Albert-Ludwigs-Universität, Inst. für Informatik
Prof. Dr. Fabian Kuhn
P. Bamberger, Y. Maus,

Theoretical Computer Science - Bridging Course
Summer Term 2017
Exercise Sheet 9

Hand in (electronically or hard copy) by 12:15 pm, January 15th, 2018

Repetition of Course Material *(0 Points)*

Let L_1, L_2 be languages (problems) over alphabets Σ_1, Σ_2. Then $L_1 \leq_p L_2$ (L_1 is polynomially reducible to L_2), iff a function $f : \Sigma_1^* \to \Sigma_2^*$ exists, that can be calculated in polynomial time and

$$\forall s \in \Sigma_1 : s \in L_1 \Longleftrightarrow f(s) \in L_2.$$

Language L is called $\mathcal{NP}$-hard, if *all* languages $L' \in \mathcal{NP}$ are polynomially reducible to L, i.e.

$$L \ \mathcal{NP}\text{-hard} \Longleftrightarrow \forall L' \in \mathcal{NP} : L' \leq_p L.$$

The reduction relation '$\leq_p$' is transitive ($L_1 \leq_p L_2$ and $L_2 \leq_p L_3 \Rightarrow L_1 \leq_p L_3$). Therefore, in order to show that L is $\mathcal{NP}$-hard, it suffices to reduce a known $\mathcal{NP}$-hard problem $\tilde{L}$ to L, i.e. $\tilde{L} \leq_p L$. Finally a language is called $\mathcal{NP}$-complete ($\Leftrightarrow$: $L \in \mathcal{NPC}$), if

1. $L \in \mathcal{NP}$ and
2. L is $\mathcal{NP}$-hard.

Exercise 1: The class $\mathcal{NPC}$ *(8 Points)*

This exercise is really (!!) important for the course.

A subset of the nodes of a graph G is a **dominating set** if every other node of G is adjacent to some node in the subset. Let

$$\text{DOMINATINGSET} = \{\langle G, k\rangle \mid \text{has a dominating set with } k \text{ nodes}\}.$$

Show that DOMINATINGSET is in $\mathcal{NPC}$. Use that

$$\text{VERTEXCOVER} := \{\langle G, k\rangle \mid \text{ Graph } G \text{ has a } \textit{vertex cover} \text{ of size at most } k\} \in \mathcal{NPC}\ .$$

Remark: A VERTEXCOVER *is a subset $V' \subseteq V$ of nodes of $G = (V, E)$ such that every edge of G is adjacent to a node in the subset.*

Exercise 2: P and NP? *(3 + (2 + 3*) Points)*

Let $CNF_k = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable cnf-formula where each variable appears in at most } k \text{ places}\}$.

(a) Assume that $P \neq NP$ holds. Decide whether CNF_2 is in $\mathcal{P}$ or in $\mathcal{NP} \setminus \mathcal{P}$. Prove your claim!

(b) Show that CNF_3 is $\mathcal{NP}$-complete.

Remark: You can gain 3 additioal points in this exercise to pass the 50% barrier.

Exercise 3: Complexity Classes: Big Picture *(2+3+2 Points)*

(a) Why is $\mathcal{P} \subseteq \mathcal{NP}$?

(b) Show that $\mathcal{P} \cap \mathcal{NPC} = \emptyset$ if $\mathcal{P} \neq \mathcal{NP}$.
Hint: Assume that there exists a $L \in \mathcal{P} \cap \mathcal{NPC}$ and derive a contradiction to $\mathcal{P} \neq \mathcal{NP}$.

(c) Give a Venn Diagram showing the sets $\mathcal{P}, \mathcal{NP}, \mathcal{NPC}$ for both cases $\mathcal{P} \neq \mathcal{NP}$ and $\mathcal{P} = \mathcal{NP}$.
Remark: Use the results of (a) and (b) even if you did not succeed in proving those.