Chapter 10
Parallel Algorithms

Algorithm Theory
WS 2017/18

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Ex. Sheet 7
hand in by Mon, Feb. 5

Sequential Algorithms

Classical Algorithm Design:

- One machine/CPU/process/… doing a computation

RAM (Random Access Machine):

- Basic standard model
- Unit cost basic operations
- Unit cost access to all memory cells

Sequential Algorithm / Program:

- Sequence of operations
  (executed one after the other)

Parallel and Distributed Algorithms

Today's computers/systems are not sequential:

- Even cell phones have several cores
- Future systems will be highly parallel on many levels
- This also requires appropriate algorithmic techniques

Goals, Scenarios, Challenges:

- Exploit parallelism to speed up computations
- Shared resources such as memory, bandwidth, …
- Increase reliability by adding redundancy
- Solve tasks in inherently decentralized environments
- …

Parallel and Distributed Systems

- Many different forms

- Processors/computers/machines/... communicate and share data through
  - Shared memory or message passing
- Computation and communication can be
  - Synchronous or asynchronous
- Many possible topologies for message passing
- Depending on system, various types of faults

Challenges

Algorithmic and theoretical challenges:

- How to parallelize computations
- Scheduling (which machine does what)
- Load balancing
- Fault tolerance
- Coordination / consistency
- Decentralized state
- Asynchrony
- Bounded bandwidth / properties of comm. channels
- …

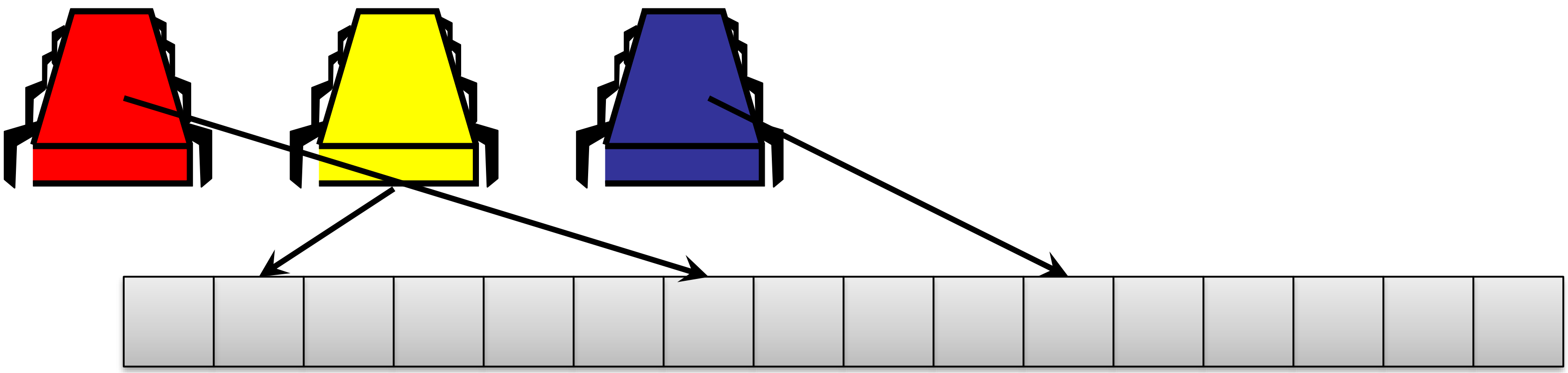
Models

- A large variety of models, e.g.:

- **PRAM** (Parallel Random Access Machine)
  - Classical model for parallel computations
- **Shared Memory**
  - Classical model to study coordination / agreement problems, distributed data structures, …
- **Message Passing** (fully connected topology)
  - Closely related to shared memory models
- Message Passing in **Networks**
  - Decentralized computations, large parallel machines, comes in various flavors…

