Chapter 2
Greedy Algorithms

Algorithm Theory
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Matroids

- Same, but more abstract…

Matroid: **pair $(\boldsymbol{E}, \boldsymbol{I})$**

- $\boldsymbol{E}$: set, called the **ground set**
- $\boldsymbol{I}$: finite family of finite subsets of E (i.e., $I \subseteq 2^E$), called **independent sets**

(E, I) needs to satisfy 3 properties:

1. Empty set is independent, i.e., $\emptyset \in \boldsymbol{I}$ (implies that $I \neq \emptyset$)
2. **Hereditary property**: For all $A \subseteq E$ and all $A' \subseteq A$,

   if $\boldsymbol{A} \in \boldsymbol{I}$, then also $\boldsymbol{A}' \in \boldsymbol{I}$

3. **Augmentation / Independent set exchange property:**
   If $A, B \in I$ and $|A| > |B|$, there exists $x \in A \setminus B$ such that

   $$\mathbf{B}' := \boldsymbol{B} \cup \{\boldsymbol{x}\} \in \boldsymbol{I}$$

Matroids and Greedy Algorithms

Weighted matroid: each $e \in E$ has a weight $w(e) > 0$

Goal: find maximum weight independent set

Greedy algorithm:

1. Start with $S = \emptyset$
2. Add max. weight $e \in E \setminus S$ to S such that $S \cup \{e\} \in I$

Claim: greedy algorithm computes optimal solution

Greedy is Optimal

- S: greedy solution A: any other solution

Matroids: Examples

Forests of a graph $G = (V, E)$:

- forest F: subgraph with no cycles (i.e., $F \subseteq E$)
- $\mathcal{F}$: set of all forests → $(E, \mathcal{F})$ is a matroid
- Greedy algorithm gives maximum weight forest (equivalent to MST problem)

Bicircular matroid of a graph $G = (V, E)$:

- $\mathcal{B}$: set of edges such that every connected subset has ≤ 1 cycle
- $(E, \mathcal{B})$ is a matroid → greedy gives max. weight such subgraph

Linearly independent vectors:

- Vector space V, E: finite set of vectors, I: sets of lin. indep. vect.
- Fano matroid can be defined like that

Forest Matroid of Graph $G = (V, E)$

Ground set: E (edges) **Independent sets:** $\mathcal{F}$ (forests of G)

Basic properties: $\emptyset \in \mathcal{F}$ + hereditary property

- Empty graph has no cycles, removing edges doesn't create cycles

Independent set exchange property:

- Given $\mathcal{F}_1, \mathcal{F}_2$ s.t. $|\mathcal{F}_1| > |\mathcal{F}_2|$
- $\mathcal{F}_1$ needs to have an edge e connecting two components of $\mathcal{F}_2$
  - Because it can only have $|\mathcal{F}_2|$ edges connecting nodes inside components

Bicircular Matroid

Bicircular Matroid

Greedoid

- Matroids can be generalized even more

- Relax hereditary property:

  Replace $A' \subseteq A \subseteq I \implies A' \in I$

  by $\emptyset \neq A \subseteq I \implies \exists a \in A,\ \text{s.t. } A \setminus \{a\} \in I$

- Augmentation property holds as before

- Under certain conditions on the weights, greedy is optimal for computing the max. weight $A \in I$ of a greedoid.
  - Additional conditions automatically satisfied by hereditary property

- More general than matroids