Chapter 9
Online Algorithms

Algorithm Theory
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Online Computations

- Sometimes, an algorithm has to start processing the input before the complete input is known
- For example, when storing data in a data structure, the sequence of operations on the data structure is not known

Online Algorithm: An algorithm that has to produce the output step-by-step when new parts of the input become available.

Offline Algorithm: An algorithm that has access to the whole input before computing the output.

- Some problems are inherently online
  - Especially when real-time requests have to be processed over a significant period of time

Competitive Ratio

- Let's again consider optimization problems
  - For simplicity, assume, we have a minimization problem

Optimal offline solution $\mathbf{OPT}(\boldsymbol{I})$:

- Best objective value that an offline algorithm can achieve for a given input sequence I

Online solution $\mathbf{ALG}(\boldsymbol{I})$:

- Objective value achieved by an online algorithm ALG on I

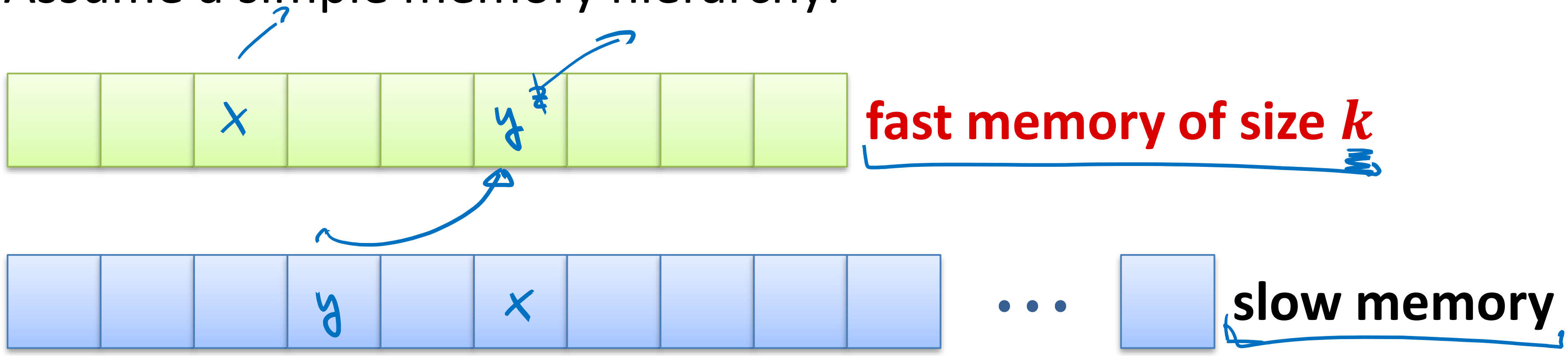
Competitive Ratio: An algorithm has competitive ratio $c \geq 1$ if

$$\mathbf{ALG}(\boldsymbol{I}) \leq \boldsymbol{c} \cdot \mathbf{OPT}(\boldsymbol{I}) + \boldsymbol{\alpha}.$$

- If $\alpha = 0$, we say that ALG is strictly c-competitive.

Paging Algorithm

Assume a simple memory hierarchy:



If a memory page has to be accessed:

- Page in fast memory (hit): take page from there
- Page not in fast memory (miss): leads to a page fault
- Page fault: the page is loaded into the fast memory and some page has to be evicted from the fast memory
- Paging algorithm: decides which page to evict
- Classical online problem: we don't know the future accesses

Paging Strategies

Least Recently Used (LRU):

- Replace the page that hasn't been used for the longest time

First In First Out (FIFO):

- Replace the page that has been in the fast memory longest

Last In First Out (LIFO):

- Replace the page most recently moved to fast memory

Least Frequently Used (LFU):

- Replace the page that has been used the least

Longest Forward Distance (LFD): optimal offline strategy

- Replace the page whose next request is latest (in the future)
- LFD is **not** an online strategy!