PRAM

- Parallel version of RAM model
- p processors, shared random access memory



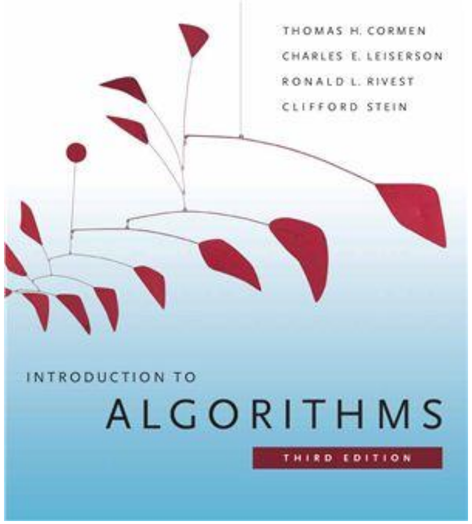
- Basic operations / access to shared memory cost 1
- Processor operations are synchronized time is divided rounds
- Focus on parallelizing computation rather than cost of communication, locality, faults, asynchrony, …

Other Parallel Models

- **Message passing:** Fully connected network, local memory and information exchange using messages

- **Dynamic Multithreaded Algorithms:** Simple parallel programming paradigm
  - E.g., used in Cormen, Leiserson, Rivest, Stein (CLRS)

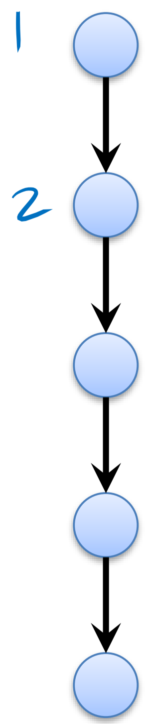
```
FIB(n)
if n < 2
    then return n
x ← spawn FIB(n − 1)
y ← spawn FIB(n − 2)
sync
return (x + y)
```



Parallel Computations

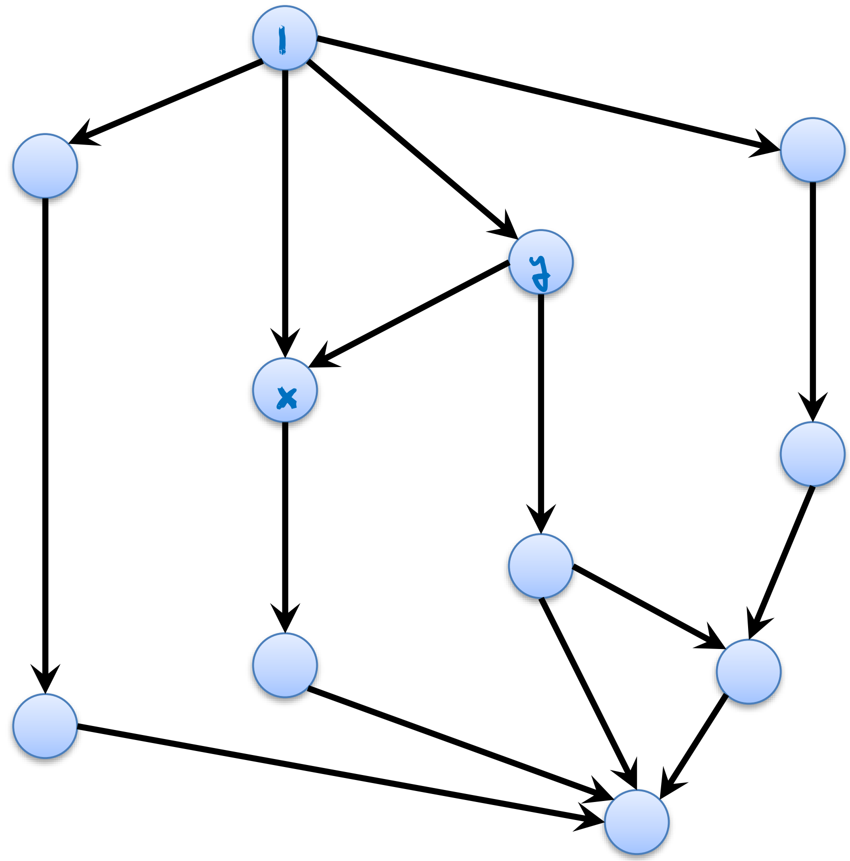
Sequential Computation:

- Sequence of operations



Parallel Computation:

- Directed Acyclic Graph (DAG)



Parallel Computations

p : # processors

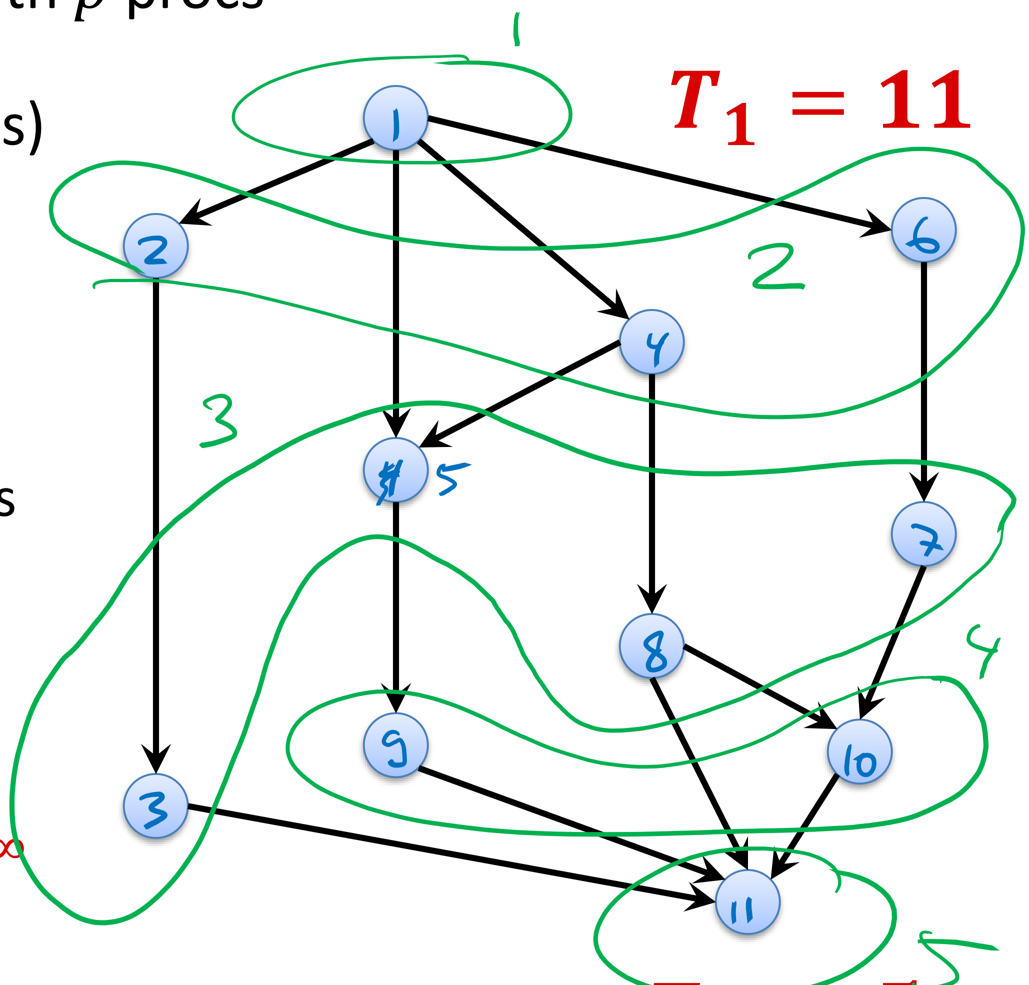
extreme cases: $p=1$, $p=\infty$

T_p: time to perform comp. with p procs

- T_1: **work** (total # operations)
  - Time when doing the computation sequentially
- T_∞: **critical path / span**
  - Time when parallelizing as much as possible
- **Lower Bounds**:

$$T_p \geq \left\lceil \frac{T_1}{p} \right\rceil, \qquad T_p \geq T_\infty$$

$$T_1 = 11$$



$$T_\infty = 5$$

Parallel Computations

T_p: time to perform comp. with p procs

- **Lower Bounds**:

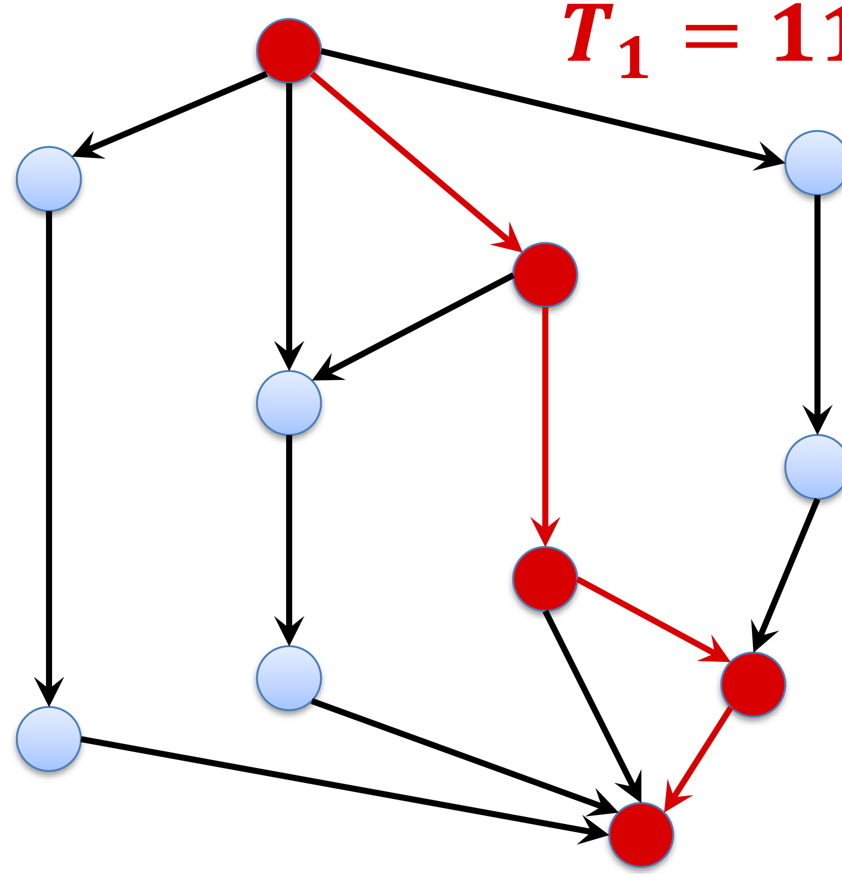
$$T_p \geq \frac{T_1}{p}, \qquad T_p \geq T_\infty$$

- **Parallelism**: $\frac{T_1}{T_\infty}$ $\qquad \frac{11}{5}$
  – maximum possible speed-up

- **Linear Speed-up**:

$$\frac{T_1}{T_p} = \Theta(p)$$

$T_1 = 11$

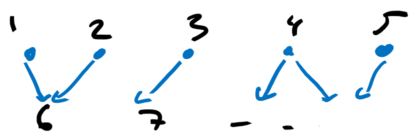


$T_\infty = 5$

Scheduling

- How to assign operations to processors?

