Chapter 9
Online Algorithms

Algorithm Theory
WS 2017/18

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Online Computations

- Sometimes, an algorithm has to start processing the input before the complete input is known
- For example, when storing data in a data structure, the sequence of operations on the data structure is not known

Online Algorithm: An algorithm that has to produce the output step-by-step when new parts of the input become available.

Offline Algorithm: An algorithm that has access to the whole input before computing the output.

- Some problems are inherently online
  - Especially when real-time requests have to be processed over a significant period of time

Competitive Ratio

- Let's again consider optimization problems
  - For simplicity, assume, we have a minimization problem

Optimal offline solution $\mathbf{OPT}(\boldsymbol{I})$:

- Best objective value that an offline algorithm can achieve for a given input sequence I

Online solution $\mathbf{ALG}(\boldsymbol{I})$:

- Objective value achieved by an online algorithm ALG on I

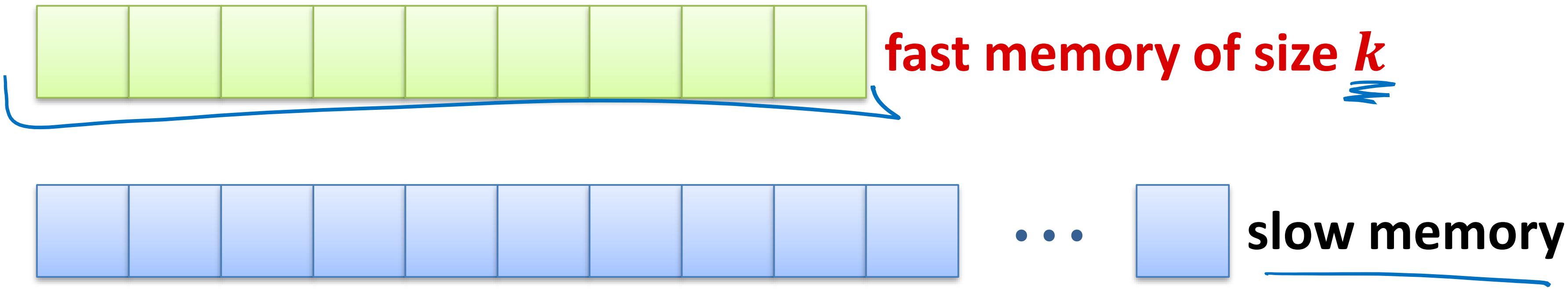
Competitive Ratio: An algorithm has competitive ratio $c \geq 1$ if

$$\mathbf{ALG}(\boldsymbol{I}) \leq \boldsymbol{c} \cdot \mathbf{OPT}(\boldsymbol{I}) + \boldsymbol{\alpha}.$$

- If $\alpha = 0$, we say that ALG is strictly c-competitive.

Paging Algorithm

Assume a simple memory hierarchy:



If a memory page has to be accessed:

- Page in fast memory (hit): take page from there
- Page not in fast memory (miss): leads to a page fault
- Page fault: the page is loaded into the fast memory and some page has to be evicted from the fast memory
- Paging algorithm: decides which page to evict
- Classical online problem: we don't know the future accesses

Paging Strategies

Least Recently Used (LRU):

- Replace the page that hasn't been used for the longest time

First In First Out (FIFO):

- Replace the page that has been in the fast memory longest

Last In First Out (LIFO):

- Replace the page most recently moved to fast memory

Least Frequently Used (LFU):

