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Algorithms Theory
Exercise Sheet 5

Due: Monday, 7th of January, 2018, 14:15 pm

Exercise 1: Pinning Paper Polygons *(4+3 Points)*

You have two rectangular sheets of paper of equal dimensions. On each sheet an adversary has drawn straight lines that form a subdivision of each sheet into n polygons such that each polygon covers an equal area. The subdivision is different for each sheet. You also receive n pins.

(a) You put *two* of the sheets directly on top of each other. Prove that your pins suffice to pierce *all* polygons on both sheets (no folding or other funny business, polygons need to be pinned through their *interior*).

(b) Show that this is not possible if we drop the condition that each polygon covers an equal area on each sheet (however, the area of each polygon is bigger than zero).

Exercise 2: Doomsday on Krypton *(6+2 Points)*

Horrible news on Krypton: The planet will be struck by a meteorite within the next m minutes and the planet, all of its n cities and all k Kryptonions inhabiting them will be destroyed. Fortunately the Kryptonian government provided interstellar escape pods for such an emergency.

In city i, where k_i Kryptonians live ($\sum_{i=1}^{n} k_i = k$), there are p_i such pods available. Each pod can carry one person. Kryptonians can either *instantly* use one of the escape pods in the city where they live, or travel to another city and use an escape pod there. It takes d_{ij} minutes to travel from city i to city j. A pod must be reached before the meteorite destroys Krypton.

Due to flight safety concerns the total number of pods that can launch from certain subsets of cities is restricted. I.e., there are subsets $S_i \subseteq [1..n]$ with $i \in [1..\ell], \ell \leq n$ and parameters r_i. Each city j is subject to exactly one flight restriction zone (that is $j \in S_i$). All cities that are part of S_i can not launch more than r_i pods taken together.

(a) You need to determine the maximum number of survivors. Describe how this problem can be reduced to a maximum flow problem.

(b) Making no additional assumptions about k, ℓ, n, m, how long does it take in the worst case to solve the maximum flow problem using the Ford-Fulkerson algorithm?

Exercise 3: Sinkless Orientation *(5+5+3 Points)*

(a) Let $B = (U \cup V, E)$ be a bipartite graph and assume that for every $A \subseteq U$, we have $|N(A)| \geq |A|$. Show that this implies that there exists a matching of size $|U|$ in B (i.e., a matching that matches every node in U).

(b) Let $G = (V, E)$ be a graph with minimum degree at least two (i.e., each node has at least two incident edges). Use the prior statement to show that there is a way to orient the edges of G such that each node of G has at least one out-going edge (this is known as a sinkless orientation).

Hint: Sinkless orientation can be seen as matching nodes and edges.

(c) Let us now assume that the graph G has minimum degree 4. Show that there exists an orientation in which each node has at least one in-coming and at least one out-going edge.

Exercise 4: Triangles in Random Graphs *(2+2+3+5 Points)*

Given a fixed vertex set $V = \{v_1, v_2, \ldots, v_n\}$ with n being an even number. Then the following (randomized) process defines the (undirected) random graph $G_p = (V, E_p)$.

For each vertex pair $\{v_i, v_j\}, i \neq j$ we independently decide with probability p whether the edge defined by this pair is part of the graph, i.e., whether $\{v_i, v_j\}$ is an element of the edge set E_p.

Furthermore we say that a subset $T = \{v_i, v_j, v_k\}$ of V of size 3 is a triangle of a graph, if all three edges $\{v_i, v_j\}$, $\{v_i, v_k\}$, $\{v_j, v_k\}$ are in the edge set of the graph.

(a) Let Z be the random variable that counts the number of edges in G_p. What is the distribution of Z? What is the probability that Z has value k, for some k?

(b) Calculate m_T, the number of all triangles that could *possibly* occur in G_p.

(c) Let X denote the number of triangles in G_p. Calculate $\mathbb{E}[X]$.

The generation of the random graphs is now changed as follows. Before edges are determined each vertex is colored either red or green; we let K be the random variable that counts the number of red vertices. Between two red vertices there is an edge with probability p_{rr}, between two green vertices with probability p_{gg} and between vertices of different color with probability p_{rg} (edges are still picked independently).

(d) Assume first that with probability $\frac{1}{7}$ all vertices are red, with probability $\frac{2}{7}$ all vertices are green and with probability $\frac{4}{7}$ each vertex independently gets color red or green with probability $1/2$ each. Also $p_{rr} = 1$, $p_{rg} = \frac{1}{\sqrt{3}}$ and $p_{gg} = 0$. Calculate $\mathbb{E}[X]$ under these conditions!