# Chapter 10
# Parallel Algorithms

## Algorithm Theory
## WS 2017/18

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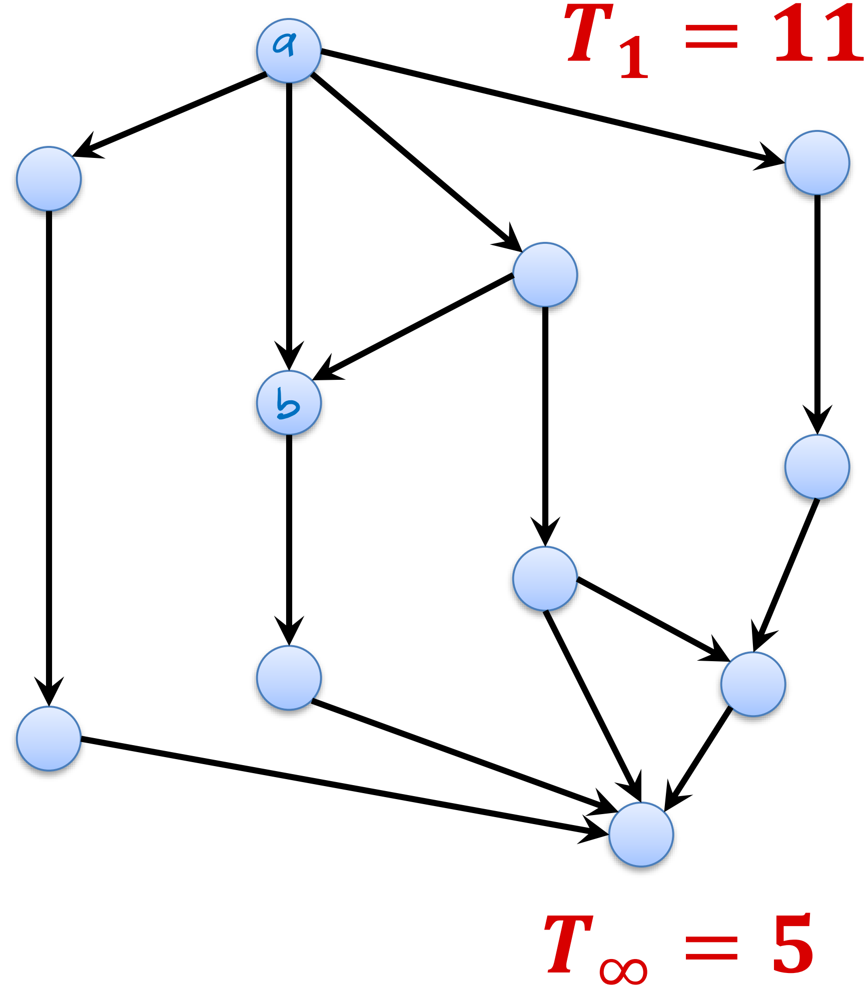
# Parallel Computations

$T_p$: time to perform comp. with $p$ procs

- $T_1$: **work** (total # operations)
  - Time when doing the computation sequentially
- $T_\infty$: **critical path** / **span**
  depth
  - Time when parallelizing as much as possible
- **Lower Bounds**:

$$T_p \geq \frac{T_1}{p}, \qquad T_p \geq T_\infty$$



# Parallel Computations

$\boldsymbol{T_p}$: time to perform comp. with $p$ procs

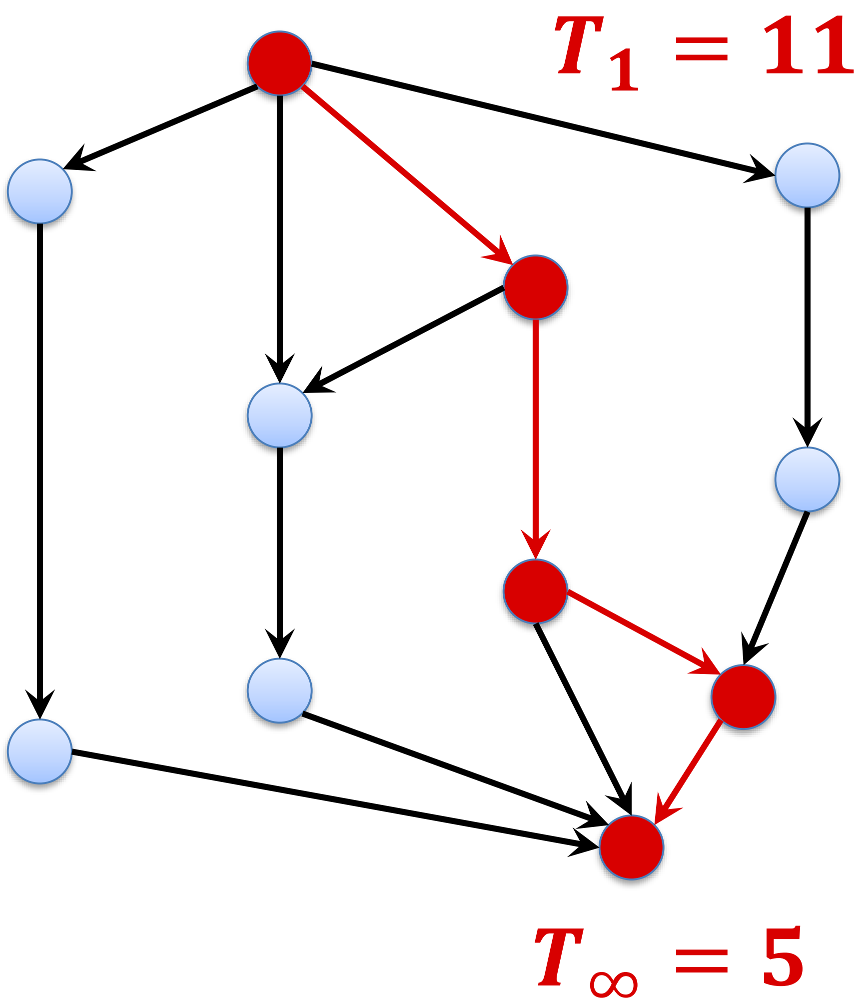
- **Lower Bounds**:

