Repetition of Course Material

Let $L_1, L_2$ be languages (problems) over alphabets $\Sigma_1, \Sigma_2$. Then $L_1 \leq_p L_2$ ($L_1$ is polynomially reducible to $L_2$), iff a function $f : \Sigma_1^* \to \Sigma_2^*$ exists, that can be calculated in polynomial time and

$$\forall s \in \Sigma_1 : s \in L_1 \iff f(s) \in L_2.$$ 

Language $L$ is called $\mathcal{NP}$-hard, if all languages $L' \in \mathcal{NP}$ are polynomially reducible to $L$, i.e.

$$L \text{ $\mathcal{NP}$-hard } \iff \forall L' \in \mathcal{NP} : L' \leq_p L.$$ 

The reduction relation '$\leq_p$' is transitive ($L_1 \leq_p L_2$ and $L_2 \leq_p L_3 \Rightarrow L_1 \leq_p L_3$). Therefore, in order to show that $L$ is $\mathcal{NP}$-hard, it suffices to reduce a known $\mathcal{NP}$-hard problem $\tilde{L}$ to $L$, i.e. $\tilde{L} \leq_p L$.

Finally a language is called $\mathcal{NP}$-complete ($\iff$) $L \in \mathcal{NPC}$, if

1. $L \in \mathcal{NP}$ and
2. $L$ is $\mathcal{NP}$-hard.

Exercise 1: The class $\mathcal{NPC}$

This exercise is really (!!) important for the course.

A subset of the nodes of a graph $G$ is a dominating set if every other node of $G$ is adjacent to some node in the subset. Let

$$\text{DOMINATINGSET} = \{ (G,k) \mid \text{has a dominating set with } k \text{ nodes} \}.$$ 

Show that DOMINATINGSET is in $\mathcal{NPC}$. Use that

$$\text{VERTEXCOVER} := \{ (G,k) \mid \text{ Graph } G \text{ has a vertex cover of size at most } k \} \in \mathcal{NPC}.$$ 

Remark: A VERTEXCOVER is a subset $V' \subseteq V$ of nodes of $G = (V,E)$ such that every edge of $G$ is adjacent to a node in the subset.

Exercise 2: $P$ and $NP$?

Let $\text{CNF}_k = \{ \langle \phi \rangle \mid \phi \text{ is a satisfiable cnf-formula where each variable appears in at most } k \text{ places} \}$.

(a) Assume that $P \neq NP$ holds. Decide whether $\text{CNF}_2$ is in $P$ or in $\mathcal{NP} \setminus P$. Prove your claim!

(b) Show that $\text{CNF}_3$ is $\mathcal{NP}$-complete.

Remark: You can gain 3 additional points in this exercise to pass the 50% barrier.
Exercise 3: Complexity Classes: Big Picture (2+3+2 Points)

(a) Why is \( P \subseteq NP \)?

(b) Show that \( P \cap NPC = \emptyset \) if \( P \neq NP \).
   
   *Hint: Assume that there exists a \( L \in P \cap NPC \) and derive a contradiction to \( P \neq NP \).*

(c) Give a Venn Diagram showing the sets \( P, NP, NPC \) for both cases \( P \neq NP \) and \( P = NP \).
   
   *Remark: Use the results of (a) and (b) even if you did not succeed in proving those.*