# Algorithm Theory - Exercise Class

## Exercise Lesson 5

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# Maximal Matching

Let $G = (V, E)$ be a graph. Show that the algorithm computes a matching $M \subseteq E$ of size $\underline{|M|} \geq \underline{\frac{1}{2}|M^*|}$ of an optimal matching $M^*$.

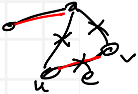
**Algorithmus 1 :** `Match`$(G)$

$M \leftarrow \emptyset$
**for** remaining $e = \{u, v\} \in E$ **do**
  $M \leftarrow M \cup \{e\}$
  $E \leftarrow E \setminus \{e' \mid e' \text{ adjacent to } u \text{ or } v\}$
**return** $M$

$v(M) \stackrel{!}{=}$ number of endpoints of edges in $M$.

$v(M) = 2 \cdot |M|$

$v(M^*) = 2 \cdot |M^*|$





Let $e \in M^*$ $e = \{u, v\}$

Then $u \in M$ or $v \in M$

$\Rightarrow v(M^*) \leq 2 \cdot v(M)$

$\Rightarrow |M^*| \leq 2 \cdot |M|$

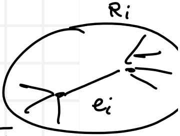
# Maximal Matching

Let $G = (V, E)$ be a graph with edge weights $w : E \to \mathbb{N}_{>0}$. Give a greedy algorithm and show that it computes a matching $M \subseteq E$ of size $w(M) = \sum_{e \in M} w(e) \geq \frac{1}{2} w(M^*)$ of an optimal matching $M^*$.

**Algorithmus 2 :** `MatchGreedy`$(G)$

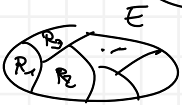
$M \leftarrow \emptyset$
**while** $E \neq \emptyset$ **do**
    $e = \{u, v\} \leftarrow$ heaviest edge in $E$
    $M \leftarrow M \cup \{e\}$
    $E \leftarrow E \setminus \{e' \mid e'$ adjacent to $u$ or $v\}$
**return** $M$

Let $e_i$ be $i$-th edge added to $M$
Let $R_i \subseteq E$ be the edges removed in the $i$-th step.




$R_1, \ldots, R_s$ partition $E$

$s = |M|$




$\forall e \in R_i: \quad w(e_i) \geq w(e)$

$$\underbrace{|M^* \cap R_i| \leq 2 \cdot |M \cap R_i|}_{}$$

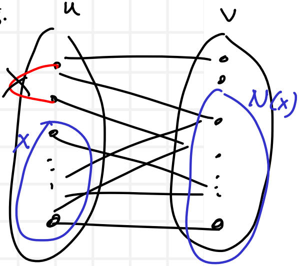
$$\Rightarrow w(M^* \cap R_i) \leq w(M \cap R_i)$$

$$\Rightarrow w(M^*) \leq \sum_{i=1}^{s} w(M^* \cap R_i) \leq 2 \sum_{i=1}^{s} w(M \cap R_i)$$
$$= 2 w(M)$$

# Maximal Matching

# Perfect Matching

An $r$-regular graph is a graph where each node has the same degree $r$. Show that any $r$**-regular bipartite** graph has a **perfect matching**.



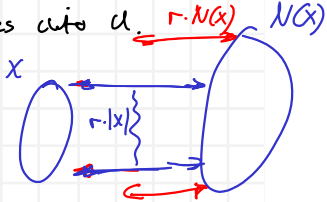

Halls Theorem: Bip. Graph $(U \cup V, E)$ with $|U| = |V|$. G has perf. matching iff $\forall X \subseteq U: |X| \le |N(X)|$.

$$r \cdot |U| = \sum_{u \in U} d(u) = |E| = \sum_{v \in V} d(v) = r \cdot |V|$$

a) $X$ has exactly $r|X|$ outgoing edges into $N(X)$.

b) $N(X)$ has exactly $r \cdot N(X)$ outgoing edges into $U$.