- Generally an online problem
  - When scheduling some jobs/operations, we do not know how the computation evolves over time



Greedy (offline) scheduling:

- Order jobs/operations as they would be scheduled optimally with ∞ processors (topological sort of DAG)
  - Easy to determine: With ∞ processors, one always schedules all jobs/ops that can be scheduled
- Always schedule as many jobs/ops as possible
- Schedule jobs/ops in the same order as with ∞ processors
  - i.e., jobs that become available earlier have priority

Brent's Theorem

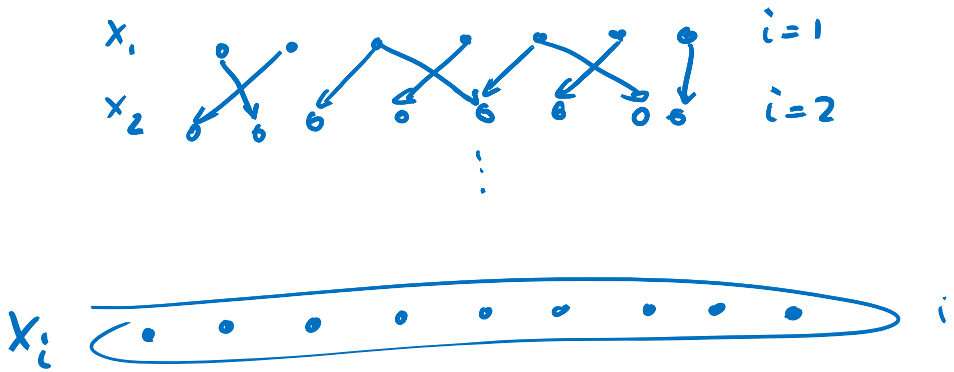
$T_p \geq \frac{T_1}{p}, T_p \geq T_\infty$

Brent's Theorem: On p processors, a parallel computation can be performed in time

$$T_p \leq \frac{T_1 - T_\infty}{p} + T_\infty.$$

Proof:

- Greedy scheduling achieves this…
- #operations scheduled with ∞ processors in round i: x_i



Brent's Theorem

Brent's Theorem: On p processors, a parallel computation can be performed in time

$$T_p \leq \frac{T_1 - T_\infty}{p} + T_\infty.$$

Proof:

- Greedy scheduling achieves this…
- #operations scheduled with ∞ processors in round i: x_i

p procs. t_i: time to schedule x_i operations

$$t_i = \left\lceil \frac{x_i}{p} \right\rceil \leq \frac{x_i}{p} + \frac{p-1}{p} = \frac{x_i - 1}{p} + 1$$

$$T_p^G \leq \sum_{i=1}^{T_\infty} t_i \leq \underbrace{\frac{1}{p}\sum_{i=1}^{T_\infty} x_i}_{\frac{T_1}{p}} - \underbrace{\sum_{i=1}^{T_\infty} \frac{1}{p}}_{\frac{T_\infty}{p}} + \underbrace{\sum_{i=1}^{T_\infty} 1}_{T_\infty} = \frac{T_1 - T_\infty}{p} + T_\infty$$

$\square$

Brent's Theorem

Brent's Theorem: On p processors, a parallel computation can be performed in time

$$T_p \leq \frac{T_1 - T_\infty}{p} + T_\infty.$$

Corollary: Greedy is a 2-approximation algorithm for scheduling.

lower bounds

$T_p^* \geq T_\infty$

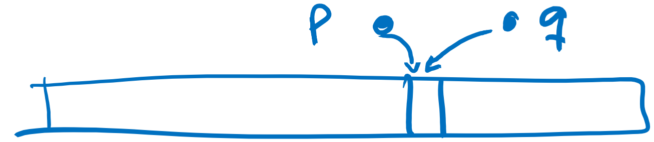
$T_p^* \geq \frac{T_1}{p}$

$$T_p^G \leq \frac{T_1}{p} + \frac{p-1}{p} \cdot T_\infty \leq \frac{2p-1}{p} \cdot T_p^* < 2 \cdot T_p^*$$

max. possible speed-up

Corollary: As long as the number of processors $p = O(T_1/T_\infty)$, it is possible to achieve a linear speed-up.