$$T_p \geq \frac{T_1}{p}, \qquad T_p \geq T_\infty$$

- **Parallelism**: $\boxed{\frac{T_1}{T_\infty}}$
  - maximum possible speed-up

- **Linear Speed-up**:

$$\frac{T_p}{T_1} = \Theta(p)$$



# Brent's Theorem

**Brent's Theorem:** On $p$ processors, a parallel computation can be performed in time

$$T_p \leq \frac{T_1 - T_\infty}{p} + T_\infty.$$

**Corollary:** Greedy is a 2-approximation algorithm for scheduling.

**Corollary:** As long as the number of processors $p = O(T_1 / T_\infty)$, it is possible to achieve a linear speed-up.

# PRAM

Back to the PRAM:

- Shared random access memory, synchronous computation steps
- The PRAM model comes in variants…

**EREW (exclusive read, exclusive write):**

- Concurrent memory access by multiple processors is not allowed
- If two or more processors try to read from or write to the same memory cell concurrently, the behavior is not specified

**CREW (concurrent read, exclusive write):**

- Reading the same memory cell concurrently is OK
- Two concurrent writes to the same cell lead to unspecified behavior
- This is the first variant that was considered (already in the 70s)

# PRAM

The PRAM model comes in variants…

**CRCW (concurrent read, concurrent write):**

- Concurrent reads and writes are both OK
- Behavior of concurrent writes has to specified
  - Weak CRCW: concurrent write only OK if all processors write $0$
  - Common-mode CRCW: all processors need to write the same value
  - Arbitrary-winner CRCW: adversary picks one of the values
  - Priority CRCW: value of processor with highest ID is written
  - Strong CRCW: largest (or smallest) value is written
- The given models are ordered in strength:

  **weak $\leq$ common-mode $\leq$ arbitrary-winner $\leq$ priority $\leq$ strong**

# Prefix Sums

- The following works for any associative binary operator $\oplus$:

**associativity:** $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

**All-Prefix-Sums:** Given a sequence of $n$ values $a_1, \ldots, a_n$, the all-prefix-sums operation w.r.t. $\oplus$ returns the sequence of prefix sums:

$$s_1, s_2, \ldots, s_n = a_1, a_1 \oplus a_2, a_1 \oplus a_2 \oplus a_3, \ldots, a_1 \oplus \cdots \oplus a_n$$

- Can be computed efficiently in parallel and turns out to be an important building block for designing parallel algorithms

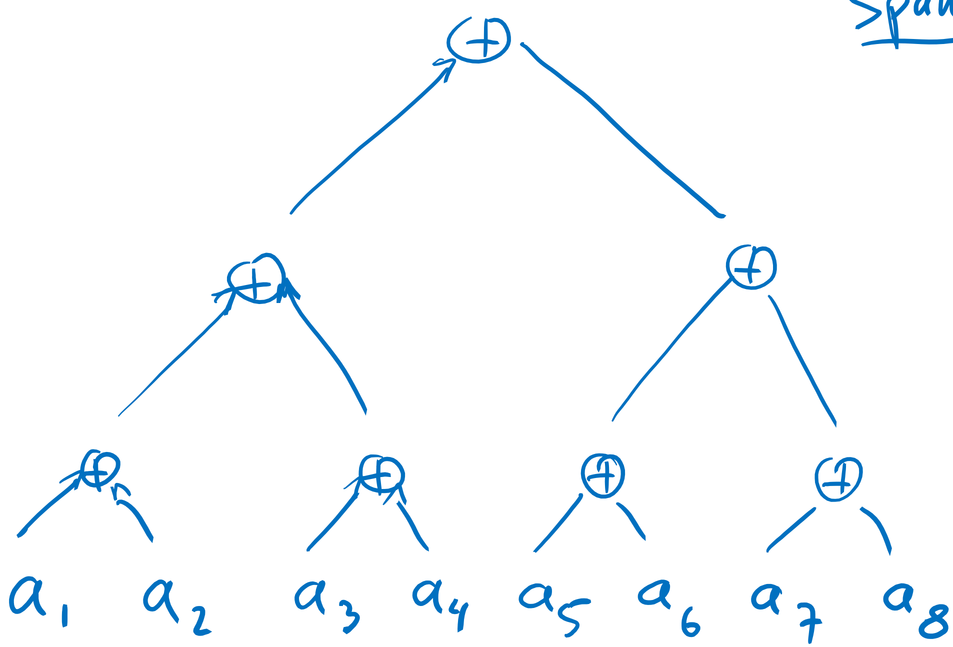
**Example:** Operator: $+$, input: $a_1, \ldots, a_8 = 3, 1, 7, 0, 4, 1, 6, 3$

$s_1, \ldots, s_8 =$ 3, 4, 11, 11, 15, 16, 22, 25

# Computing the Sum

- Let's first look at $s_n = a_1 \oplus a_2 \oplus \cdots \oplus a_n$

- Parallelize using a binary tree:




Work: $O(n)$

Span: $O(\log n)$

Using Brent's Thm:

can compute $s_n$ in time $O(\log n)$ using $O(n/\log n)$ processors

# Computing the Sum

**Lemma:** The sum $s_n = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ can be computed in time $O(\log n)$ on an EREW PRAM. The total number of operations (total work) is $O(n)$.

