Chapter 1
Divide and Conquer

Algorithm Theory
WS 2018/19

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Divide-And-Conquer Principle

- Important algorithm design method

- Examples from basic alg. & data structures class (Informatik 2):
  - Sorting: Mergesort, Quicksort
  - Binary search

- Further examples
  - Median
  - Compairing orders
  - Convex hull / Delaunay triangulation / Voronoi diagram
  - Closest pairs
  - Line intersections
  - Polynomial multiplication / FFT
  - ...

Formulation of the D&C principle

Divide-and-conquer method for solving a problem instance of size n:

1. Divide

$n \leq c$: Solve the problem directly.

$n > c$: Divide the problem into k subproblems of sizes $n_1, \dots, n_k < n$ ($k \geq 2$).

2. Conquer

Solve the k subproblems in the same way (recursively).

3. Combine

Combine the partial solutions to generate a solution for the original instance.

Running Time

Recurrence relation:

$$T(n) = 2 \cdot T(n/2) + c \cdot n, \qquad T(1) = a$$

Solution:

- Same as for computing number of number of inversions, merge sort (and many others…)

$$T(n) = O(n \cdot \log n)$$

Recurrence Relations: Master Theorem

Recurrence relation

$$\boldsymbol{T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \qquad T(n) = O(1) \text{ for } n \leq n_0}$$

Cases

- $f(n) = O(n^c),\ c < \log_b a$

$$\boldsymbol{T(n) = \Theta\left(n^{\log_b a}\right)}$$

- $f(n) = \Omega(n^c),\ c > \log_b a$

$$\boldsymbol{T(n) = \Theta\left(f(n)\right)}$$

- $f(n) = \Theta\left(n^c \cdot \log^k n\right),\ c = \log_b a$

$$\boldsymbol{T(n) = \Theta\left(n^c \cdot \log^{k+1} n\right)}$$

Polynomials

Real polynomial p in one variable x:

$$p(x) = a_{n-1}x^{n-1} + \dots + a_1x^1 + a_0$$

Coefficients of p: $a_0, a_1, \dots, a_n \in \mathbb{R}$

Degree of p: largest power of x in p ($n-1$ in the above case)

Example:

$$p(x) = 3x^3 - 15x^2 + 18x$$

Set of all real-valued polynomials in x: $\mathbb{R}[x]$ (polynomial ring)

Divide-&-Conquer Polynomial Multiplication

- Multiplication is slow $\left(\Theta(n^2)\right)$ when using the standard coefficient representation
- Try divide-and-conquer to get a faster algorithm
- Assume: degree is $n-1$, n is even
- Divide polynomial $p(x) = a_{n-1}x^{n-1} + \cdots + a_0$ into 2 polynomials of degree $n/2 - 1$:

$$p_0(x) = a_{n/2-1}x^{n/2-1} + \cdots + a_0$$
$$p_1(x) = a_{n-1}x^{n/2-1} + \cdots + a_{n/2}$$
$$p(x) = p_1(x) \cdot x^{n/2} + p_0(x)$$

- Similarly: $q(x) = q_1(x) \cdot x^{n/2} + q_0(x)$

Divide-&-Conquer Polynomial Multiplication

- **Divide:**
  $p(x) = p_1(x) \cdot x^{n/2} + p_0(x), \qquad q(x) = q_1(x) \cdot x^{n/2} + q_0(x)$

- **Multiplication:**
  $$p(x)q(x) = p_1(x)q_1(x) \cdot x^n + (p_0(x)q_1(x) + p_1(x)q_0(x)) \cdot x^{n/2} + p_0(x)q_0(x)$$

- 4 multiplications of degree $n/2 - 1$ polynomials:

$$T(n) = 4T\left(n/2\right) + O(n)$$

- Leads to $T(n) = \Theta(n^2)$ like the naive algorithm…
  - follows immediately by using the master theorem