$\overset{a)+b)}{\Rightarrow} \quad r|X| \le r \cdot N(X)$



# Large Chromatic Number without Cliques

**Want to show:** *For any $k$ and $l$ there is a graph $G$ with $\chi(G) \geq k$ and no cycle shorter than $l$.*

A $c$-**coloring** of a graph $G = (V, E)$ is an assignment $\phi : V \to \{1, \ldots, c\}$ such that for each $\{u, v\} \in E$, $\phi(u) \neq \phi(v)$.

The **chromatic number** $\chi(G)$ is the smallest $c$ such that $G$ has a $c$-coloring.

An **independent set** $I \subset V$ of $G$ contains no nodes that are neighbors in $G$.

The **independence number** $\alpha(G)$ is the size of the largest ind. set of $G$.

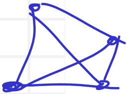
**(a)** Show that $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$

Nodes of one color are independent.

$\Rightarrow$ No color can have more than $\alpha(G)$ nodes.

$\Rightarrow$ At least $\frac{|V(G)|}{\alpha(G)}$ colors.

$\chi(G) = 4$



# Large Chromatic Number without Cliques

*For any $k$ and $l$ there is a graph $G$ with $\chi(G) \geq k$ and no cycle shorter than $l$.*

**"Roll the dice" to obtain a random graph and hope to get a good one.**

# Large Chromatic Number without Cliques

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**"Roll the dice" to obtain a random graph and hope to get a good one.**

Let $G = G_{n,p}$ with $n$ nodes. Edge $\{u, v\}$ exists with prob. $p = n^{\frac{1}{2\ell}-1} = \frac{\sqrt[2\ell]{n}}{n}$. $\xrightarrow{n \to \infty} 0$

# Large Chromatic Number without Cliques

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**(b)** Show that for $a = \lceil \frac{3}{p} \ln n \rceil$ we have: $\mathbb{P}[\alpha(G) \geq a] \longrightarrow 0 \ (n \to \infty)$.

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$$
\begin{aligned}
\mathbb{P}[\alpha(G) \geq a] &= \mathbb{P}[\exists W \subseteq V, W \text{ independent set}, |W| \geq a] \\
&= \mathbb{P}[\exists W \subseteq V, W \text{ independent set}, |W| = a] = \mathbb{P}\Big(\bigcup_{W \subseteq V} W \text{ ind.}, |W|=a\Big) \\
&\overset{\text{union bound}}{\leq} \sum_{W \subseteq V, |W|=a} \mathbb{P}[W \text{ is an independent set}] \\
&\leq \binom{n}{a} (1-p)^{\binom{a}{2}} \\
&\leq n^a e^{-pa(a-1)/2} \\
&\leq \frac{n^a}{n^{\frac{3}{2}(a-1)}} \xrightarrow{a \geq 2} 0 \ \ (n \to \infty).
\end{aligned}
$$

aus n wähle a

aus a wähle 2

$=$

$e^x \geq 1 + x$

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# Large Chromatic Number without Cliques

Let $p = n^{\frac{1}{2\ell}-1} = \frac{\sqrt[2\ell]{n}}{n}$ and **let $X$ be the number of cycles of length at most $\ell$.**

**(c)** For large $n$, show that $\mathbb{E}[X]$ can be upper bounded by $\frac{n}{4}$.

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Choose $(v_1, \ldots, v_j)$. The probability these nodes form a cycle in exactly that order is $p^j$. The number of series of nodes of length $j$ is at most $n^j$.

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$\mathbb{E}(\sum X_j)$ $\quad X_j :=$ number of cycles of length exactly $j$

$\mathbb{E}(X_j) \leq n^j p^j$

$$\mathbb{E}[X] \overset{=}{\leq} \sum_{j=3}^{\ell} n^j p^j = \sum_{j=3}^{\ell} n^{\frac{1}{2\ell}j} \leq \sum_{j=0}^{\ell} n^{\frac{1}{2\ell}j}$$

$$\overset{\text{geo. series}}{=} \frac{1-n^{\frac{\ell+1}{2\ell}}}{1-n^{\frac{1}{2\ell}}} = \frac{n^{-\frac{1}{2\ell}}-n^{\frac{\ell}{2\ell}}}{n^{-\frac{1}{2\ell}}-1} = \frac{n^{\frac{1}{2}}-n^{-\frac{1}{2\ell}}}{1-n^{-\frac{1}{2\ell}}} \leq \frac{n^{\frac{1}{2}}}{1-n^{-\frac{1}{2\ell}}} = \frac{n}{n^{\frac{1}{2}}(1-\frac{1}{\sqrt[2\ell]{n}})}$$

$\to 0$, $n \to \infty$

$$\sum_{i=0}^{\ell} r^{\ell} = \frac{1-r^{\ell+1}}{1-r}$$

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**(c)** For large $n$, show that $\mathbb{E}[X]$ can be upper bounded by $\frac{n}{4}$.