**Proof:**

**Corollary:** The sum $s_n$ can be computed in time $O(\log n)$ using $O(n/\log n)$ processors on an EREW PRAM.

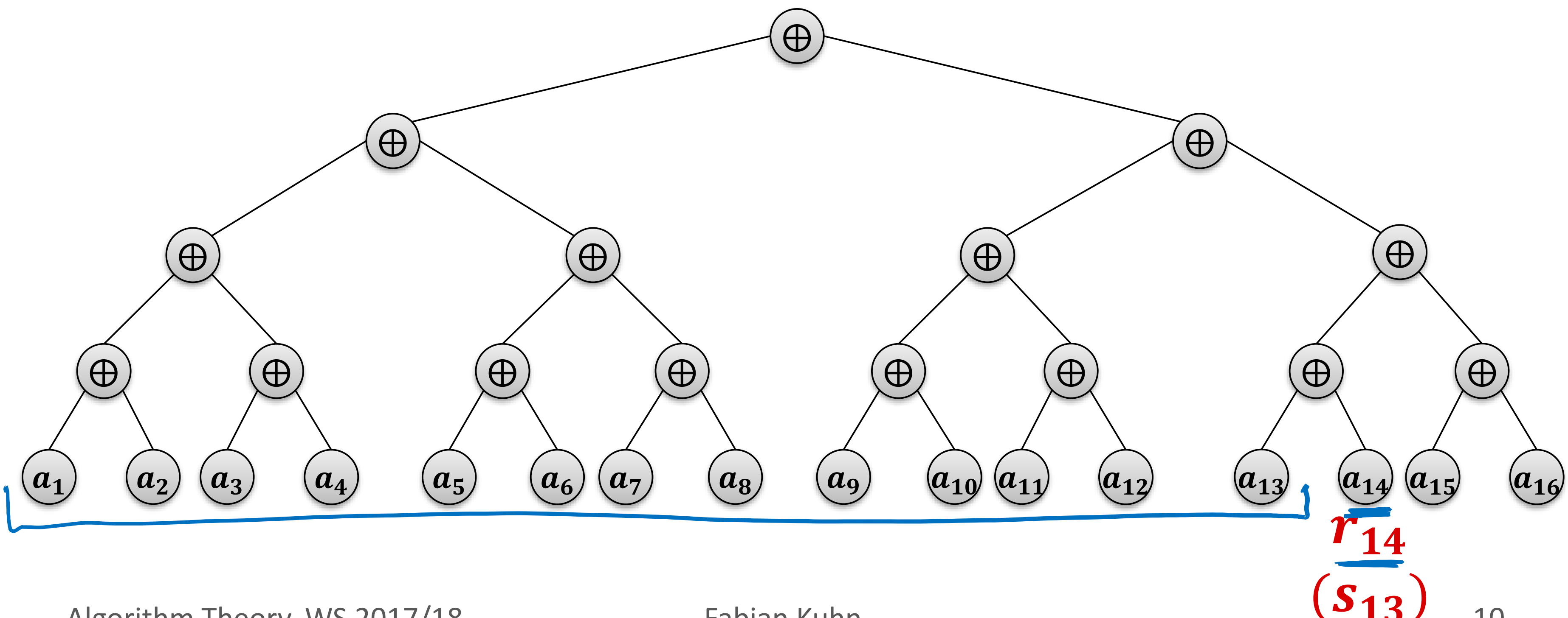
**Proof:**

- Follows from Brent's theorem ($T_1 = O(n)$, $T_\infty = O(\log n)$)

# Getting The Prefix Sums

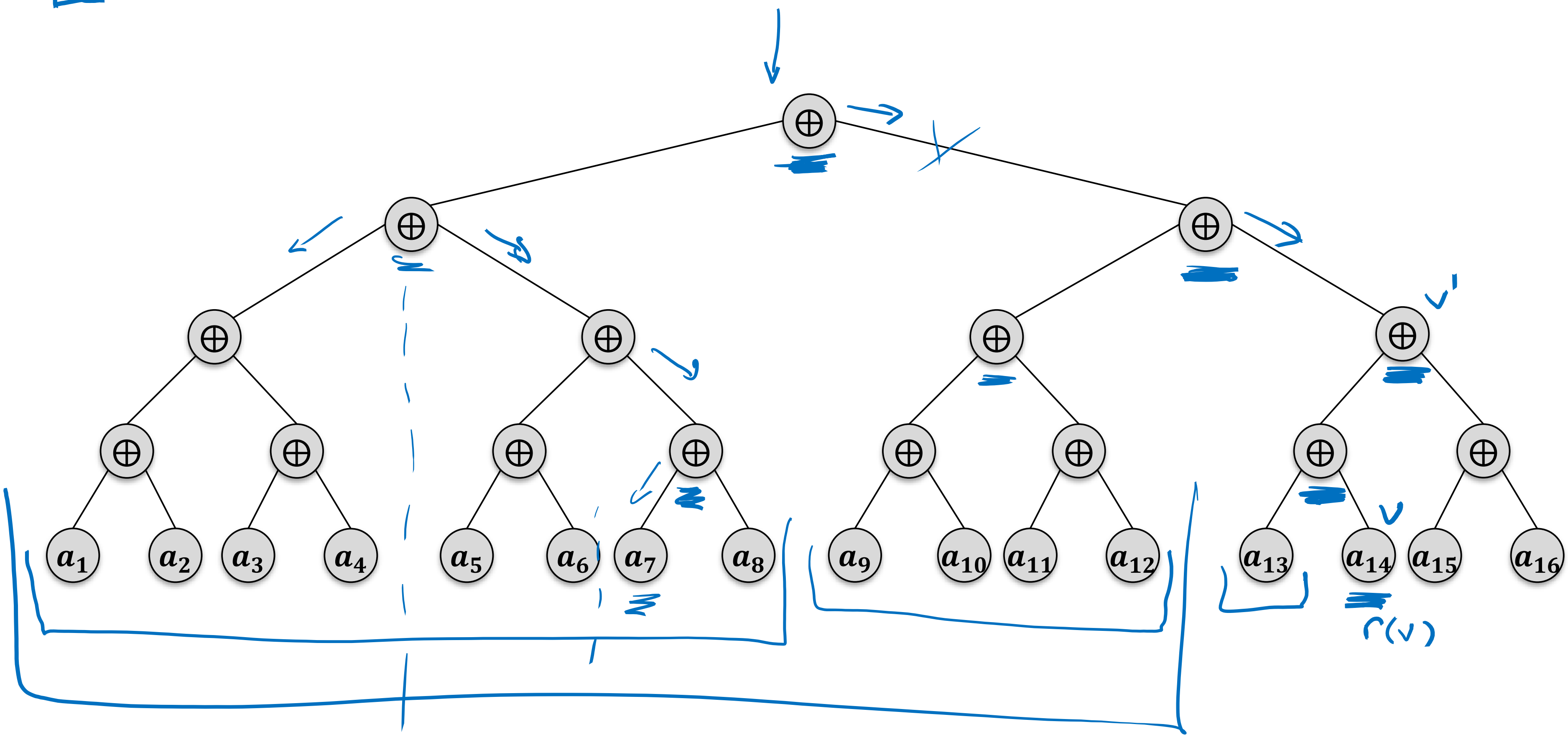
$s_1, \dots, s_n$ $\boxed{s_i = r_i \oplus a_i}$

