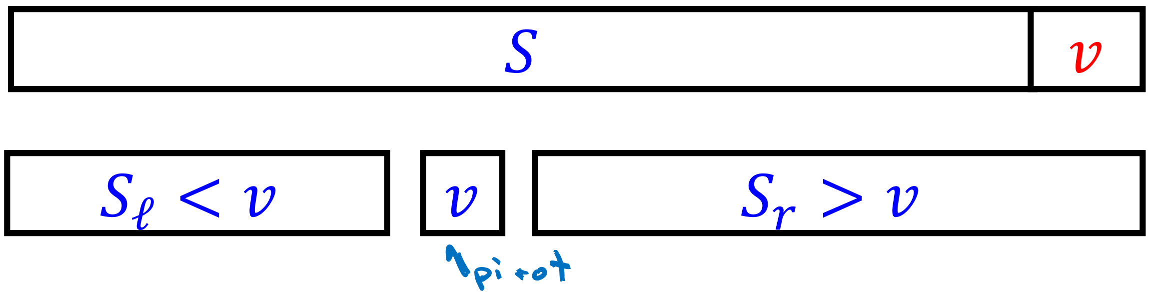
# Chapter 7
# Randomization

## Algorithm Theory
## WS 2017/18


## Fabian Kuhn

# Randomized Quicksort

**Quicksort:**




**function** Quick ($S$: sequence): sequence;  worst case time: $O(n^2)$

{returns the sorted sequence $S$}

**begin**

  **if** $\#S \leq 1$ then **return** $S$

  **else** { choose pivot element $v$ in $S$;

    partition $S$ into $S_\ell$ with elements $< v$,

    and $S_r$ with elements $> v$

    **return** $\boxed{\text{Quick}(S_\ell) \mid v \mid \text{Quick}(S_r)}$

**end**;

# Randomized Quicksort Analysis

**Randomized Quicksort:** pick uniform random element as pivot

**Running Time** of sorting $n$ elements:

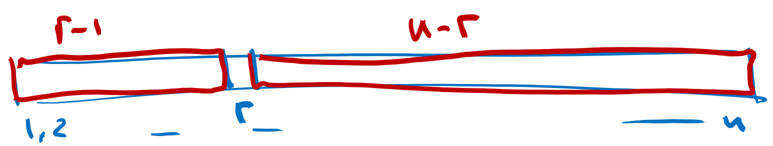
- Let's just count the number of comparisons
- In the partitioning step, all $n-1$ non-pivot elements have to be compared to the pivot
- **Number of comparisons:**

depends on partition

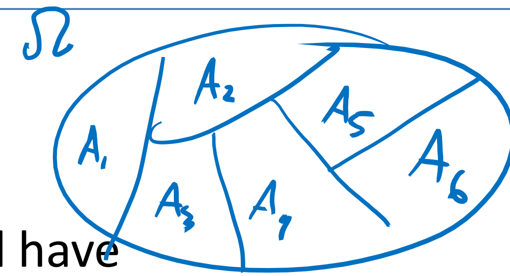
random variable

$$\boldsymbol{n-1} + \boldsymbol{\#\text{comparisons in recursive calls}}$$

- **If rank of pivot is $r$:**
  recursive calls with $\boldsymbol{r-1}$ and $\boldsymbol{n-r}$ elements



# Law of Total Expectation




- Given a random variable $X$ and
- a set of events $A_1, \dots, A_k$ that partition $\Omega$
  - E.g., for a second random variable $Y$, we could have

$$A_i \coloneqq \{\omega \in \Omega : Y(\omega) = i\}$$

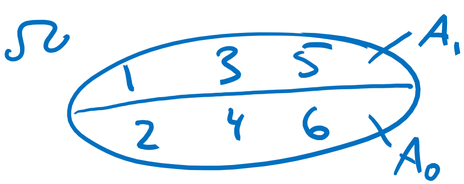
**Law of Total Expectation**

$$\mathbb{E}[X] = \sum_{i=1}^{k} \mathbb{P}(A_i) \cdot \mathbb{E}[X \mid A_i] = \sum_{y} \mathbb{P}(Y = y) \cdot \mathbb{E}[X \mid Y = y]$$

**Example:**

- $X$: outcome of rolling a die
- $A_0 = \{X \text{ is even}\}, \quad A_1 = \{X \text{ is odd}\}$