- Replace the page that has been used the least

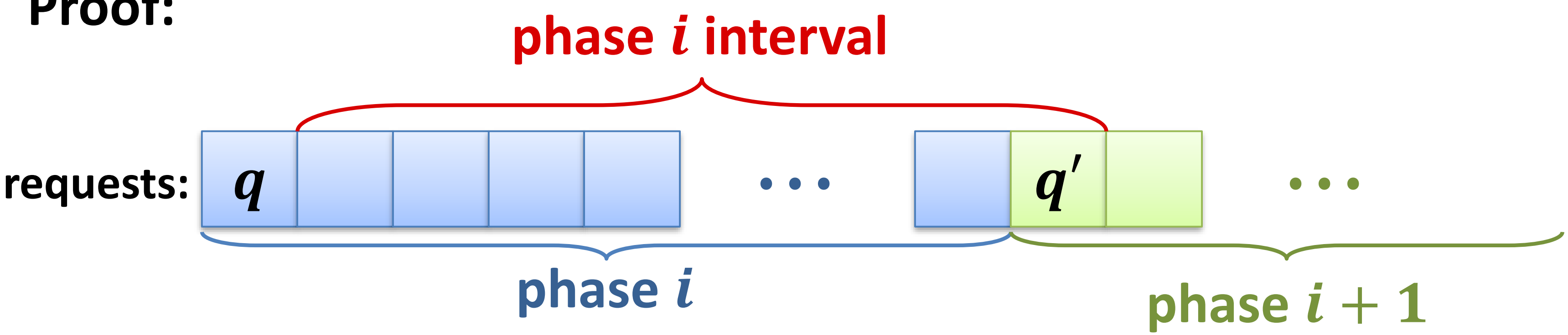
Longest Forward Distance (LFD): optimal offline algorithm

- Replace the page whose next request is latest (in the future)
- LFD is **not** an online strategy!

Optimal Algorithm

Lemma: Algorithm LFD has at least one page fault in each phase i interval (for $i = 1, \dots, p-1$, where p is the number of phases).

Proof:



- q is in fast memory after first request of phase i
- Number of distinct requests in phase i: k
- By maximality of phase i: q' does not occur in phase i
- Number of distinct requests $\neq q$ in phase interval i: k

→ at least one page fault

LRU and FIFO Algorithms

Lemma: Algorithm LFD has at least one page fault in each phase i interval (for $i = 1, \dots, p-1$, where p is the number of phases).

Corollary: The number of page faults of an optimal offline algorithm is at least $p - 1$, where p is the number of phases

Theorem: The LRU and the FIFO algorithms both have a competitive ratio of at most k.

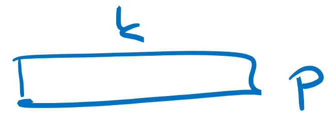
Proof:

- We will show that both have at most k page faults per phase
- We then have (for every input I):

$$\mathrm{LRU}(I), \mathrm{FIFO}(I) \leq k \cdot p \leq k \cdot \mathrm{OPT}(I) + k$$

Lower Bound

Theorem: Even if the slow memory contains only $k + 1$ pages, any deterministic algorithm has competitive ratio at least k.



Proof:

- Consider some given deterministic algorithm ALG
- Because ALG is deterministic, the content of the fast memory after the first i requests is determined by the first i requests.
- Construct a request sequence inductively as follows:
  - Assume some initial slow memory content
  - The $(i+1)^{\text{st}}$ request is for the page which is not in fast memory after the first i requests (throughout we only use $k + 1$ different pages)
- There is a page fault for every request
- OPT has a page fault at most every k requests
  - There is always a page that is not required for the next $k - 1$ requests

Randomized Algorithms

- We have seen that deterministic paging algorithms cannot be better than k-competitive

- Does it help to use randomization?

Competitive Ratio: A randomized online algorithm has competitive ratio $c \geq 1$ if for all inputs I,

$$\underbrace{\mathbb{E}[\mathbf{ALG}(\boldsymbol{I})]} \leq \boldsymbol{c} \cdot \mathbf{OPT}(\boldsymbol{I}) + \underline{\boldsymbol{\alpha}}.$$

- If $\alpha \leq 0$, we say that ALG is strictly c-competitive.

