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Theoretical Computer Science - Bridging Course
Winter Term 2017
Exercise Sheet 1

hand in (electronically or hard copy) by 12:15 pm, Monday, October 30th, 2017

Exercise 1: Induction *(6 Points)*

Prove the following summation formula by induction on n:

For all natural numbers $n \geq 1$ it holds: $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$

Sample Solution

For $n = 1$ we have to show that $\sum_{k=1}^{1} k^2 = \frac{1(1+1)(2+1)}{6}$, which is obviously true because both sides are equal 1.

Now assume the statement is true for n. It follows that

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 &= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 && (1)\\
&= \frac{1}{6}n(n+1)(2n+1) + n^2 + 2n + 1 = \frac{1}{6}(2n^3 + 3n^2 + n) + n^2 + 2n + 1 && (2)\\
&= \frac{1}{6}(2n^3 + 9n^2 + 13n + 6) && (3)\\
&= \frac{1}{6}(n+1)(n+2)(2n+3). && (4)
\end{aligned}$$

which shows that the statement also holds for $n+1$. The second equality in the equation above comes from the assumption for n.

Exercise 2: Sets & More *(8 Points)*

Either proof the following statements or show that they are not true.

1. $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x + y = 0\} \subseteq \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \cdot y = 0\}$
2. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
3. $A \subsetneq B$ implies $C \cup A \subsetneq C \cup B$.

Sample Solution

1. $(5,-5) \in \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x+y=0\}$ but $(5,-5) \notin \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x+y=0\}$.

2. $\subseteq$: Let $x \in A \setminus (B \cap C)$, which implies that $x \in A$. Assume that x is not contained in the right hand side, i.e., $x \notin (A \setminus B) \cup (A \setminus C)$ which implies

$$(1): \quad x \notin (A \setminus B) \text{ and}$$

$$(2): \quad x \notin (A \setminus C).$$

   As $x \in A$ it (1) implies $x \in B$ and (2) implies $x \in C$. That means $x \in B \cap C$. But this is a contradiction to $x \in A \setminus (B \cap C)$.

   $\supseteq$: Let $x \in A \setminus B \cup A \setminus C$ which implies $x \in A$. Assume that x is not contained in the right hand side, i.e., $x \notin A \setminus (B \cap C)$ which implies $x \in B \cap C$. But then $x \notin A \setminus B$ and $x \notin A \setminus C$ which implies $x \notin A \setminus (B \cap C)$, a contradiction.

3. This statement does not hold. Let $A = \{1,2\}$ $B = \{1,2,3\}$ and $C = \{3,4,5\}$. Then $A \subsetneq B$ but $A \cup C \subsetneq B \cup C$ is equivalent to $\{1,2,3,4,5\} \subsetneq \{1,2,3,4,5\}$ which is not true.

Exercise 3: Where is the even degree node? *(6 Points)*

A *simple graph* is a graph without self loops, i.e., every edge of the graph is an edge between two distinct nodes. The degree $d(v)$ of a node $v \in V$ in an undirected graph $G = (V,E)$ is the number of its neighbors, i.e,

$$d(v) = |\{u \in V \mid \{v,u\} \in E\}|.$$

Show that every simple graph with an odd number of nodes contains a node with even degree.

Hint: Consider the sum $D = \sum_{v \in V} d(v)$ of all degrees. Is D odd or even?

Sample Solution

Let $G = (V,E)$ be a graph.
The degree $d(v)$ of a node v is the number of incident edges, i.e., edges that contain v. Let D be the sum of all degrees.
Every edge contributes two to D, hence $D = 2|E|$, i.e., D is even.
We can also write D as the sum of the degrees of the nodes in V, i.e., $D = \sum_{v \in V} d(v)$. The sum of an odd number of odd numbers is odd. If all $d(v)$ were odd also D would be odd as $|V|$ is odd. Thus there has to be a node with even degree.