$\mathbb{E}[X] = 3.5$




$$\mathbb{E}[X] = \mathbb{P}(A_0) \cdot \mathbb{E}[X \mid A_0] + \mathbb{P}(A_1) \cdot \mathbb{E}[X \mid A_1]$$

$$= \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 3 = 3.5$$

# Randomized Quicksort Analysis

$C(n)$

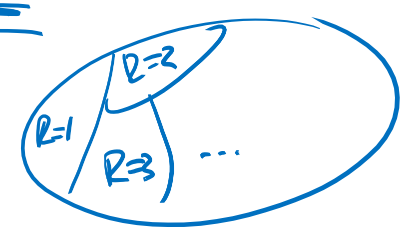
**Random variables:**

$\mathbb{E}[C] = \mathbb{E}[n-1+C_\ell+C_r]$

- $C$: total number of comparisons (for a given array of length $n$)
- $R$: rank of first pivot
- $C_\ell, C_r$: number of comparisons for the 2 recursive calls

$$\mathbb{E}[C] = n - 1 + \mathbb{E}[C_\ell] + \mathbb{E}[C_r]$$

**Law of Total Expectation:**




$$\mathbb{E}[C] = \sum_{r=1}^{n} \mathbb{P}(R=r) \cdot \mathbb{E}[C \mid R=r]$$

$$= \sum_{r=1}^{n} \mathbb{P}(R=r) \cdot \left(n - 1 + \underbrace{\mathbb{E}[C_\ell \mid R=r]}_{\text{\# comp. when sorting an array } r-1} + \underbrace{\mathbb{E}[C_r \mid R=r]}_{\text{\# comp ... ... } n-r}\right)$$

# Randomized Quicksort Analysis

We have seen that: $\mathbb{P}(R=r) = \frac{1}{n}$

$$\underbrace{\mathbb{E}[C]}_{T(n)} = \sum_{r=1}^{n} \mathbb{P}(R=r) \cdot (n - 1 + \underbrace{\mathbb{E}[C_\ell | R = r]}_{T(r-1)} + \underbrace{\mathbb{E}[C_r | R = r]}_{T(n-r)})$$

**Define:**

- $\boldsymbol{T(n)}$: expected number of comparisons when sorting $n$ elements

$$\mathbb{E}[C] = T(n)$$
$$\mathbb{E}[C_\ell | R = r] = T(r-1)$$
$$\mathbb{E}[C_r | R = r] = T(n-r)$$

**Recursion:**

$$\begin{cases} \boldsymbol{T(n) = \sum_{r=1}^{n} \frac{1}{n} \cdot \big(n - 1 + T(r-1) + T(n-r)\big)} \\ \boldsymbol{T(0) = T(1) = 0} \end{cases}$$

# Randomized Quicksort Analysis

$(T(0)+T(n-1)) + (T(1)+T(n-2)) + (T(2)+T(n-3)) + \ldots + (T(n-1)+T(0))$

**Theorem:** The expected number of comparisons when sorting $n$ elements using randomized quicksort is $T(n) \leq \underline{2n\ln n}$.

**Proof:** (by induction on n)

$T(1) = 0$

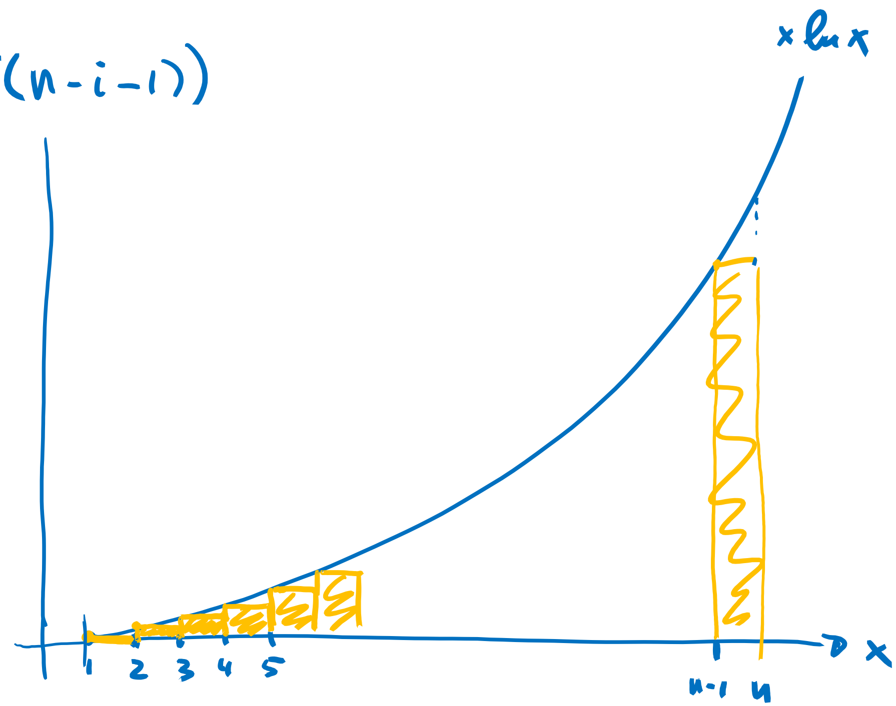
$$T(n) = \sum_{r=1}^{n} \frac{1}{n} \cdot \big(n - 1 + T(r-1) + T(n-r)\big), \qquad T(0) = 0$$