Choose $(v_1, \ldots, v_j)$. The probability these nodes form a cycle in exactly that order is $p^j$. The number of series of nodes of length $j$ is at most $n^j$.

$$\mathbb{E}[X] \leq \sum_{j=3}^{\ell} n^j p^j = \sum_{j=3}^{\ell} n^{\frac{1}{2\ell} j} \leq \sum_{j=0}^{\ell} n^{\frac{1}{2\ell} j}$$

$$= \frac{1 - n^{\frac{\ell+1}{2\ell}}}{1 - n^{\frac{1}{2\ell}}} = \frac{n^{-\frac{1}{2\ell}} - n^{\frac{\ell}{2\ell}}}{n^{-\frac{1}{2\ell}} - 1} = \frac{n^{\frac{1}{2}} - n^{-\frac{1}{2\ell}}}{1 - n^{-\frac{1}{2\ell}}} \leq \frac{n^{\frac{1}{2}}}{1 - n^{-\frac{1}{2\ell}}} = \frac{n}{n^{\frac{1}{2}}(1 - \frac{1}{\sqrt[2\ell]{n}})}$$

For large enough $n$, this is smaller than $\frac{n}{4}$.

# Large Chromatic Number without Cliques

From $(b)$ and $(c)$ we get $\mathbb{P}[X \geq n/2 \text{ or } \alpha(G) \geq a] < 1$.

$$\implies 1 - \mathbb{P}[X \geq n/2 \text{ or } \alpha(G) \geq a] > 0$$

$\mathbb{P}(X < n/2 \text{ and } \alpha(G) < a) > 0$

*Probability that $G$ has number of cycles with length less equal $\ell$ is less than $n/2$ and the independence number is smaller than $a$ is not zero.*

$$\implies \text{Such a graph exists! Call it } H.$$

$H$ has a small independence number but it might contain some short cycles.

**(d)** Construct $H'$ with no cycles of length $\leq \ell$, $\alpha(H') < a$ and $|V(H')| \geq n/2$.

**(e)** Show that $H'$ has chromatic number at least $k$.

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*Remove a node from each cycle in H.*

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$H$ has a small independence number but it might contain some short cycles.

**(d)** Construct $H'$ with no cycles of length $\leq \ell$, $\alpha(H') < a$ and $|V(H')| \geq n/2$.

*Remove a node from each cycle in $H$.*

**(e)** Show that $H'$ has chromatic number at least $k$.

$$\chi(H') \overset{a)}{\geq} \frac{|V(H')|}{\alpha(H')} \geq \frac{n/2}{3n^{1-\frac{1}{2\ell}} \ln n} = \frac{\sqrt[2\ell]{n}}{6 \ln n}. \qquad (a = \lceil \tfrac{3}{p} \ln n \rceil, p = n^{\frac{1}{2\ell}-1})$$

*Choose $n$ sufficiently large to obtain $\chi(H') > k$.*

# J. Ernovs' Poison Darts

The paranoid super villain Doctor Meta wants to improve the security of his secret laboratory to protect his ingenious research from trouble making secret agents. He designed a mechanism that shoots poisonous darts at potential intruders and instructed his forgetful chief chemist Joe Ernov to create a potent poison.

Unfortunately Mr. Ernov forgot to mix the two final chemicals $A$ and $B$ of the poison and instead filled these directly into the darts. Now exactly $p \cdot 100\%$ with $p \in (0,1)$ of the (many) darts contain chemical $A$ while the rest contain chemical $B$ and cannot be distinguished. The liquids $A$ and $B$ are still poisonous when injected separately such that the portion of $A$ is less than $p(1+\varepsilon)$ $(0 < \varepsilon < 1)$, as poor Joe had to find out.