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof: σ: sequence of requests

- For contradiction, assume that LFD is not optimal
- Then there exists a finite input sequence σ on which LFD is not optimal (assume that the length of σ is $|\sigma| = n$)
- Let OPT be an optimal solution for σ such that
  - OPT processes requests $1, \dots, i$ in exactly the same way as LFD
  - OPT processes request $i + 1$ differently than LFD
  - Any other optimal strategy processes one of the first $i + 1$ requests differently than LFD
- Hence, OPT is the optimal solution that behaves in the same way as LFD for as long as possible → we have $i < n$
- Goal: Construct OPT′ that is identical with LFD for req. $1, \dots, i + 1$

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

Case 1: Request $i+1$ does **not** lead to a page fault

- LFD does not change the content of the fast memory
- OPT behaves differently than LFD
  → OPT replaces some page in the fast memory
  - As up to request $i+1$, both algorithms behave in the same way, they also have the same fast memory content
  - OPT therefore does not require the new page for request $i+1$
  - Hence, OPT can also load that page later (without extra cost) → OPT′

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

Case 2: Request $i + 1$ does lead to a **page fault**

- LFD and OPT move the same page into the fast memory, but they evict different pages
  - If OPT loads more than one page, all pages that are not required for request $i + 1$ can also be loaded later
- Say, LFD evicts page p and OPT evicts page p'
- By the definition of LFD, p' is required again before page p

LFD is Optimal

Theorem: LFD (longest forward distance) is an optimal offline alg.

Proof:

Case 2: Request $i+1$ does lead to a **page fault**

$j > j'$

p' $\quad$ p $\quad$ p'

$i+1$ $\quad$ $\ell' < \ell$: OPT evicts p $\quad$ j': next req. for p' $\quad$ j: next req. for p

LFD evicts p
OPT evicts p'

$\ell \leq j'$: OPT loads p' (for first time after $i+1$)

a) OPT keeps p in fast memory until request ℓ
   - Evict p at request $i+1$, keep p' instead and load p (instead of p') back into the fast memory at request ℓ

b) OPT evicts p at request $\ell' < \ell$
   - Evict p at request $i+1$ and p' at request ℓ' (switch evictions of p and p')

Phase Partition

We partition a given request sequence σ into phases as follows:

- **Phase $\mathbf{0}$**: empty sequence
- **Phase $\boldsymbol{i}$** : maximal sequence that immediately follows phase $i-1$ and contains at most k distinct page requests

Example sequence ($\boldsymbol{k=4}$):

phase 1

$$2, 5, 12, 5, 4, 2, 10, 8, 3, 6, 2, 2, 6, 6, 8, 3, 2, 6, 9, 10, 6, 3, 10, 2, 1, 3, 5$$

phase 1   phase 2   phase 3   phase 4   phase 5

phase 1 interval   phase 2 interval   ph. 3 int.   phase 4 int.

Phase $\boldsymbol{i}$ Interval: interval starting with the second request of phase i and ending with the first request of phase $i+1$

- If the last phase is phase p, phase i interval is defined for $i = 1, \ldots, p-1$

Optimal Algorithm

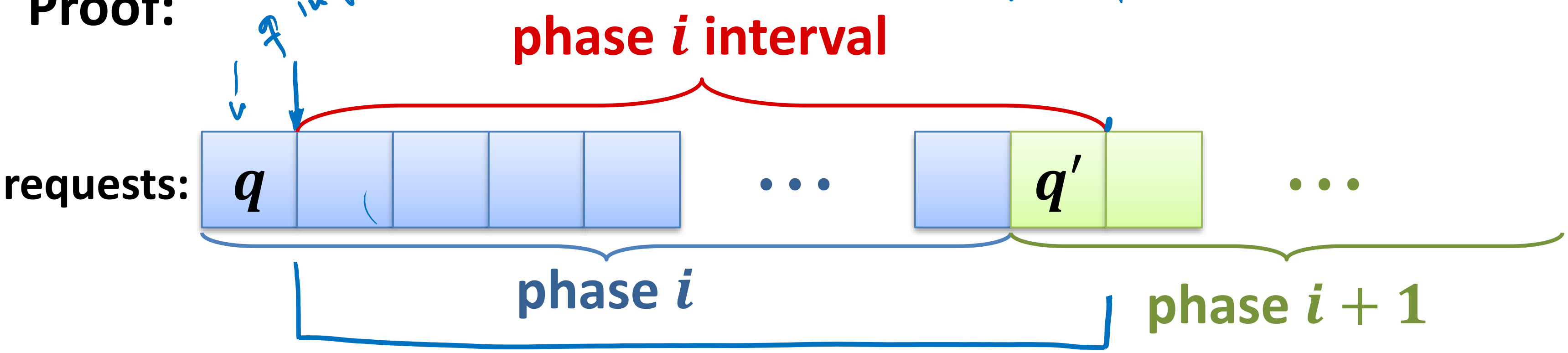
cost of LFD/OPT $\geq p-1$

Lemma: Algorithm LFD has at least one page fault in each phase i interval (for $i = 1, \dots, p-1$, where p is the number of phases).