$$= n-1+\frac{1}{n}\cdot\sum_{i=0}^{n-1}\big(T(i)+T(n-i-1)\big)$$

$$= n-1+\frac{2}{n}\sum_{i=1}^{n-1}T(i)$$

$$\overset{(I.H.)}{\leq} n-1+\frac{4}{n}\sum_{i=1}^{n-1} i\ln i$$

$$\leq n-1+\frac{4}{n}\int_1^n x\ln x\,dx$$



# Randomized Quicksort Analysis

**Theorem:** The expected number of comparisons when sorting $n$ elements using randomized quicksort is $T(n) \le 2n \ln n$.

**Proof:**

$$T(n) \le n - 1 + \frac{4}{n} \cdot \int_1^n x \ln x \, dx$$

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4}$$

$$T(n) \le n - 1 + \frac{4}{n}\left[\frac{n^2 \ln n}{2} - \frac{n^2}{4} + \frac{1}{4}\right]$$

$$= n - 1 + 2n \ln n - n + \frac{1}{n}$$

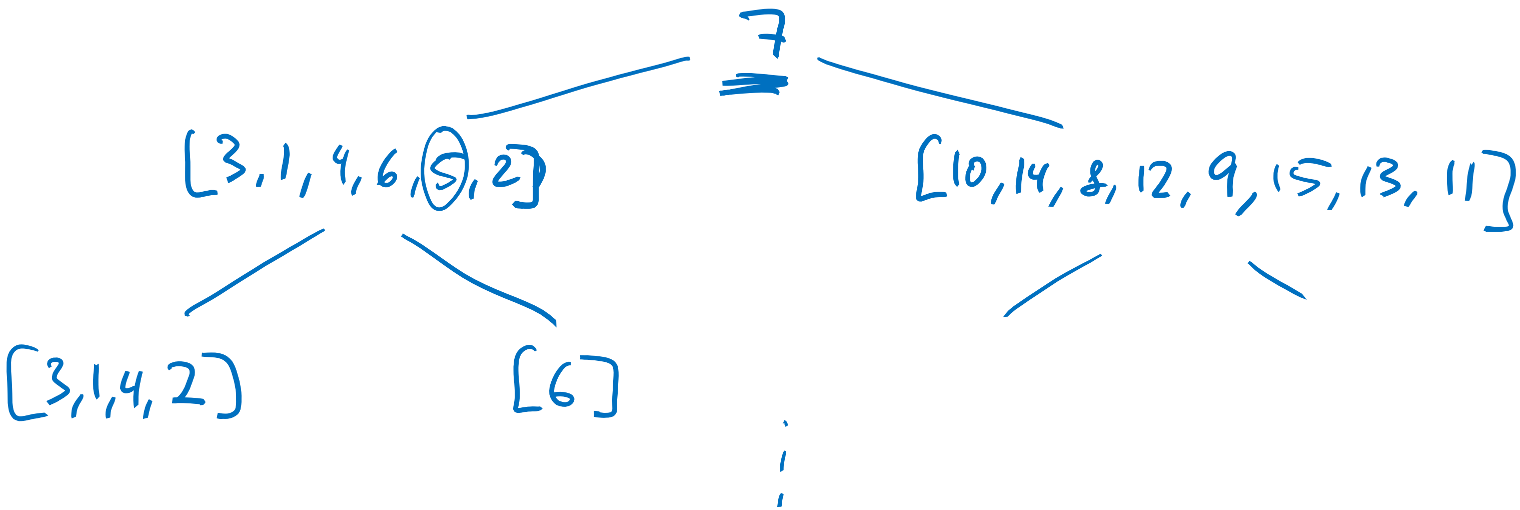
$$= 2n \ln n + \underbrace{\frac{1}{n} - 1}_{<0} < 2n \ln n \qquad \square$$

$$\mathbb{E}[C] \le 2n \ln n$$

# Alternative Analysis

Array to sort: [ 7 , 3 , 1 , 10 , 14 , 8 , 12 , 9 , 4 , 6 , 5 , 15 , 2 , 13 , 11 ]

