Theoretical Computer Science - Bridging Course
Summer Term 2017
Exercise Sheet 4

Hand in (electronically or hard copy) by 12:15 pm, November 20th, 2017

Exercise 1: Constructing Pushdown Automata

Consider the language \( L = \{a^n b^{2m} ba^n | m, n > 0 \} \) over the alphabet \( \Sigma = \{a, b\} \).
Construct a PDA \( A \) with \( L(A) = L \).

Sample Solution

The formal definition of the automaton is implicitly given.

Exercise 2: Understanding PDAs

Consider the PDA \( A = (\{q_0, q_1, q_2\}, \{a, b\}, \{$, Z\}, q_0, \delta, \{q_2\}) \) with the following transition relation \( \delta \)

\[
\begin{align*}
(q_0, a, \$) &\rightarrow \{(ZZ\$, q_0)\} \\
(q_0, b, Z) &\rightarrow \{(\epsilon, q_1)\} \\
(q_1, \epsilon, \$) &\rightarrow \{(\epsilon, q_2)\} \\
(q_0, a, Z) &\rightarrow \{(ZZZ, q_0)\} \\
(q_1, b, Z) &\rightarrow \{(\epsilon, q_1)\}
\end{align*}
\]

Remark: Assume that the stack contains the symbol \( \$ \) at the start.

1. Decide which of the words \( b, aabbb \) and \( abb \) are accepted by \( A \). Explain your answers by either giving an accepting sequence of configurations or by explaining why non sequence of configurations is accepting.
2. Which language is recognized by \( A \)?
Sample Solution

The automaton looks as follows.

1. It is clear that the word $b$ cannot be accepted as a $b$ is only processed by the automaton if the top of the stack contains a $Z$ and the stack can only contain a $Z$ if an $a$ has been read.

The word $aabbba$ is accepted by the language by the following sequence of configurations.

$$
(aabbba, q_0, \$) \rightarrow (abbb, q_0, ZZZ) \rightarrow (bbaa, q_0, ZZZZ) \rightarrow (bb, q_1, ZZZZ) \\
\rightarrow (b, q_1, ZZZ) \rightarrow (b, q_1, Z) \rightarrow (b, q_1, \$) \rightarrow (b, q_2, \$)
$$

The word $abbb$ is not accepted by the automaton. The only deterministicly occurring sequence of configurations when reading the word is

$$
(abbb, q_0, \$) \rightarrow (bbaa, q_0, ZZZZ) \rightarrow (bb, q_1, ZZZ) \rightarrow (b, q_1, \$) \rightarrow (b, q_1, \$).
$$

In this last configuration there is no transition which can be executed and the state $q_1$ is not accepting.

2. The automaton recognizes the language $\{a^n b^{2n} \mid n \geq 1\}$

Exercise 3: Context Free Grammar

Give a contextfree grammar for each of the following languages.

1. $L_1 = \{a^k b^{3k} \mid k \geq 0\}$
2. $L_2 = \{a^i b^j \mid 0 < i \leq j\}$
3. $L_1 \cdot L_2$
4. $L_1 \cup L_2$

Sample Solution

$S_1$, $S_2$, $T_1$ and $T_2$ are the respective start symbols. The rest of the formal definition of the grammars is given implicitly.

1.

$$S_1 \rightarrow aS_1 bbb \mid \epsilon$$

2.

$$S_2 \rightarrow aRb$$

$$R \rightarrow aRb \mid B$$

$$B \rightarrow Bb \mid b \mid \epsilon$$
Exercise 4: Chomsky Normal Form. \( (4 \text{ Points}) \)

Convert the following grammar into Chomsky normal form along the procedure given in the lecture.

\[
S \rightarrow AB \mid A \mid B \\
A \rightarrow aAA \mid aA \mid a \\
B \rightarrow bBB \mid bB \mid b
\]

It is not sufficient to just state the final grammar without intermediate steps.

Which language is recognized by the grammar?

Sample Solution

The grammar recognized the language which corresponds to \( L(a^*b^*) \setminus \{e\} \).

Remove rules of the form \( S \rightarrow B \) according to the lecture and obtain.

\[
S \rightarrow AB \mid aAA \mid aA \mid bBB \mid bB \mid b \\
A \rightarrow aAA \mid aA \mid a \\
B \rightarrow bBB \mid bB \mid b
\]

Transform according to step four in the lecture:

\[
S \rightarrow AB \mid aAA \mid aA \mid bBB \mid bB \mid b \\
A \rightarrow aT_1 \mid aA \mid a \\
B \rightarrow bT_2 \mid bB \mid b \\
T_1 \rightarrow AA \\
T_2 \rightarrow BB
\]

Substitute terminal symbols \( (a \text{ and } b) \) with new non-terminals \( (U_a \text{ and } U_b) \) in terms of length two:

\[
S \rightarrow AB \mid U_aT_1 \mid U_aA \mid a \mid U_bT_2 \mid U_bB \mid b \\
A \rightarrow U_aT_1 \mid U_aA \mid a \\
B \rightarrow U_bT_2 \mid U_bB \mid b \\
T_1 \rightarrow AA \\
T_2 \rightarrow BB \\
U_a \rightarrow a \\
U_b \rightarrow b.
\]