- Instead of computing the sequence $s_1, s_2, \dots, s_n$ let's compute
  $r_1, \dots, r_n = 0, s_1, s_2, \dots, s_{n-1}$ (0: neutral element w.r.t. $\oplus$)
  $$r_1, \dots, r_n = 0, a_1, a_1 \oplus a_2, \dots, a_1 \oplus \cdots \oplus a_{n-1}$$
- Together with $s_n$, this gives all prefix sums
- Prefix sum $r_i = s_{i-1} = a_1 \oplus \cdots \oplus a_{i-1}$:



# Getting The Prefix Sums

**Claim:** The prefix sum $r_i = a_1 \oplus \cdots \oplus a_{i-1}$ is the sum of all the leaves in the left sub-tree of ancestor $u$ of the leaf $v$ containing $a_i$ such that $v$ is in the right sub-tree of $u$.



# Computing The Prefix Sums

$r_i$

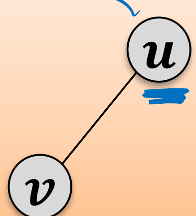
**For each node $v$ of the binary tree, define $r(v)$ as follows:**

- $r(v)$ is the sum of the values $a_i$ at the leaves in all the left sub-trees of ancestors $u$ of $v$ such that $v$ is in the right sub-tree of $u$.

For a leaf node $v$ holding value $a_i$: $\boldsymbol{r(v) = r_i = s_{i-1}}$

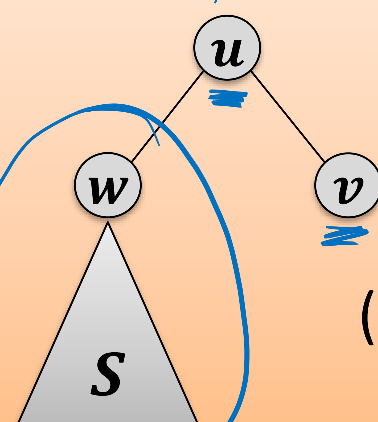
For the root node: $\boldsymbol{r(\text{root}) = 0}$

For all other nodes $v$:




$v$ is the left child of $u$:

$$r(v) = r(u)$$



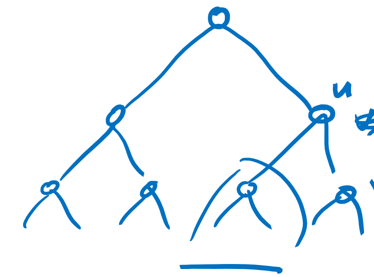

$v$ is the right child of $u$:
($u$ has left child $w$)

$$r(v) = r(u) + S$$

($S$: sum of values in sub-tree of $w$)

# Computing The Prefix Sums

- leaf node $v$ holding value $a_i$: $\boldsymbol{r(v) = r_i = s_{i-1}}$
- root node: $\boldsymbol{r(\mathbf{root}) = 0}$
- Node $v$ is the left child of $u$: $r(v) = r(u)$
- Node $v$ is the right child of $u$: $r(v) = r(u) + S$
  - Where: $S =$ sum of values in left sub-tree of $u$