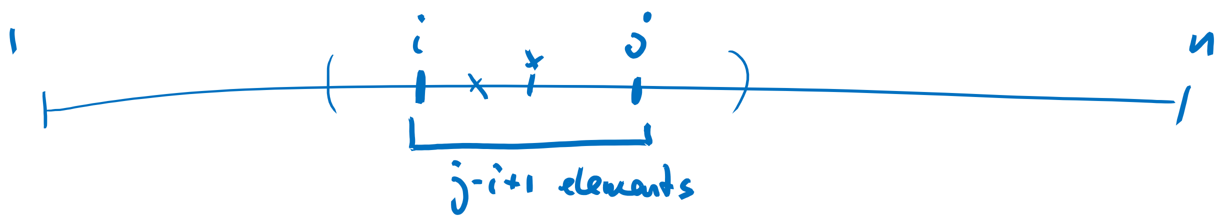
**Viewing quicksort run as a tree:**



# Comparisons

- Comparisons are only between pivot and non-pivot elements
- Every element can only be the pivot once:
  → every 2 elements can only be compared once!

- W.l.o.g., assume that the elements to sort are $1, 2, \ldots, n$
- Elements $i$ and $j$ are compared if and only if either $i$ or $j$ is a pivot before any element $h: i < h < j$ is chosen as pivot
  - i.e., iff $i$ is an ancestor of $j$ or $j$ is an ancestor of $i$




$$\mathbb{P}(\text{comparison betw. } i \text{ and } j) = \frac{2}{j - i + 1}$$

# Counting Comparisons

Random variable for every pair of elements $(i, j)$:

$$\boldsymbol{X_{ij}} = \begin{cases} 1, & \text{if there is a comparison between } i \text{ and } j \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X_{ij} = 1) = \frac{2}{j-i+1} \qquad \mathbb{E}[X_{ij}] = \frac{2}{j-i+1}$$

Number of comparisons: $\boldsymbol{X}$

$$X = \sum_{i<j} X_{ij}$$

- What is $\mathbb{E}[X]$?

# Randomized Quicksort Analysis

**Theorem:** The expected number of comparisons when sorting $n$ elements using randomized quicksort is $T(n) \leq 2n \ln n$.

**Proof:**

- **Linearity of expectation:**
  For all random variables $X_1, \dots, X_n$ and all $a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}\left[\sum_{i}^{n} a_i X_i\right] = \sum_{i}^{n} a_i \mathbb{E}[X_i].$$

$$X = \sum_{i<j} X_{ij}$$

$$\mathbb{E}[X] = \mathbb{E}\Big[\sum_{i<j} X_{ij}\Big] = \sum_{i<j} \mathbb{E}[X_{ij}] = \sum_{i<j} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

# Randomized Quicksort Analysis

**Theorem:** The expected number of comparisons when sorting $n$ elements using randomized quicksort is $T(n) \leq 2n \ln n$.

**Proof:**

$$\mathbb{E}[X] = 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{j-i+1} = 2\sum_{i=1}^{n-1}\sum_{k=2}^{n-i+1}\frac{1}{k}$$

$$\leq 2\sum_{i=1}^{n-1}\sum_{k=2}^{n}\frac{1}{k}$$

$$= 2\sum_{i=1}^{n-1}\left(H(n)-1\right)$$

$$= 2(n-1)(H(n)-1)$$

$$\leq \underline{\underline{2n\ln n}}$$

Harmonic Series

$$H(n) = \sum_{i=1}^{n}\frac{1}{i}$$

$$\underline{H(n) \leq 1 + \ln n}$$

# Quicksort: High Probability Bound $1 - \frac{1}{n^c}$

- We have seen that the number of comparisons of randomized quicksort is $O(n \log n)$ in expectation.

