University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
M. Ahmadi, P. Schneider

Algorithms Theory
Exercise Sheet 7

Due: Monday, 4th of February, 2019, 14:15 pm

Exercise 1: The Modified Contraction Algorithm   *(3+6 Points)*

Let us consider the following modified version of the contraction algorithm presented in the lecture. Instead of choosing an edge uniformly at random and merging its endpoints, in each step the modified algorithm chooses a pair of nodes in graph G uniformly at random and merges the two nodes into a single node.

(a) Give an example graph of size at least n where the above algorithm does not work well, that is, where the probability of finding a minimum cut is exponentially small in n.

(b) Prove that property for the example you gave in (a), i.e., show that the modified contraction algorithm has probability of finding a minimum cut at most c^n for some constant $c < 1$.

Exercise 2: Metric TSP with Small Edge Weights   *(3+6 Points)*

Consider the family of complete, weighted, undirected graphs $G = (V, E, w)$ in which all edges have weight either 1 or 2.

Remark: TSP is the Traveling Salesperson Problem. The goal is to find a tour, i.e., a permutation $v_1, \ldots, v_n$ of nodes, that minimizes the total weight of edges on that tour $w(v_1, v_n) + \sum_{1=1}^{n} w(v_i, v_{i+1})$.

(a) Assume you have a subroutine that computes a *minimum* 2-matching for the above family of graphs in polynomial time. Describe an *efficient* algorithm that computes a 4/3-approximation for the TSP problem for graphs of this family.

   Remark: A 2-matching is a subset $M \subseteq E$, so that every $v \in V$ is incident to exactly 2 edges in M.

(b) Prove that your algorithm computes a 4/3-approximation of TSP on the this family of graphs.

Exercise 3: Covering Paths   *(4+4+4 Points)*

Consider an undirected, unweighted Graph $G = (V, E)$ with n nodes. We are given a set $\mathcal{P}$ of p *simple* paths in G, where each path has exactly ℓ nodes. We say a path $P \in \mathcal{P}$ is *covered* by a set Q of nodes if P has a node in Q. Then, the goal is to find a set $Q \subseteq V$ of nodes with minimum cardinality such that *every* path in $\mathcal{P}$ is covered by Q.

Now consider the following simple greedy algorithm: It starts with $Q = \emptyset$. As long as there is a path not covered by Q, the node that covers the most uncovered paths is added to Q.

(a) Argue why the above greedy algorithm provides a $(1+\ln p)$-approximation of an minimal solution.

(b) Show that after selecting i nodes, there are at most $p(1-\frac{\ell}{n})^i$ uncovered paths left.

(c) Show that for $i > \frac{n}{\ell} \ln p$ all paths are covered.   *Hint: $1-x < e^{-x}$ for all $x > 0$.*

Exercise 4: Resilience to Edge Failures   *(10 Points)*

In the lecture, we showed that every (undirected) graph with edge connectivity λ has at most $n^{2\alpha}$ cuts of size at most $\alpha \cdot \lambda$. Use this fact to prove the following statement:

Let $G = (V, E)$ be an undirected graph with constant edge connectivity $\lambda \geq 1$ and let $p := \min\left\{1, \frac{c \ln n}{\lambda}\right\}$, where $c > 0$ is a constant. Assume that edge $e \in E$ is sampled independently with probability p. Let E_p be the set of sampled edges and let $G_p = (V, E_p)$ be the graph induced by the sampled edges. Show that if the constant c is chosen sufficiently large, the graph G_p is connected with high probability.

Hint: A graph is connected if and only if there is an edge across each of the $2^{n-1} - 2$ possible cuts. Analyze the probability that for a cut of a given size k in G, at least one edge is sampled in E_p. Then, use the upper bound from the lecture on the number of cuts of a given size and a union bound over all cuts of a given size in G. Finally, one can do a union bound over all possible cut sizes.