Karatsuba Algorithm

- Recursive multiplication:

$$r(x) = \big(p_0(x) + p_1(x)\big) \cdot \big(q_0(x) + q_1(x)\big)$$

$$\begin{aligned} p(x)q(x) = {} & p_1(x)q_1(x) \cdot x^n \\ & + \big(r(x) - p_0(x)q_0(x) + p_1(x)q_1(x)\big) \cdot x^{n/2} \\ & + p_0(x)q_0(x) \end{aligned}$$

- Recursively do 3 multiplications of degr. $(n/2 - 1)$-polynomials

$$T(n) = 3T\big(n/2\big) + O(n)$$

- Gives: $T(n) = O(n^{1.58496\ldots})$ (see Master theorem)

Representation of Polynomials

Coefficient representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree $n-1$ is given by its n coefficients $a_0, \dots, a_{n-1}$:

$$p(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

- Coefficient vector $\boldsymbol{a} = (a_0, a_1, \dots a_{n-1})$

- Example:

$$p(x) = 3x^3 - 15x^2 + 18x$$

- The most typical (and probably most natural) representation of polynomials

Representation of Polynomials

Point-value representation:

- Polynomial $p(x) \in \mathbb{R}[x]$ of degree $n-1$ is given by n point-value pairs:

$$p = \{(x_0, p(x_0)), (x_1, p(x_1)), \dots, (x_{n-1}, p(x_{n-1}))\}$$

where $x_i \neq x_j$ for $i \neq j$.

- Example: The polynomial

$$p(x) = 3x(x-2)(x-3)$$

is uniquely defined by the four point-value pairs $(0,0), (1,6), (2,0), (3,0)$.

Operations: Coefficient Representation

$$\boldsymbol{p(x) = a_{n-1}x^{n-1} + \cdots + a_0}, \qquad \boldsymbol{q(x) = b_{n-1}x^{n-1} + \cdots + b_0}$$

Evaluation: Horner's method: Time $O(n)$

Addition:

$$p(x) + q(x) = (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_0 + b_0)$$

- Time: $O(n)$

Multiplication:

$$p(x) \cdot q(x) = c_{2n-2}x^{2n-2} + \cdots + c_0, \qquad \text{where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

- Naive solution: Need to compute product $a_i b_j$ for all $0 \leq i, j \leq n$
- Time: $O(n^2)$

Operations: Point-Value Representation

$$p = \{(x_0, p(x_0)), \dots, (x_{n-1}, p(x_{n-1}))\}$$
$$q = \{(x_0, q(x_0)), \dots, (x_{n-1}, q(x_{n-1}))\}$$

- Note: we use the **same points $x_0, \dots, x_n$** for both polynomials

Addition:

$$p + q = \{(x_0, p(x_0) + q(x_0)), \dots, (x_{n-1}, p(x_{n-1}) + q(x_{n-1}))\}$$

- Time: $O(n)$

Multiplication:

$$p \cdot q = \{(x_0, p(x_0) \cdot q(x_0)), \dots, (x_{2n-2}, p(x_{2n-2}) \cdot q(x_{2n-2}))\}$$