- Can we also show that the number of comparisons is $O(n \log n)$ with high probability?

- **Recall:**

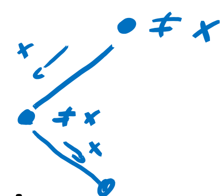
  On each recursion level, each pivot is compared once with each other element that is still in the same "part"

# Counting Number of Comparisons

- We looked at 2 ways to count the number of comparisons
  - recursive characterization of the expected number
  - number of different pairs of values that are compared

Let's consider yet another way:

- Each comparison is between a pivot and a non-pivot
- How many times is a specific array element $x$ compared as a non-pivot?




Value $x$ is compared as a non-pivot to a pivot once in every recursion level until one of the following two conditions apply:

1. $x$ is chosen as a pivot
2. $x$ is alone

#comp. of x as non-pivot
= depth when x becomes pivot / alone

# Successful Recursion Level

- Consider a specific recursion level $\ell$

$$\overbrace{[\qquad x \qquad]}^{K_\ell}$$

- Assume that at the beginning of recursion level $\ell$, element $x$ is in a sub-array of length $K_\ell$ that still needs to be sorted.

- If $x$ has been chosen as a pivot before level $\ell$, we set $K_\ell := 1$

**Definition:** We say that recursion level $\ell$ is successful for element $x$ iff the following is true:

$$K_{\ell+1} = 1 \quad \text{or} \quad K_{\ell+1} \leq \frac{2}{3} \cdot K_\ell$$

# Successful Recursion Level

**Lemma:** For every recursion level $\ell$ and every array element $x$, it holds that level $\ell$ is successful for $x$ with probability at least $1/3$, independently of what happens in other recursion levels.

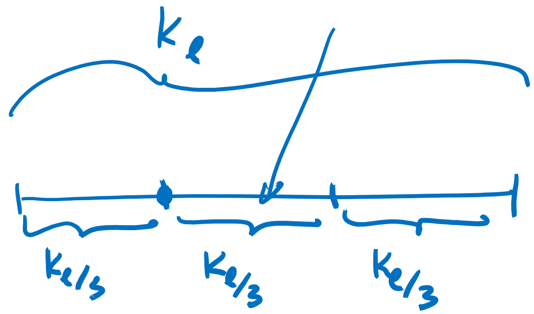
$\geq$

**Proof:**

$K_{\ell+1} \leq K_\ell$

$K_\ell = 1 \Longrightarrow K_{\ell+1} = 1$ ✓

$\underline{\underline{K_\ell > 1}}$




if pivot is in the middle part

$\Longrightarrow$ both remaining arrays have size $\leq \frac{2}{3} \cdot K_\ell$

$\Longrightarrow$ probability for this $\geq \frac{1}{3}$

# Number of Successful Recursion Levels

**Lemma:** If among the first $\ell$ recursion levels, at least $\log_{3/2}(n)$ are successful for element $x$, we have $K_{\ell+1} = 1$.

**Proof:**

$K_1 = n$, $\quad K_{i+1} \leq K_i \quad$ if level $i$ succ. : $K_{i+1} \leq \frac{2}{3} K_i$

$$K_{\ell+1} \leq n \cdot \left(\frac{2}{3}\right)^{\#\text{succ. levels}} \leq n \cdot \left(\frac{2}{3}\right)^{\log_{3/2} n} = n \cdot \frac{1}{n} = \underline{\underline{1}}$$

# Chernoff Bounds

- Let $X_1, \dots, X_n$ be independent 0-1 random variables and define $p_i := \mathbb{P}(X_i = 1)$.
- Consider the random variable $X = \sum_{i=1}^{n} X_i$
- We have $\mu := \mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} p_i$

**Chernoff Bound (Lower Tail):**

$$\forall \boldsymbol{\delta} > \mathbf{0}: \ \mathbb{P}(\boldsymbol{X} < (\mathbf{1} - \boldsymbol{\delta})\boldsymbol{\mu}) < \boldsymbol{e}^{-\boldsymbol{\delta}^2\boldsymbol{\mu}/2}$$

**Chernoff Bound (Upper Tail):**

$$\forall \boldsymbol{\delta} > \mathbf{0}: \ \mathbb{P}(\boldsymbol{X} > (\mathbf{1} + \boldsymbol{\delta})\boldsymbol{\mu}) < \left(\frac{\boldsymbol{e}^{\boldsymbol{\delta}}}{(\mathbf{1} + \boldsymbol{\delta})^{\mathbf{1}+\boldsymbol{\delta}}}\right)^{\boldsymbol{\mu}} < \boldsymbol{e}^{-\boldsymbol{\delta}^2\boldsymbol{\mu}/3}$$