Adversaries

- For randomized algorithm, we need to distinguish between different kinds of adversaries (providing the input)

Oblivious Adversary:

- Has to determine the complete input sequence before the algorithm starts
  - The adversary cannot adapt to random decisions of the algorithm

Adaptive Adversary:

- The input sequence is constructed during the execution
- When determining the next input, the adversary knows how the algorithm reacted to the previous inputs
- Input sequence depends on the random behavior of the alg.
- Sometimes, two adaptive adversaries are distinguished
  - offline, online : different way of measuring the adversary cost

Lower Bound

The adversaries can be ordered according to their strength

$$\text{oblivious} < \text{online adaptive} < \text{offline adaptive}$$

- An algorithm that achieves a given comp. ratio with an adaptive adversary is at least as good with an oblivious one
- A lower bound that holds against an oblivious adversary also holds for the two adaptive adversaries
- …

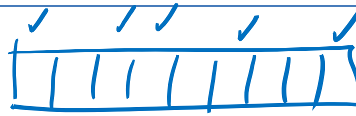
Theorem: No randomized paging algorithm can be better than k-competitive against an adaptive adversary.

Proof: The same proof as for deterministic algorithms works.

- Are there better algorithms with an oblivious adversary?

The Randomized Marking Algorithm

- Every entry in fast memory has a marked flag



- Initially, all entries are unmarked.
- If a page in fast memory is accessed, it gets marked
- When a page fault occurs:
  – If all k pages in fast memory are marked, all marked bits are set to 0
  – The page to be evicted is chosen uniformly at random among the unmarked pages
  – The marked bit of the new page in fast memory is set to 1

Example

Input Sequence (k=6):

$$\underbrace{2, 5, 3, 3, 6, 8, 2, 9, 5}_{\text{phase 1}}, \underbrace{7, 1, 2, 5, 2, 3, 7, 4}_{\text{phase 2}}, \underbrace{8, 1, 2, 7, 5, 3}_{\text{phase 3}}, \underbrace{6, 9, 6, 10, 4, 1, 2}_{\text{phase 4}} \ldots$$

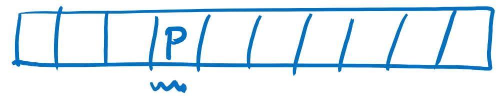
Fast Memory:

10	6	1	9	4	2

Observations:

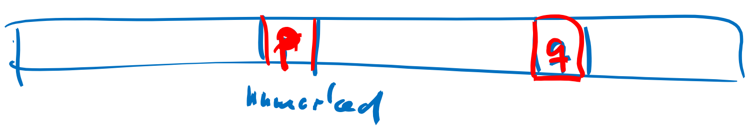
- At the end of a phase, the fast memory entries are exactly the k pages of that phase
- At the beginning of a phase, all entries get unmarked
- #page faults depends on #new pages in a phase

Page Faults per Phase



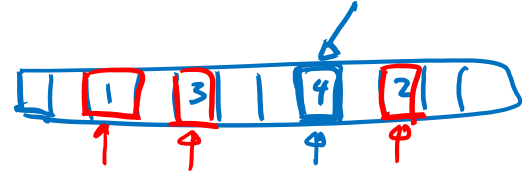
Consider a fixed phase i:

- Assume that of the k pages of phase i, m_i are new and $k - m_i$ are old (i.e., they already appear in phase $i-1$)
- All m_i new pages lead to page faults (when they are requested for the first time)
- When requested for the first time, an old page leads to a page fault, if the page was evicted in one of the previous page faults



- We need to count the number of page faults for old pages

Page Faults per Phase



$j = 4$

Phase i, j^{th} old page that is requested (for the first time):

- There is a page fault if the page has been evicted
- There have been at most $m_i + j - 1$ distinct requests before
- The old places of the $j - 1$ first old pages are occupied
- The other $\leq m_i$ pages are at uniformly random places among the remaining $k - (j - 1)$ places (oblivious adv.)
- Probability that the old place of the j^{th} old page is taken:

$$\boxed{\leq \frac{m_i}{k - (j - 1)}}$$

m_i

Page Faults per Phase

$F_{ij} = \begin{cases} 1 & \text{if } j^{th} \text{ old page} \rightarrow \text{page fault} \\ 0 & \text{otherwise} \end{cases}$

Phase $i > 1$, j^{th} old page that is requested (for the first time):