PRAM



Back to the PRAM:

- Shared random access memory, synchronous computation steps
- The PRAM model comes in variants…

EREW (exclusive read, exclusive write):

- Concurrent memory access by multiple processors is not allowed
- If two or more processors try to read from or write to the same memory cell concurrently, the behavior is not specified

CREW (concurrent read, exclusive write):

- Reading the same memory cell concurrently is OK
- Two concurrent writes to the same cell lead to unspecified behavior also concurrent read & write not allowed
- This is the first variant that was considered (already in the 70s)

PRAM

The PRAM model comes in variants…

CRCW (concurrent read, concurrent write):

- Concurrent reads and writes are both OK
- Behavior of concurrent writes has to specified
  - Weak CRCW: concurrent write only OK if all processors write 0
  - Common-mode CRCW: all processors need to write the same value
  - Arbitrary-winner CRCW: adversary picks one of the values
  - Priority CRCW: value of processor with highest ID is written
  - Strong CRCW: largest (or smallest) value is written

(ID, x)

- The given models are ordered in strength:

weak $\leq$ common-mode $\leq$ arbitrary-winner $\leq$ priority $\leq$ strong

Some Relations Between PRAM Models

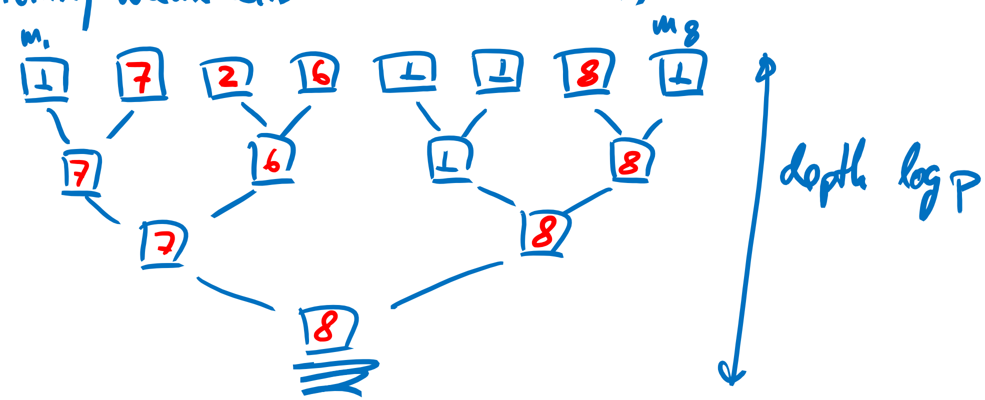
Theorem: A parallel computation that can be performed in time t, using p proc. on a strong CRCW machine, can also be performed in time $O(t \log p)$ using p processors on an EREW machine.

- Each (parallel) step on the CRCW machine can be simulated by $O(\log p)$ steps on an EREW machine

concurrent writes: memory cell c, coordinate write access to cell c

cell c: auxiliary mem. cells (initialized to $\perp$)

say: $p=8$



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Theorem: A parallel computation that can be performed in time t, using p proc. on a strong CRCW machine, can also be performed in time $O(t \log p)$ using p processors on an EREW machine.

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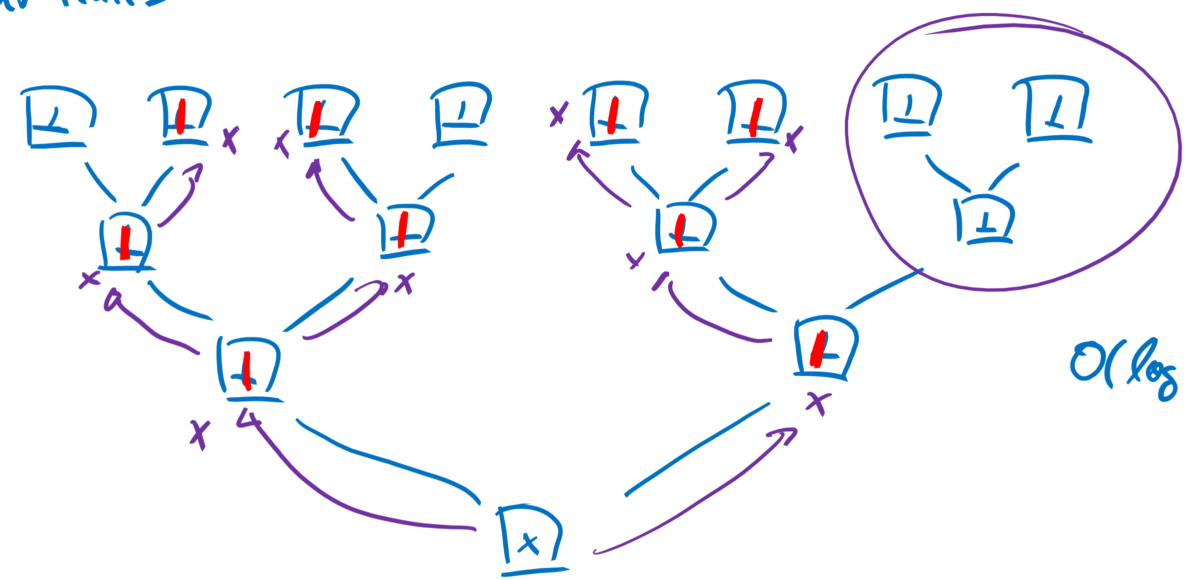
concurrent reads