After the demise of your predecessor you became the new chief chemist and are tasked to fix the problem. Your hunch is that if the victim is hit by sufficiently many ($d$) randomly selected darts, then the ratio $d_A/d$ ($d_A$ is the number of darts filled with $A$), should be below the effective threshold $d_A/d < p(1+\varepsilon)$.

Each dart is a guaranteed hit and delivers an equal amount of its chemical. Each dart is filled with chemical $A$ with uniform probability $p$ and else with chemical $B$. Use as few darts as possible such that the trap kills any intruding secret agents with high probability (w.h.p.), i.e. with probability $1 - \frac{1}{n^c}$ for a given constant $c > 0$ and number of darts $n \geq 2$. How many do you need? Of course Doctor Meta demands proof.

# J. Ernovs' Poison Darts

Determine number of darts $d$ s.t. $d_A/d < p(1+\varepsilon)$ w.h.p. $1-\frac{1}{n^c}$.

Chernoff:

$$\mathbb{P}(X \geq \mathbb{E}(X)(1+\varepsilon)) \leq \exp\left(\frac{-\varepsilon^2 \mathbb{E}(X)}{3}\right) \qquad d\cdot p$$

$$X = \sum X_i \,, \quad 0 \leq \varepsilon \leq 1$$

$A = H_2O$

$d$

$$\Leftarrow \quad \frac{d_A}{d} < p(1+\varepsilon)$$

Define $X_i = 1$ if dart $i$ contains $A$, $X_i = 0$ else. $\qquad \mathbb{P}(X_i = 1) = p$

$$X = \sum_{i=1}^{d} X_i \qquad \mathbb{E}(X) = dp$$

$$\Rightarrow \mathbb{P}\left(\frac{X}{d} \geq p(1+\varepsilon)\right) = \mathbb{P}\left(X \geq \overset{\mathbb{E}(X)}{dp}(1+\varepsilon)\right) \overset{!}{\leq} \frac{1}{n^c}$$

choose $d = \frac{c\cdot 3\cdot \ln n}{\varepsilon^2 \cdot p}$

$$\Rightarrow \mathbb{P}(X \geq d\cdot p(1+\varepsilon)) \leq \exp\left(\frac{-\varepsilon^2 d\cdot p}{3}\right) = \frac{1}{n^c}$$

# U. Borunds' Faulty Sensors

Doctor Metas incompetent chief engineer Unir Borund designed $n$ sensors to detect possible intruders. It turns out that the sensor design is flawed and experiments show that in a given time slot (round) of scanning a sensor fails to detect an intruder with rather high probability $\frac{1}{2}$.

(a) Having heard of your work with poison darts, Mr. Borund asks you to help him make his sensors more reliable by repeated rounds of scanning. He wants you to prove that the probability of failure of a *single* sensor can be decreased to $\frac{1}{n^c}$ for given constant $c > 0$ with as few rounds $r$ as possible.

$\mathbb{P}(\text{"Failure until round } r\text{"}) = \left(\frac{1}{2}\right)^r$

choose $r \geq c \cdot \log_2 n$

$\Rightarrow \left(\frac{1}{2}\right)^r \leq \frac{1}{n^c}$

# U. Borunds' Faulty Sensors

Doctor Meta demands that an intruder is detected by *all* $n$ sensors with probability at least $1-\frac{1}{n^c}$ for given $c>0$. Metas chief engineer was tasked to devise a method that still requires only $r \in \mathcal{O}(\log n)$ rounds of scanning, but failed.

(b) After the chief engineer demised in a tragic accident (that may or may not have involved sharks), Doctor Meta approaches you to solve the problem.

$$P\Big(\bigcup_{i\le n} \text{"Failure of sensor } i\text{"}\Big) \le \sum_{i=n} \left(\frac{1}{2}\right)^r = n\cdot\left(\frac{1}{2}\right)^r \overset{!}{\le} \frac{1}{n^c}$$

choose $r = \underline{\underline{2}}\cdot x\cdot \log_2 n$

$$n\cdot\frac{1}{2^r} = n\frac{1}{n^{2x}} = \frac{1}{n^{2x-1}}$$

choose $x$ s.t. $2x-1>c$