- Probability that there is a page fault:

$F_i = \sum_{j=1}^{k-m_i} F_{ij}$

$$\mathbb{E}[F_{ij}] = \mathbb{P}(F_{ij} = 1) \leq \frac{m_i}{k-(j-1)}$$

Number of page faults for old pages in phase i: F_i

$$\mathbb{E}[F_i] = \sum_{j=1}^{k-m_i} \overbrace{\mathbb{P}\left(j^{\text{th}} \text{ old page incurs page fault}\right)}^{=\mathbb{E}[F_{ij}]}$$

$$\leq \sum_{j=1}^{k-m_i} \frac{m_i}{k-(j-1)} = m_i \cdot \sum_{\ell=m_i+1}^{k} \frac{1}{\ell}$$

$$= m_i \cdot \big(H(k) - H(m_i)\big) \leq m_i \cdot (H(k) - 1)$$

$H(k) = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$

$m_i \geq 1$

Competitive Ratio

Theorem: Against an oblivious adversary, the randomized marking algorithm has a competitive ratio of at most $2H(k) \leq 2\ln(k) + 2$.

Proof:

- Assume that there are p phases
- #page faults of rand. marking algorithm in phase i: $F_i + m_i$
- We have seen that

$$\mathbb{E}[F_i] \leq m_i \cdot (H(k) - 1) \leq m_i \cdot \ln(k)$$

- Let F be the total number of page faults of the algorithm:

$$\mathbb{E}[F] \leq \sum_{i=1}^{p} \underbrace{(\mathbb{E}[F_i] + m_i)}_{\leq m_i \cdot H(k)} \leq H(k) \cdot \sum_{i=1}^{p} m_i$$

Competitive Ratio

Theorem: Against an oblivious adversary, the randomized marking algorithm has a competitive ratio of at most $2H(k) \leq 2\ln(k) + 2$.

phase $i-1$ phase i

k m_i

Proof:

$k + m_i$

- Let F_i^* be the number of page faults in phase i in an opt. exec.
- Phase 1: m_1 pages have to be replaced → $F_1^* \geq m_1$
- Phase $i > 1$:
  - Number of distinct page requests in phases $i-1$ and i: $\boldsymbol{k + m_i}$
  - Therefore, $\boldsymbol{F_{i-1}^* + F_i^* \geq m_i}$
- Total number of page ~~requests~~ faults F^*:

$$F^* = \sum_{i=1}^{p} F_i^* \geq \underbrace{\frac{1}{2} \cdot \left(F_1^* + \sum_{i=2}^{p} (F_{i-1}^* + F_i^*) \right)}_{\sum F_i^* - \frac{F_p^*}{2}} \geq \frac{1}{2} \cdot \sum_{i=1}^{p} m_i$$

Competitive Ratio

Theorem: Against an oblivious adversary, the randomized marking algorithm has a competitive ratio of at most $2H(k) \leq 2\ln(k) + 2$.

Proof:

- Randomized marking algorithm:

$$\mathbb{E}[F] \leq H(k) \cdot \sum_{i=1}^{p} m_i$$

- Optimal algorithm:

$$F^* \geq \frac{1}{2} \cdot \sum_{i=1}^{p} m_i$$

Remark: We next show that no randomized algorithm has a competitive ratio better than $H(k)$ (against an obl. adversary)

Randomized Lower Bound

Yao's Principle (more precisely Yao's Minimax Principle)**:**

exp. cost of best randomized alg. for worst-case input

$\cancel{\leq}$ $\geq$

exp. cost of best deterministic alg. for a given random input distr.