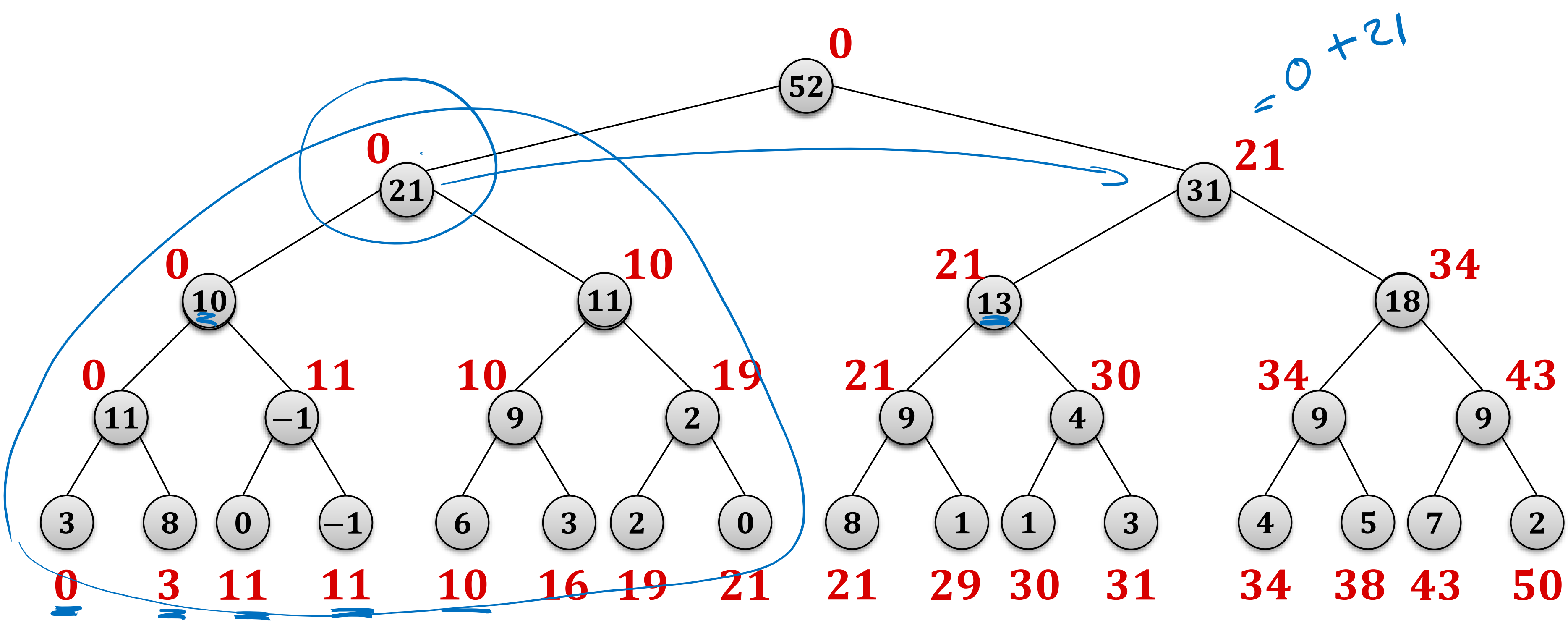



**Algorithm to compute values $r(v)$:**

1. Compute sum of values in each sub-tree (bottom-up)
   - Can be done in parallel time $O(\log n)$ with $O(n)$ total work
2. Compute values $r(v)$ top-down from root to leaves:
   - To compute the value $r(v)$, only $r(u)$ of the parent $u$ and the sum of the left sibling (if $v$ is a right child) are needed
   - Can be done in parallel time $O(\log n)$ with $O(n)$ total work

# Example

1. Compute sums of all sub-trees
   – Bottom-up (level-wise in parallel, starting at the leaves)
2. Compute values $r(v)$
   – Top-down (starting at the root)



# Computing Prefix Sums

**Theorem:** Given a sequence $a_1, \dots, a_n$ of $n$ values, all prefix sums $s_i = a_1 \oplus \cdots \oplus a_i$ (for $1 \leq i \leq n$) can be computed in time $O(\log n)$ using $O(n/\log n)$ processors on an EREW PRAM.

**Proof:**

- Computing the sums of all sub-trees can be done in parallel in time $O(\log n)$ using $O(n)$ total operations.
- The same is true for the top-down step to compute the $r(v)$
- The theorem then follows from Brent's theorem:

$$T_1 = O(n), \qquad T_\infty = O(\log n) \qquad \Longrightarrow \qquad T_p < T_\infty + \frac{T_1}{p}$$

**Remark:** This can be adapted to other parallel models and to different ways of storing the value (e.g., array or list)

# Parallel Quicksort

≤ 16 | 16 | > 16

- Key challenge: parallelize partition

pivot

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 14 | 18 | 8 | 19 | 21 | 3 | 1 | 25 | 17 | 11 | 4 | 20 | 10 | 26 | 2 | 9 | 13 | 23 | 16 |

partition

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 14 | 8 | 3 | 1 | 11 | 4 | 10 | 2 | 9 | 13 | 16 | 18 | 19 | 21 | 25 | 17 | 20 | 26 | 23 |

- How can we do this in parallel?
- For now, let's just care about the values $\leq$ pivot
- What are their new positions

# Using Prefix Sums

- Goal: Determine positions of values $\leq$ pivot after partition

pivot

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 14 | 18 | 8 | 19 | 21 | 3 | 1 | 25 | 17 | 11 | 4 | 20 | 10 | 26 | 2 | 9 | 13 | 23 | 16 |

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 |

prefix sums

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 | 5 | 6 | 7 | 7 | 8 | 8 | 9 | 10 | 11 | 11 | 12 |

partition

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 14 | 8 | 3 | 1 | 11 | 4 | 10 | 2 | 9 | 13 | 16 | 18 | 19 | 21 | 25 | 17 | 20 | 26 | 23 |

# Applying to Quicksort

**Theorem:** On an EREW PRAM, using $p$ processors, randomized quicksort can be executed in time $T_p$ (in expectation and with high probability), where

$$T_p = O\left(\frac{n \log n}{p} + \log^2 n\right).$$

**Proof:**

work per partition step: $O(n)$, span of partition step: $O(\log n)$

total work: $O(n \log n)$, total span: $O(\log^2 n)$

**Remark:**

- We get optimal (linear) speed-up w.r.t. to the sequential algorithm for all $p = O(n/\log n)$.

# Partition Using Prefix Sums

- The positions of the entries $>$ pivot can be determined in the same way
- **Prefix sums:** $T_1 = O(n), \quad T_\infty = O(\log n)$
- **Remaining computations:** $T_1 = O(n), \quad T_\infty = O(1)$
- **Overall:** $T_1 = O(n), \quad T_\infty = O(\log n)$

**Lemma:** The partitioning of quicksort can be carried out in parallel in time $O(\log n)$ using $O\left(\frac{n}{\log n}\right)$ processors.