Proof:

q in fast mem.

"#distinct req. $\neq q$" $= k$



- q is in fast memory after first request of phase i
- Number of distinct requests in phase i: k
- By maximality of phase i: q' does not occur in phase i
- Number of distinct requests $\neq q$ in phase interval i: k

→ at least one page fault

LRU and FIFO Algorithms

Lemma: Algorithm LFD has at least one page fault in each phase i interval (for $i = 1, \dots, p-1$, where p is the number of phases).

Corollary: The number of page faults of an optimal offline algorithm is at least $p - 1$, where p is the number of phases

Theorem: The LRU and the FIFO algorithms both have a competitive ratio of at most k.

Proof:

$k \cdot p \leq k(p-1) + k$

- We will show that both have at most k page faults per phase
- We then have (for every input I):

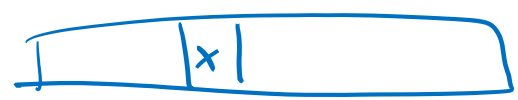
$$\mathrm{LRU}(I), \mathrm{FIFO}(I) \leq k \cdot p \leq \underbrace{k \cdot \mathrm{OPT}(I) + k}$$

LRU and FIFO Algorithms

Theorem: The LRU and the FIFO algorithms both have a competitive ratio of at most k.

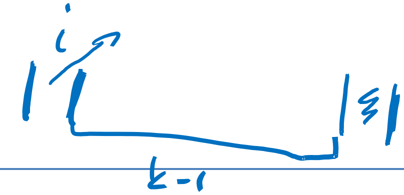
Proof:

- Need to show that both have at most k page faults per phase
- LRU:
  - The k last pages used are the k least recently used
  - Throughout a phase i, the k distinct pages of phase i are the l.r.u.
  - Once in the fast memory, these pages are therefore not evicted until the end of the phase

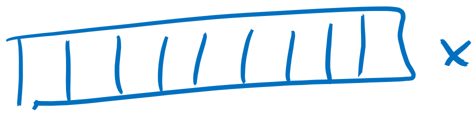


- FIFO:
  - In each page fault in phase i, one of the k pages of phase i is loaded into fast memory
  - Once a page is loaded in a page fault of phase i it belongs to the least k pages loaded into fast memory throughout the rest of the phase
  - Hence: Each of the k pages leads to ≤ 1 page fault in phase i

Lower Bound



Theorem: Even if the slow memory contains only $k+1$ pages, any deterministic algorithm has competitive ratio at least k.



Proof:

- Consider some given deterministic algorithm ALG
- Because ALG is deterministic, the content of the fast memory after the first i requests is determined by the first i requests.
- Construct a request sequence inductively as follows:
  – Assume some initial slow memory content
  – The $(i+1)^{\text{st}}$ request is for the page which is not in fast memory after the first i requests (throughout we only use $k+1$ different pages)
- There is a page fault for every request
- OPT has a page fault at most every k requests
  – There is always a page that is not required for the next $k-1$ requests

Randomized Algorithms

- We have seen that deterministic paging algorithms cannot be better than k-competitive
- Does it help to use randomization?

Competitive Ratio: A randomized online algorithm has competitive ratio $c \geq 1$ if for all inputs I,

$$\mathbb{E}[\mathbf{ALG}(\boldsymbol{I})] \leq \boldsymbol{c} \cdot \mathbf{OPT}(\boldsymbol{I}) + \boldsymbol{\alpha}.$$

- If $\alpha \leq 0$, we say that ALG is strictly c-competitive.

Adversaries

- For randomized algorithm, we need to distinguish between different kinds of adversaries (providing the input)

Oblivious Adversary:

- Has to determine the complete input sequence before the algorithm starts
  - The adversary cannot adapt to random decisions of the algorithm

Adaptive Adversary:

- The input sequence is constructed during the execution
- When determining the next input, the adversary knows how the algorithm reacted to the previous inputs
- Input sequence depends on the random behavior of the alg.
- Sometimes, two adaptive adversaries are distinguished
  - offline, online : different way of measuring the adversary cost