Proving a lower bound using Yao's principle:

- Design a random input distribution
- Show that every deterministic algorithm has a bad expected competitive ratio if the input is chosen at random according to this distribution
- Yao's principle then implies that every randomized algorithm is at least equally bad for worst-case input
  - worst-case fixed input: holds even for oblivious adversary

Randomized Paging Lower Bound

Input Distribution

- There are $k + 1$ different pages ~~in the slow memory~~
- In each step, a uniformly random page is requested

Deterministic Online Algorithms

- Consider some request i

p

  – Current state of the fast memory depends on requests $i - 1$ and on the algorithm, assume that page p is **not** in fast memory

  – $\mathbb{P}(\text{page fault}) = \mathbb{P}(\text{request for page } p) = \frac{1}{k+1}$

- Expected #page faults after n requests:

$$\boxed{\frac{n}{k + 1}}$$

Randomized Paging Lower Bound

Best Offline Algorithm: Longest Forward Distance

- After each page fault, optimal offline algorithm ~~loads~~ evicts the page that will not be used for the longest possible time
- After a page fault, all $k+1$ pages are requested at least once before the next page fault

time between two page faults $=$ time to request each page at least once -1

Claim: If $T =$ time to request each page once, then coupon collector process

$$\mathbb{E}[T] = (k+1) \cdot H(k+1)$$

- For $i \in \{0, \dots, k+1\}$: T_i time to request i^{th} page after requesting $i-1$ different pages $T_1 = 1$ T_2 T_3 $T = \sum_{i=1}^{k+1} T_i$
- Probability for req. i^{th} page after requesting $i-1$ diff. pages:

$$p_i = \frac{k+1-(i-1)}{k+1}$$

Randomized Paging Lower Bound

Claim: If T = time to request each page once, then

$$\mathbb{E}[\boldsymbol{T}] = (\boldsymbol{k}+\mathbf{1}) \cdot \boldsymbol{H}(\boldsymbol{k}+\mathbf{1})$$

- For $i \in \{0, \dots, k+1\}$: T_i time to request i^{th} page after requesting $i-1$ different pages
- Prob. for req. i^{th} page after req. $i-1$ diff. pages: $p_i = \frac{k+1-(i-1)}{k+1}$

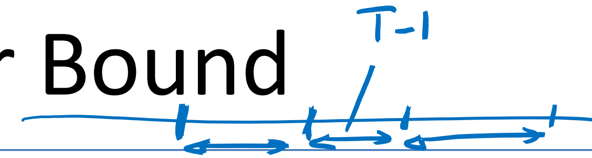
$$T = T_1 + T_2 + T_3 + T_4 + \dots + T_{k+1}$$

$$T_i \sim \mathrm{Geom}(p_i) \qquad E[T_i] = \frac{1}{p_i} = \frac{k+1}{k+1-(i-1)} = (k+1) \cdot \frac{1}{k+1-(i-1)}$$

$$E[T] = \sum_{i=1}^{k+1} E[T_i] = (k+1) \cdot \sum_{i=1}^{k+1} \frac{1}{k+1-(i-1)} = (k+1) \sum_{j=1}^{k+1} \frac{1}{j}$$

$$= (k+1) \cdot H(k+1)$$

Randomized Paging Lower Bound



Claim: For $k+1$ pages and n uniformly random requests, the optimal expected number of page faults is at most

$$\frac{n}{(k+1)\cdot H(k)} - 1$$

Proof Sketch

- Average time $\bar{T}$ between page faults

$$\mathbb{E}[\bar{T}] = \mathbb{E}[T] - 1 = (k+1)H(k+1) - 1 = (k+1)\cdot H(k)$$

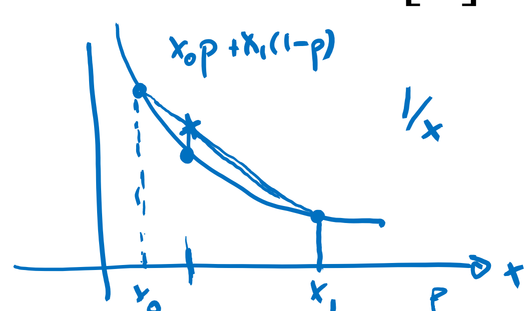
- Number of page faults $X = \lfloor n/\bar{T} \rfloor \geq \frac{n}{\bar{T}} - 1$