- Time: $O(n)$

Evaluation: Polynomial interpolation can be done in $O(n^2)$

Operations on Polynomials

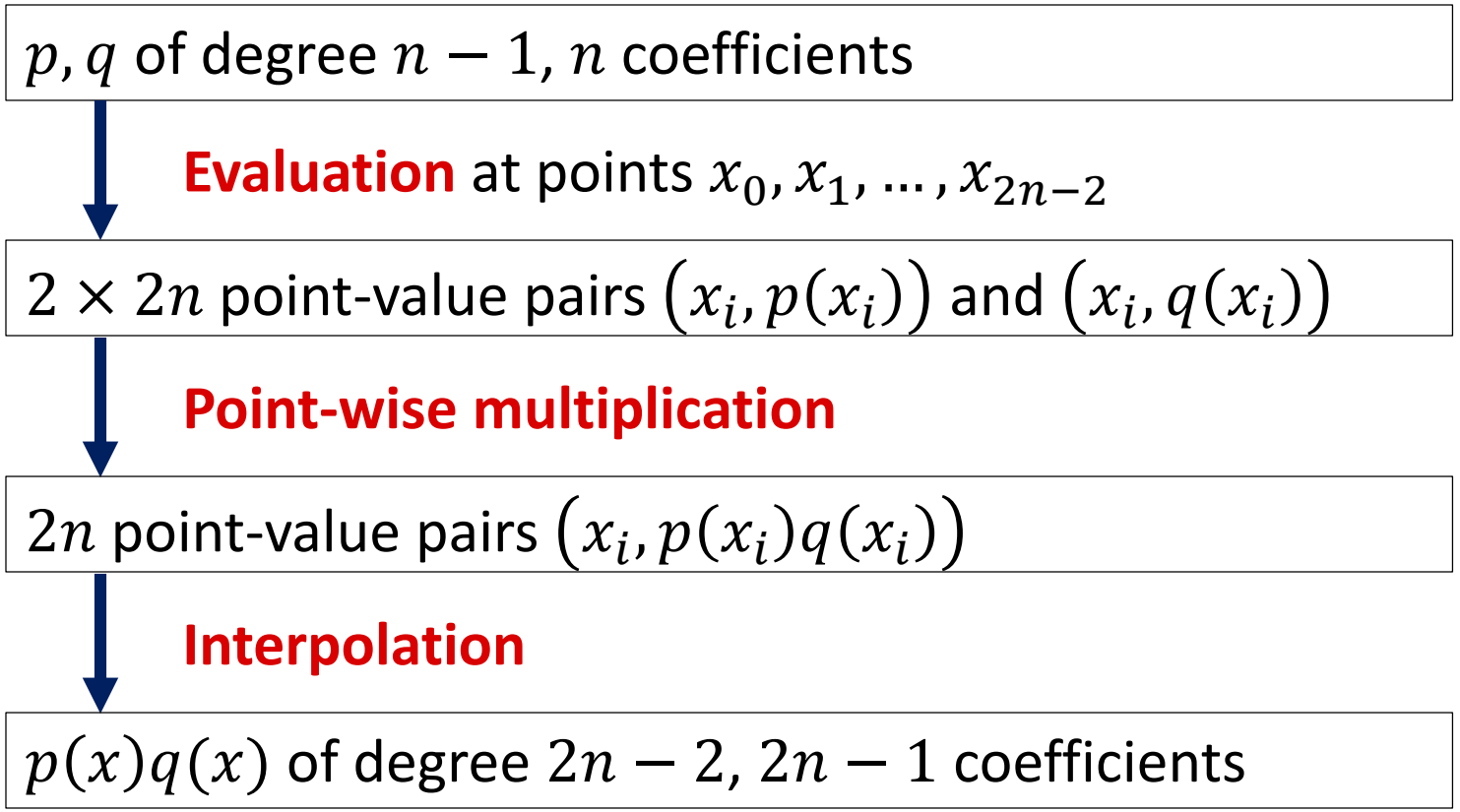
Cost depending on representation:

	Coefficient	Roots	Point-Value
Evaluation	$O(n)$	$O(n)$	$O(n^2)$
Addition	$O(n)$	∞	$O(n)$
Multiplication	$O(n^{1.58})$	$O(n)$	$O(n)$

Faster Polynomial Multiplication?

Multiplication is fast when using the point-value representation

Idea to compute $p(x) \cdot q(x)$ (for polynomials of degree $< n$):



Coefficients to Point-Value Representation

Given: Polynomial $p(x)$ by the coefficient vector $(a_0, a_1, \dots, a_{N-1})$

Goal: Compute $p(x)$ for all x in a given set X

- Where X is of size $|X| = N$
- Assume that N is a power of 2

Divide and Conquer Approach

- Divide $p(x)$ of degree $N-1$ (N is even) into 2 polynomials of degree ${}^{N}/_{2} - 1$ differently than in Karatsuba's algorithm

- $p_0(y) = a_0 + a_2 y + a_4 y^2 + \cdots + a_{N-2} y^{{}^{N}/_{2}-1}$ (even coeff.)
  $p_1(y) = a_1 + a_3 y + a_5 y^2 + \cdots + a_{N-1} y^{{}^{N}/_{2}-1}$ (odd coeff.)