**holds for $\delta \leq 1$**

# Chernoff Bounds, Example

Assume that a fair coin is flipped $n$ times. What is the probability to have

$\mu = \mathbb{E}[X] = \frac{n}{2}$

1. less than $n/3$ heads?

coin $i$ : rand. var $X_i$ , $X_i = 1 \Leftrightarrow$ heads $\quad p_i = \mathbb{P}(X_i = 1) = \frac{1}{2}$ $\quad$ #heads $X = \sum X_i$

$$\mathbb{P}(X < n/3) = \mathbb{P}\left(X < \left(1 - \frac{1}{3}\right)\frac{n}{2}\right) < e^{-\frac{1}{9}\cdot\frac{1}{2}\cdot\frac{n}{2}} = e^{-n/36}$$

2. more than $0.51n$ tails?

$$\mathbb{P}\left(X > (1 + 0.02)\cdot\frac{n}{2}\right) < e^{-\frac{0.02^2}{2}\cdot\frac{n}{2}}$$

3. less than $n/2 - \sqrt{c \cdot n \ln n}$ tails?

$$\mathbb{P}\left(X < \left(1 - \frac{2\sqrt{cn\ln n}}{n}\right)\frac{n}{2}\right) \le e^{-\frac{\cancel{4}cn\ln n}{n^{\cancel{2}}}\cdot\frac{\cancel{n}}{\cancel{4}}} = \frac{1}{n^c}$$

$1 - \frac{1}{n^c}$

w.h.p. #tails $= \frac{n}{2} \pm O(\sqrt{n\log n})$

# Proof of Chernoff Bound

- Independent Bernoulli random variables $X_1, X_2, \dots, X_n$
- $\mathbb{P}(X_i = 1) \geq p_i$, $X := \sum_{i=1}^{n} X_i$, $\mu := \sum_{i=1}^{n} p_i \overset{\leq}{\cancel{=}} \mathbb{E}[X]$

**Chernoff Lower Tail:** $\mathbb{P}(X < (1-\delta)\mu) < e^{-\delta^2\mu/2}$

Recall

- Markov Inequality: Given non-negative rand. var. $Z \geq 0$

$$\forall t > 0:\ \mathbb{P}(Z > t) < \frac{\mathbb{E}[Z]}{t}$$

$\mathbb{P}(Z \geq c \cdot \mathbb{E}[Z]) \leq \frac{1}{c}$

- Independent random variables $Y, Z$:

$$\mathbb{E}[Y \cdot Z] = \mathbb{E}[Y] \cdot \mathbb{E}[Z]$$

# Proof of Chernoff Bound

$s>0$ $\qquad$ $e^x$

- $\mathbb{P}(X_i = 1) \geq p_i$, $X \coloneqq \sum_{i=1}^{n} X_i$, $\mu \coloneqq \sum_{i=1}^{n} p_i \overset{\leq}{\geq} \mathbb{E}[X]$

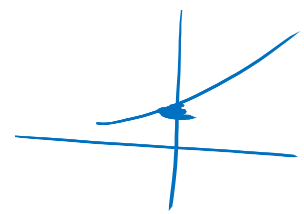
$e^{-sX}$

**Chernoff Lower Tail:** $\mathbb{P}(\boldsymbol{X} < (\mathbf{1}-\boldsymbol{\delta})\boldsymbol{\mu}) < \boldsymbol{e}^{-\boldsymbol{\delta}^2\boldsymbol{\mu}/2}$

$$\mathbb{P}(X < (1-\delta)\mu) = \mathbb{P}(-X > -(1-\delta)\mu)$$

$$\mathbb{P}(Z \geq t) \leq \frac{\mathbb{E}[Z]}{t}$$

$$= \mathbb{P}\big(\underbrace{e^{-sX}}_{Z} > e^{-s(1-\delta)\mu}\big)$$



$$\overset{\text{Markov}}{<} \frac{\mathbb{E}[e^{-sX}]}{e^{-s(1-\delta)\mu}}$$

$$\mathbb{E}[e^{-sX}] = \mathbb{E}\big[e^{-s\sum_{i=1}^{n} X_i}\big] = \mathbb{E}\Big[\prod_{i=1}^{n} e^{-sX_i}\Big] \overset{\text{indep. of } X_i}{=} \prod_{i=1}^{n} \mathbb{E}[e^{-sX_i}]$$