**Proof:**

- By Brent's theorem: $T_p \le \frac{T_1}{p} + T_\infty$

# Other Applications of Prefix Sums

- Prefix sums are a very powerful primitive to design parallel algorithms.
  - Particularly also by using other operators than "+"

**Example Applications:**

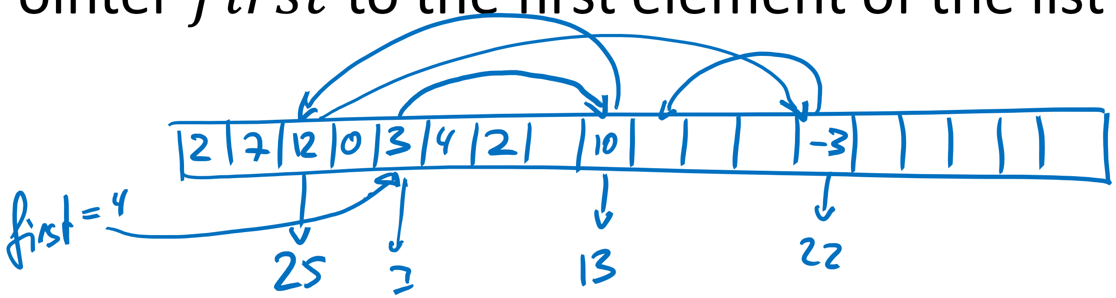
- Lexical comparison of strings
- Add multi-precision numbers
- Evaluate polynomials
- Solve recurrences
- Radix sort / quick sort
- Search for regular expressions
- Implement some tree operations
- …

# Prefix Sums in Linked Lists

**Given:** Linked list $L$ of length $n$ in the following way

- Elements are in an array $A$ of length $n$ in an unordered way
- Each array element $A[i]$ also contains a next pointer
- Pointer $first$ to the first element of the list

first = 4
A[4].next = 8
A[8].next = 2

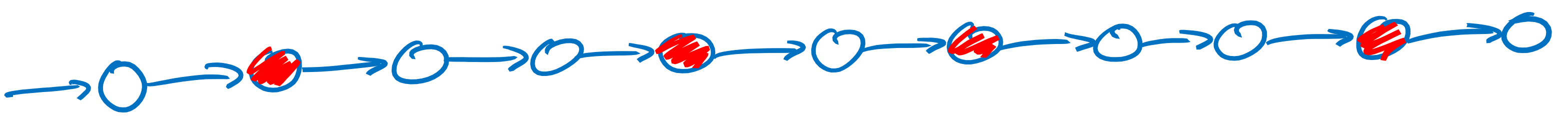



**Goal:** Compute all prefix sums w.r.t. to the order given by the list

# 2-Ruling Set of a Linked List

Given a linked list, select a subset of the entries such that

- No two neighboring entries are selected
- For every entry that is not selected, either the predecessor or the successor is selected
  - i.e., between two consecutive selected entries there are at least one and at most two unselected entries

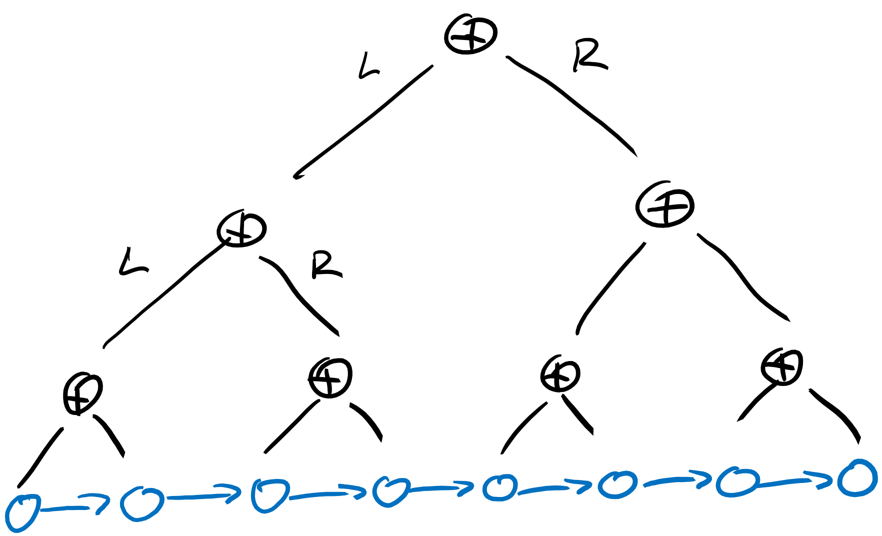


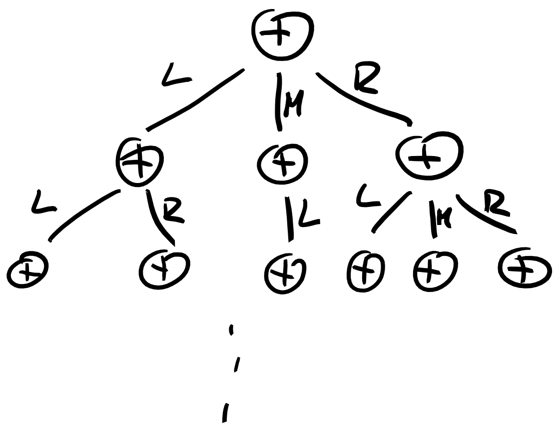
- We will see that a 2-ruling set of a linked list can be computed efficiently in parallel

# Using 2-Ruling Sets to Get Prefix Sums

**Observations:**

- To compute the prefix sums of an array/list of numbers, we need a binary tree such that the numbers are at the leaves and an in-order traversal of the tree gives the right order
- The algorithm can be generalized to non-binary trees