Coefficients to Point-Value Representation

Goal: Compute $p(x)$ for all x in a given set X of size $|X| = N$

- Divide $p(x)$ of degr. $N-1$ into 2 polynomials of degr. ${}^{N}/_{2} - 1$

$$p_0(y) = a_0 + a_2 y + a_4 y^2 + \cdots + a_{N-2} y^{{}^{N}/_{2}-1} \quad \text{(even coeff.)}$$
$$p_1(y) = a_1 + a_3 y + a_5 y^2 + \cdots + a_{N-1} y^{{}^{N}/_{2}-1} \quad \text{(odd coeff.)}$$

Let's first look at the "combine" step:

$$\forall x \in X: \quad p(x) = p_0(x^2) + x \cdot p_1(x^2)$$

- Recursively compute $p_0(y)$ and $p_1(y)$ for all $y \in X^2$
  - Where $X^2 := \{x^2 : x \in X\}$
- Generally, we have $|X^2| = |X|$

Analysis

Recurrence formula for the given algorithm:

Faster Algorithm?

- In order to have a faster algorithm, we need $|X^2| < |X|$

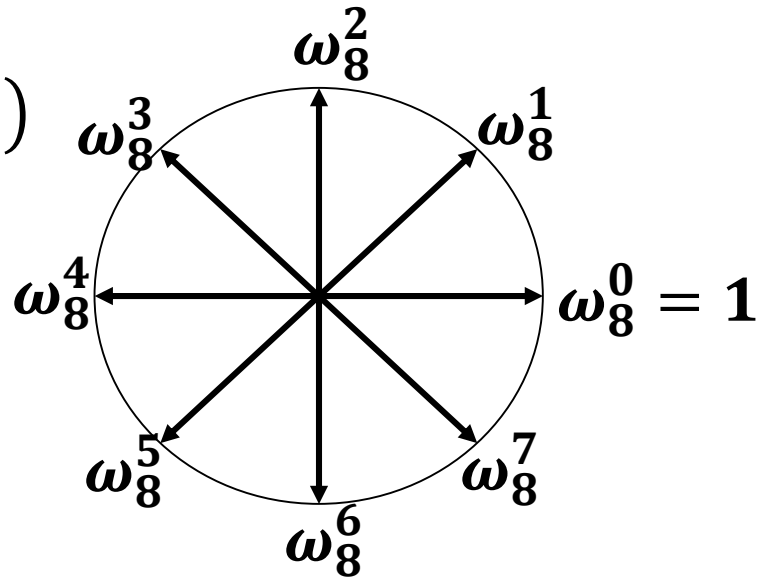
Choice of X

- Select points $x_0, x_1, \dots, x_{N-1}$ to evaluate p and q in a clever way

Consider the N complex roots of unity:

Principle root of unity: $\omega_N = e^{2\pi i/N}$

$$\left(i = \sqrt{-1}, \qquad e^{2\pi i} = 1\right)$$



Powers of ω_n (roots of unity):

$$1 = \omega_N^0, \omega_N^1, \dots, \omega_N^{N-1}$$

Note: $\omega_N^k = e^{2\pi ik/N} = \cos\frac{2\pi k}{N} + i \cdot \sin\frac{2\pi k}{N}$

Properties of the Roots of Unity

- **Cancellation Lemma:**

  For all integers $n > 0$, $k \geq 0$, and $d > 0$, we have:

$$\omega_{dn}^{dk} = \omega_n^k \,, \qquad \omega_n^{k+n} = \omega_n^k$$

- **Proof:**

Properties of the Roots of Unity

Claim: If $X = \left\{\omega_{2k}^i : i \in \{0, \dots, 2k-1\}\right\}$, we have

$$\boldsymbol{X^2 = \left\{\omega_k^i : i \in \{0, \dots, k-1\}\right\}, \qquad |X^2| = \frac{|X|}{2}}$$

Analysis

New recurrence formula:

$$T(N, |X|) \leq 2 \cdot T\left(N/2, |X|/2\right) + O(N + |X|)$$