$O(\log p)$ parallel steps

Some Relations Between PRAM Models

Theorem: A parallel computation that can be performed in time t, using p proc. on a strong CRCW machine, can also be performed in time $O(t \log p)$ using p processors on an EREW machine.

- Each (parallel) step on the CRCW machine can be simulated by $O(\log p)$ steps on an EREW machine

$p = \frac{T_1}{T_\infty}$

Theorem: A parallel computation that can be performed in time t, using p probabilistic processors on a strong CRCW machine, can also be performed in expected time $O(t \log p)$ using $O(p/\log p)$ processors on an arbitrary-winner CRCW machine.

- The same simulation turns out more efficient in this case

Some Relations Between PRAM Models

Theorem: A computation that can be performed in time t, using p processors on a strong CRCW machine, can also be performed in time $O(t)$ using $O(p^2)$ processors on a weak CRCW machine

Proof:

- Strong: largest value wins, weak: only concurrently writing 0 is OK

Simulate 1 step of a strong CRCW PRAM on a weak CRCW PRAM

processors: strong CRCW: $1, \dots, p$

additional procs: $q_{i,j}$ for every pair (i,j), $i,j \in \{1, \dots, p\}$ $(i<j)$

additional memory cells

for all $i \in \{1, \dots, p\}$: f_i, v_i, a_i (initialized to 0)

if proc. $i \in \{1, \dots, p\}$ wants to write x to memory cell c (in strong CRCW PRAM)

$f_i := 1, \ v_i := x, \ a_i := c$

Some Relations Between PRAM Models

Theorem: A computation that can be performed in time t, using p processors on a strong CRCW machine, can also be performed in time $O(t)$ using $O(p^2)$ processors on a weak CRCW machine

Proof:

- Strong: largest value wins, weak: only concurrently writing 0 is OK

proc i wants to write x to cell c : $f_i := 1$, $v_i := x$, $a_i := c$

$\forall i,j$: $q_{i,j}$ reads $f_i, f_j, v_i, v_j, a_i, a_j$ (assume $i < j$)