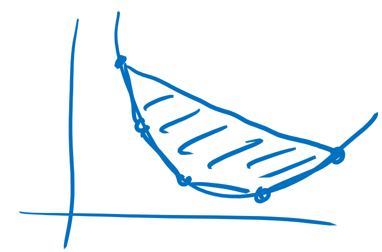
$$\mathbb{E}[X] = \mathbb{E}\left[\left\lfloor \frac{n}{\bar{T}} \right\rfloor\right] \geq \mathbb{E}\left[\frac{n}{\bar{T}}\right] - 1 \geq \frac{n}{\mathbb{E}[\bar{T}]} - 1$$

pos. rand. var. $X > 0$

$$\mathbb{E}\left[\frac{1}{X}\right] \geq \frac{1}{\mathbb{E}[X]}$$

Jensen's Ineq.





$$\mathbb{E}\left[\frac{1}{X}\right] = \underbrace{\mathbb{P}(X=x_0)}_{p}\cdot\frac{1}{x_0} + \underbrace{\mathbb{P}(X=x_1)}_{1-p}\cdot\frac{1}{x_1}$$

Randomized Paging Lower Bound

Theorem: Every randomized paging algorithm has competitive ratio at least $H(k)$ even for an oblivious adversary.

1. Assume we $k + 1$ pages and uniformly random page requests
2. Expected number of page faults of best deterministic algorithm
$$= \frac{n}{k+1}$$
3. Expected number of page faults of optimal algorithm
$$\geq \frac{n}{(k+1) \cdot H(k)} - 1$$
4. Yao's principle now proves the theorem
   – not really necessary here, step 2 also works directly for randomized alg.

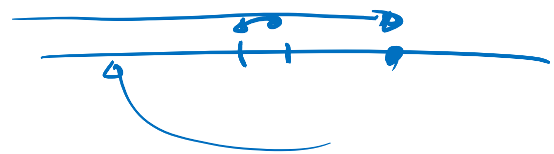
Self-Adjusting Lists

- Linked lists are often inefficient
  - Cost of accessing an item at position i is linear in i
- But, linked lists are extremely simple
  - And therefore nevertheless interesting
- Can we at least improve the behavior of linked lists?
- In practical applications, not all items are accessed equally often and not equally distributed over time
  - The same items might be used several times over a short period of time
- **Idea:** rearrange list after accesses to optimize the structure for future accesses
- **Problem:** We don't know the future accesses
  - The list rearrangement problems is an online problem!

Model

- Only find operations (i.e., access some item)
  - Let's ignore insert and delete operations
  - Results can be generalized to cover insertions and deletions

Cost Model:



- Accessing item at position i costs i
- The only operation allowed for rearranging the list is swapping two adjacent list items
- Swapping any two adjacent items costs 1

Rearranging The List

Frequency Count (FC):

- For each item keep a count of how many times it was accessed
- Keep items in non-increasing order of these counts
- After accessing an item, increase its count and move it forward past items with smaller count

Move-To-Front (MTF):

- Whenever an item is accessed, move it all the way to the front

Transpose (TR):

- After accessing an item, swap it with its predecessor

Cost

Cost when accessing item at position i:

- Frequency Count (FC): between i and $2i - 1$
- Move-To-Front (MTF): $2i - 1$
- Transpose (TR): $i + 1$

Random Accesses:

- If each item x has an access probability p_x and the items are accessed independently at random using these probabilities, FC and TR are asymptotically optimal

n requests

avg. cost per req. $n \to \infty$

Real access patterns are not random, TR usually behaves badly and the much simpler MTF often beats FC

Move-To-Front

- We will see that MTF is competitive
- To analyze MTF we need competitive analysis and amortized analysis

Operation k:

- Assume, the operation accesses item x at position i
- c_k: actual cost of the MTF algorithm

$$c_k = 2i - 1$$

- a_k: amortized cost of the MTF algorithm
- c_k^*: actual cost of an optimal offline strategy
  - Let's call the optimal offline strategy OPT

$$\boxed{\begin{array}{c}\sum c_i \leq \sum a_i \\ \\ a_k \leq \alpha \cdot c_k^*\end{array}}$$

$$\sum c_i \leq \sum a_i \leq \alpha \sum c_i^*$$

Potential Function

- For the analysis, we think of running the MTF and OPT at the same time
- The state of the system is determined by the two lists of MTF and OPT
- Similarly to amortized analysis for data structures, we use a potential function which maps the system state to a real number
- If the MTF list and the list of OPT are similar, the actual cost of both algorithms for most requests is roughly the same
- If the lists are very different, the costs can be very different and the potential function should have a large value to be able to compensate for the potentially high cost difference
- We therefore use a **potential function** which measures the **difference between** the **MTF list** and the **optimal offline list**

Potential Function

Potential Function $\boldsymbol{\Phi_k}$:

- **Inversion:** pair of items x and y such that x precedes y in one list and y precedes x in the other list
- ***Twice*** the number of ***inversions*** between the lists of MTF and OPT after the first k operations
- Measure for the difference between the lists after k operations

Initially, the two lists are identical: $\boldsymbol{\Phi_0 = 0}$

For all k, it holds that $\boldsymbol{0 \leq \Phi_k \leq 2 \cdot \binom{n}{2} = n(n-1)}$

Potential Function

$\boxed{a_k \le \alpha \cdot c_k^*}$

$a_k = c_k + \phi_k - \phi_{k-1}$

Potential Function Φ_k:

- **Inversion:** pair of items x and y such that x precedes y in one list and y precedes x in the other list
- ***Twice*** the number of ***inversions*** between the lists of MTF and OPT after the first k operations
- Measure for the difference between the lists after k operations

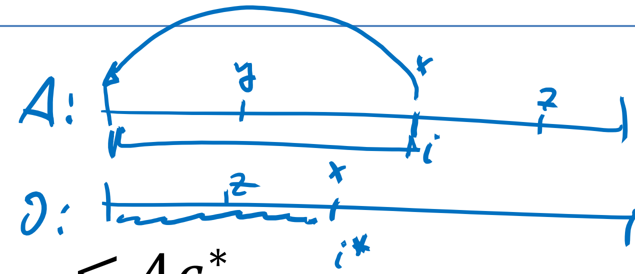
To show that MTF is α-competitive, we will show that

$$\forall k: \; a_k = c_k + \Phi_k - \Phi_{k-1} \le \alpha \cdot c_k^*$$

$$\sum_{i=1}^{T} a_k = \sum_{i=1}^{T} (c_k + \phi_k - \phi_{k-1}) = \sum_{i=1}^{T} c_k + \phi_T \ge \sum_{i=1}^{T} c_k$$

Competitive Analysis

Theorem: **MTF** is **4-competitive**.



Proof:

- Need that $a_k = c_k + \Phi_k - \Phi_{k-1} \leq 4c_k^*$
- Position of x in list of OPT: i^*
- Number of swaps of OPT: s^*
- In MTF list, position of x is changed w.r.t. to the $i-1$ preceding items (nothing else is changed)
- For each of these items, either an inversion is created or one is destroyed (before the s^* swaps of OPT)

$i-1-(i^*-1) = i-i^*$

- Number of new inversions (before OPT's swaps) $\leq i^* - 1$:
  - Before op. k, only $i^* - 1$ items are before x in OPT's list
  - With all other items, x is ordered the same as in OPT's list after moving it to the front

Competitive Analysis

Theorem: **MTF** is **4-competitive**.

Proof:

- Need that $a_k = c_k + \Phi_k - \Phi_{k-1} \leq 4c_k^*$
- $c_k = 2i - 1$, $c_k^* = i^* + s^*$
- Number of inversions created: $\leq i^* - 1 + s^*$
- Number of inversions destroyed: $\geq i - i^*$

$$a_k = 2i - 1 + 2(i^* - 1 + s^* - i + i^*)$$

$$= 2i - 1 + 4i^* - 2 + 2s^* - 2i$$

$$= 4i^* + 2s^* - 3 \leq 4(i^* + s^*) = 4c_k^*$$