$$\mathbb{E}[e^{-sX_i}] = \mathbb{P}(X_i=1)\cdot e^{-s} + 1 - \mathbb{P}(X_i=1) = 1 + \mathbb{P}(X_i=1)\underbrace{(e^{-s}-1)}_{<0}$$

$$\leq 1 + p_i(e^{-s}-1)$$

# Proof of Chernoff Bound

$1 + x \le e^{x}$

- $\mathbb{P}(X_i = 1) \ge p_i$, $X \coloneqq \sum_{i=1}^{n} X_i$, $\mu \coloneqq \sum_{i=1}^{n} p_i \ge \mathbb{E}[X]$

**Chernoff Lower Tail:** $\mathbb{P}(\boldsymbol{X} < (\mathbf{1} - \boldsymbol{\delta})\boldsymbol{\mu}) < \boldsymbol{e}^{-\boldsymbol{\delta}^2\boldsymbol{\mu}/2}$

$$\underline{\mathbb{P}(X < (1-\delta)\mu)} < \frac{\mathbb{E}[e^{-sX}]}{e^{-s(1-\delta)\mu}} \le \underline{e^{\mu(e^{-s}-1) + s(1-\delta)\mu}}$$

(holds for all $s > 0$)

$$\mathbb{E}[e^{-sX}] = \prod_{i=1}^{n} \mathbb{E}[e^{-sX_i}] \le \prod_{i=1}^{n} \left(1 + p_i(e^{-s}-1)\right)$$

$$\le \prod_{i=1}^{n} e^{p_i(e^{-s}-1)}$$

$$= e^{\sum_{i=1}^{n} p_i(e^{-s}-1)} = e^{\mu(e^{-s}-1)}$$

<u>set $s = \delta$:</u>

$$\mathbb{P}(X < (1-\delta)\mu) < e^{\mu(e^{\delta} - 1 + \delta - \delta^2)}$$

# Proof of Chernoff Bound

- $\mathbb{P}(X_i = 1) \geq p_i,\ X \coloneqq \sum_{i=1}^{n} X_i,\ \mu \coloneqq \sum_{i=1}^{n} p_i \geq \mathbb{E}[X]$

**Chernoff Lower Tail:** $\mathbb{P}(\boldsymbol{X} < (\boldsymbol{1}-\boldsymbol{\delta})\boldsymbol{\mu}) < \boldsymbol{e^{-\delta^2\mu/2}}$

$$\mathbb{P}(X < (1-\delta)\mu) < e^{\mu(e^{-\delta}-1+\delta-\delta^2)}$$

$$= e^{\mu\left(1-\delta+\frac{\delta^2}{2}+\delta-\delta^2-1\right)}$$

$$= e^{-\frac{\delta^2}{2}\cdot\mu}$$

$\square$

$$e^{-\delta} = 1-\delta+\frac{\delta^2}{2}\underbrace{\left[-\frac{\delta^3}{6}+\frac{\delta^4}{24}-\ldots\right]}_{\leq 0}$$

$(\delta \leq 1)$

$$\leq 1-\delta+\frac{\delta^2}{2}$$

# Number of Comparisons for $x$

**Lemma:** For every array element $x$, with high probability, as a non-pivot, $x$ is compared to a pivot at most $O(\log n)$ times.

**Proof:**

$X_i = 1$ iff level $i$ is successful

$\mathbb{P}(X_i = 1) \geq \frac{1}{3}$

Chernoff: # levels needed $O(\log n)$ w.h.p.

# Number of Comparisons for $x$

**Lemma:** For every array element $x$, with high probability, as a non-pivot, $x$ is compared to a pivot at most $O(\log n)$ times.

**Proof:**

# Number of Comparisons

**Theorem:** With high probability, the total number of comparisons is at most $O(n \log n)$.

**Proof:**

union bound over all elements

# Types of Randomized Algorithms

## Las Vegas Algorithm:

- always a correct solution
- running time is a random variable

- **Example:** randomized quicksort, contention resolution

## Monte Carlo Algorithm:

- probabilistic correctness guarantee (**m**ostly **c**orrect)
- fixed (deterministic) running time

- **Example:** primality test