if $f_i = f_j = 1$ and $a_i = a_j$ then

if $v_j \geq v_i$ then $f_i := 0$

else $f_j := 0$

mem. cell c

proc		
2	:	10
3	:	8
5	:	12
7	:	7

$f_2 = 0$

$f_3 = 0$, $f_3 = 0$

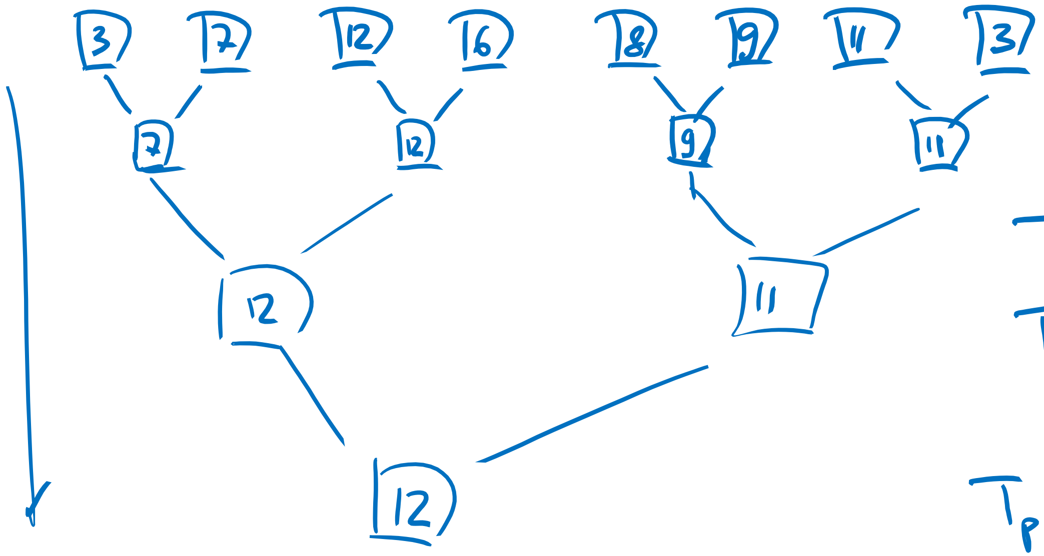
proc. i writes v_i to cell a_i $\Longleftrightarrow$ $f_i = 1$

Computing the Maximum

Given: n values

Goal: find the maximum value

Observation: The maximum can be computed in parallel by using a binary tree.



Can be done on an EREW PRAM

$T_1 = O(n)$

$T_\infty = O(\log n)$

$T_p = O\left(\frac{n}{p} + \log n\right)$

Computing the Maximum

Observation: On a strong CRCW machine, the maximum of a n values can be computed in $O(1)$ time using n processors

- Each value is concurrently written to the same memory cell

Lemma: On a weak CRCW machine, the maximum of n integers between 1 and $\sqrt{n}$ can be computed in time $O(1)$ using $O(n)$ proc.

Proof:

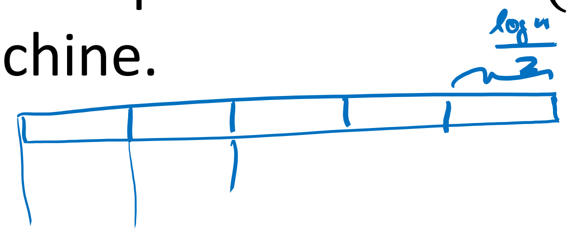
$\boxed{f_1}\ \boxed{0}\ \cdots\ \boxed{f_{\sqrt{n}}}$

- We have $\sqrt{n}$ memory cells $f_1, \dots, f_{\sqrt{n}}$ for the possible values
- Initialize all $f_i := 1$
- For the n values $x_1, \dots, x_n$, processor j sets $f_{x_j} := 0$
  - Since only zeroes are written, concurrent writes are OK
- Now, $f_i = 0$ iff value i occurs at least once
- Strong CRCW machine: max. value in time $O(1)$ w. $O(\sqrt{n})$ proc.
- Weak CRCW machine: time $O(1)$ using $O(n)$ proc. (prev. lemma)

Computing the Maximum

values are integers $\leq n^c$

Theorem: If each value can be represented using $O(\log n)$ bits, the maximum of n (integer) values can be computed in time $O(1)$ using $O(n)$ processors on a weak CRCW machine.



Proof:

- First look at $\frac{\log_2 n}{2}$ highest order bits
- The maximum value also has the maximum among those bits
- There are only $\sqrt{n}$ possibilities for these bits
- max. of $\frac{\log_2 n}{2}$ highest order bits can be computed in $O(1)$ time
- For those with largest $\frac{\log_2 n}{2}$ highest order bits, continue with next block of $\frac{\log_2 n}{2}$ bits, …