# Using 2-Ruling Sets to Get Prefix Sums

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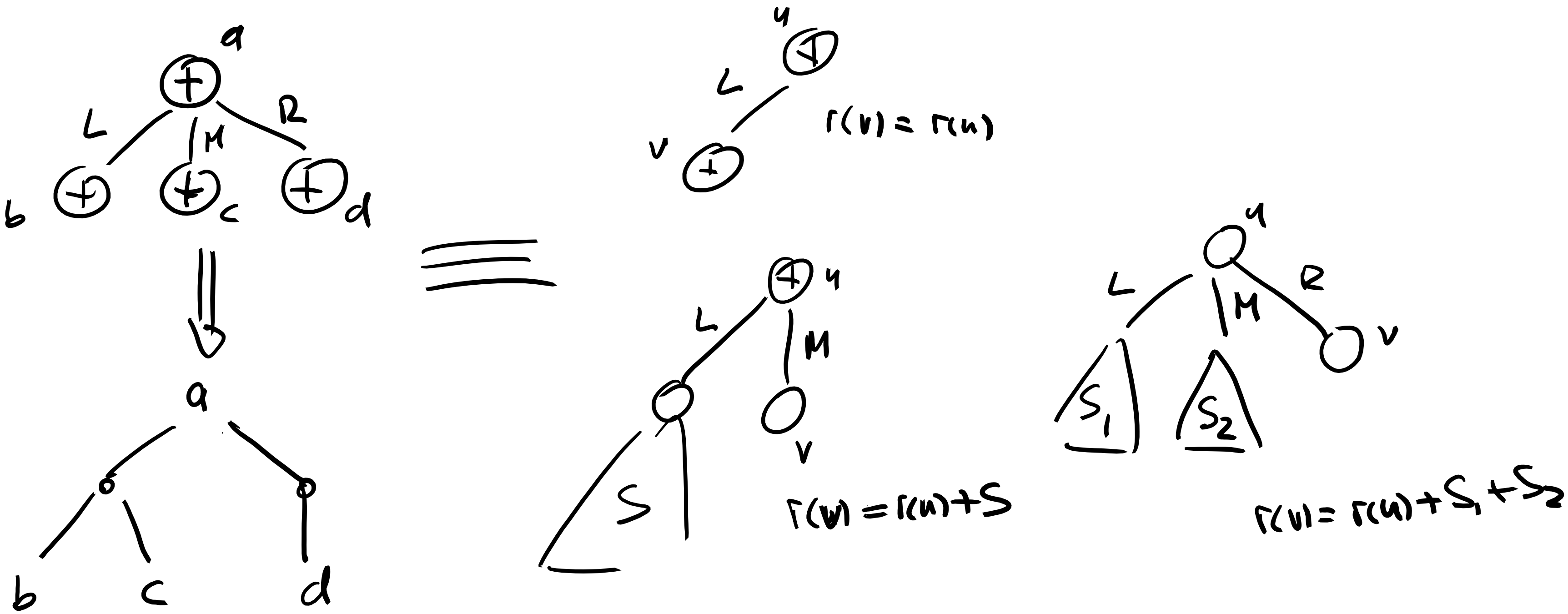
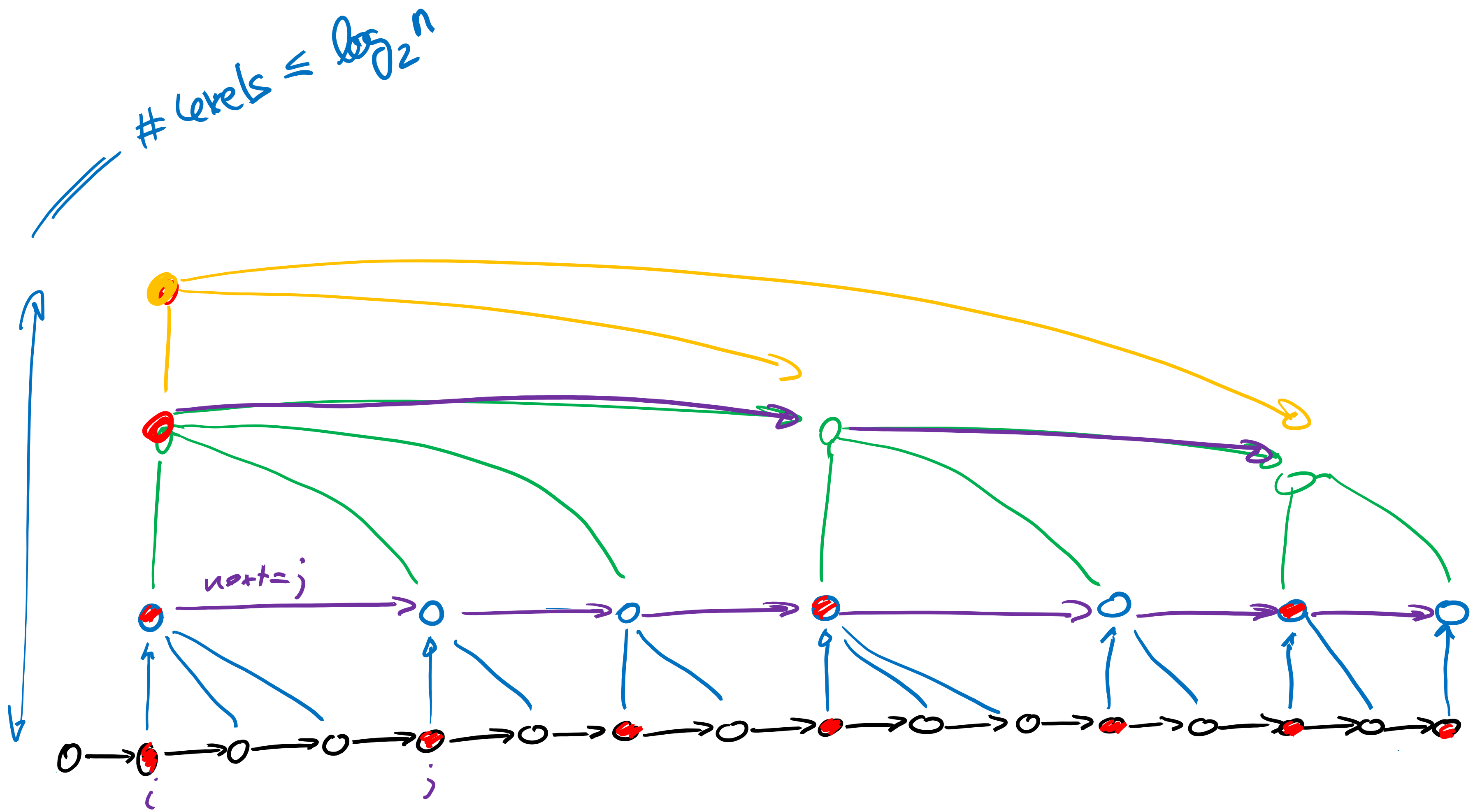


# Using 2-Ruling Sets to Get Prefix Sums

**Basic Idea:**

- Use 2-ruling sets to build a tree of logarithmic depth



# Using 2-Ruling Sets to Get Prefix Sums

$w(n) \geq n$

**Lemma:** If a 2-Ruling Set of a list of length $N$ can be computed in parallel with $w(N)$ work and $d(N)$ depth, all prefix sums of a list of length $n$ can be computed in parallel with

- Work $O\big(w(n) + w(n/2) + w(n/4) + \cdots + w(1)\big)$ $+ O(n)$
- Depth $O\big(d(n) + d(n/2) + d(n/4) + \cdots + d(1)\big)$ $+ O(\log n)$

**Proof Sketch:**

$d(n) \geq 1$

build ruling sets : bottom level : $w(n)$

2nd level : list of length $\leq n/2$ : $w(n/2)$

$\vdots$

(also for the depth / span)

additional work : $O(n)$ additional span: $O(\log n)$

$w(n) = n \cdot \log^* n$ , $d(n) = \log^* n$

# Prefix Sums in Linked Lists

$\log^{(2)} x = \log(\log(x))$

**Log-Star Function:**

- For $i \geq 1$: $\log_2^{(i)} x = \log_2\left(\log_2^{(i-1)} x\right)$, and $\log_2^{(0)} x = x$
- For $x > 2$: $\log^* x := \min\{i : \log^{(i)} x \leq 2\}$, for $x \leq 2$: $\log^* x := 1$

\# times to apply $\log_2$ to get value $\leq 2$

$\log^* 4 = 2$

\# atoms $\approx 10^{80}$

$\log^* 10^{80} = 5$

**Lemma:** A 2-ruling set of a linked list of length $n$ can be computed in parallel with work $O(n \cdot \log^* n)$ and span $O(\log^* n)$.

- i.e., in time $O(\log^* n)$ using $O(n)$ processors
  - We will first see how to apply this and prove it afterwards…

# Prefix Sums in Linked Lists

**Lemma:** A 2-ruling set of a linked list of length $n$ can be computed in parallel with work $O(n \cdot \log^* n)$ and span $O(\log^* n)$.

**Theorem:** All prefix sums of a linked list of length $n$ can be computed in parallel with total work $O(n \cdot \log^* n)$ and span $O(\log n \cdot \log^* n)$.

- i.e., in time $O(\log n \cdot \log^* n)$ using $O(n / \log n)$ processors.

work : $W(n) + W(n/2) + \ldots + W(1)$

$\leq O\left(\log^* n \cdot \left(n + \frac{n}{2} + \frac{n}{4} + \ldots + 1\right)\right) = O(n \log^* n)$

span : $O(\log n \cdot \log^* n)$

# Computing 2-Ruling Sets

- Instead of computing a 2-ruling set, we first compute a coloring of the list:
  - each list element gets a color s.t. adjacent elements get different colors

- Each element initially has a unique $\log n$-bit label in $\{1, \dots, N\}$
  - can be interpreted as an initial coloring with $N$ colors

**Algorithm runs in phases:**

- Each phase: compute new coloring with smaller number of colors

We will show that

- #phases to get to $O(1)$ colors is $O(\log^* n)$
- each phase has $O(n)$ work and $O(1)$ depth

# Reducing the number of colors

Assume that we start with a coloring with colors $\{0, \dots, x-1\}$

$a \to b \to c$

$a' = f(a,b)$ $b' = f(b,c)$

should guarantee that $f(a,b) \neq f(b,c)$

position 5 (first) diff of a & b)

9 8 7 6 5 4 3 2 1 0

$a = 0010110000 \to 0101.1$ value of pos. 5 bit of a

$b = 1010010000 \to 1000.0$

$c = 0110010000 \to 1000.1$

$d = 1010010000$

# Reducing the number of colors

Assume that we start with a coloring with colors $\{0, \dots, x-1\}$

get valid new coloring

initial coloring : $\lfloor \log_2 x \rfloor$ bits

largest new color $\leq \lfloor \log_2 x \rfloor \cdot 2 + 1$

$\# \text{bits}: \lfloor \log_2 (\lfloor \log_2 x \rfloor \cdot 2 + 1) \rfloor \approx \log_2 \log_2 x + 1$

need to repeat $O(\log^* n)$ times to get to $O(1)$ colors

# Reducing the number of colors

Assume that we start with a coloring with colors $\{0, \dots, x-1\}$

Stops when colors are $\in \{0, \dots, 5\}$ $5 = (101)_2$

$\underline{11}x \rightarrow \leq \underline{10.1}$

$1000 \rightarrow \leq 11.1$

as long as the old color $> 5$, the new color is strictly smaller