Chapter 2
Greedy Algorithms

Algorithm Theory
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Matroids

- Same, but more abstract…

Matroid: **pair $(\boldsymbol{E}, \boldsymbol{I})$**

- $\boldsymbol{E}$: set, called the **ground set** set of elements
- $\boldsymbol{I}$: finite family of finite subsets of E (i.e., $I \subseteq 2^E$), called **independent sets**

(E, I) needs to satisfy 3 properties:

1. Empty set is independent, i.e., $\emptyset \in \boldsymbol{I}$ (implies that $I \neq \emptyset$)
2. **Hereditary property**: For all $A \subseteq E$ and all $A' \subseteq A$,

   if $\boldsymbol{A} \in \boldsymbol{I}$, then also $\boldsymbol{A}' \in \boldsymbol{I}$

3. **Augmentation / Independent set exchange property:**
   If $A, B \in I$ and $|A| > |B|$, there exists $x \in A \setminus B$ such that

   $$\mathbf{B}' \coloneqq \boldsymbol{B} \cup \{\boldsymbol{x}\} \in \boldsymbol{I}$$

Matroids and Greedy Algorithms

Weighted matroid: each $e \in E$ has a weight $w(e) > 0$

Goal: find maximum weight independent set

Greedy algorithm:

1. Start with $S = \emptyset$
2. Add max. weight $e \in E \setminus S$ to S such that $S \cup \{e\} \in I$

Claim: greedy algorithm computes optimal solution

Greedy is Optimal

Matroid (E, I), weights $w(e) \geq 0$ for all $e \in E$

- S: greedy solution $\quad$ A: any other solution (ind. set)

$S \subseteq E,\ S \in I \qquad A \subseteq E,\ A \in I$

$|S| \geq |A|$: $(s \geq a)$

for contradiction, assume $|A| > |S|$: exch. prop: $\exists x \in A \setminus S$ s.t. $S \cup \{x\} \in I$

greedy would have added x

$w(S) \geq w(A)$:

for contradiction, assume $\boxed{w(S) < w(A)}$

$S = \{x_1, x_2, \dots, x_s\} \quad w(x_1) \geq w(x_2) \geq \dots \geq w(x_s)$

$A = \{y_1, y_2, \dots, y_a\} \quad w(y_1) \geq w(y_2) \geq \dots \geq w(y_a)$

will show that $(*)$

$\forall i \in \{1, \dots, a\} : w(x_i) \geq w(y_i)$

$\hookrightarrow w(S) \geq w(A)$

$\neg(*) \Longrightarrow$ there is a smallest $k \leq a$ s.t. $w(x_k) < w(y_k)$

$S' = \{x_1, \dots, x_{k-1}\}$ $\quad$ augm. prop. : $\exists y \in A' \setminus S'$ s.t. $S' \cup \{y\} \in I$

$w(y) \geq w(y_k) > w(x_k)$

$A' = \{y_1, \dots, y_k\}$ $\quad$ greedy considers y before x_k

$|A'| > |S'|$ $\quad$ greedy would add y

$\lightning$

Matroids: Examples

Forests of a graph $G = (V, E)$: $(E, \mathcal{F})$

- forest F: subgraph with no cycles (i.e., $F \subseteq E$)
- $\mathcal{F}$: set of all forests → $(E, \mathcal{F})$ is a matroid
- Greedy algorithm gives maximum weight forest (equivalent to MST problem)

Bicircular matroid of a graph $G = (V, E)$:

- $\mathcal{B}$: set of edges such that every connected subset has ≤ 1 cycle
- $(E, \mathcal{B})$ is a matroid → greedy gives max. weight such subgraph

Linearly independent vectors:

- Vector space V, E: finite set of vectors, I: sets of lin. indep. vect.
- Fano matroid can be defined like that

Forest Matroid of Graph $G = (V, E)$

Ground set: E (edges) **Independent sets:** $\mathcal{F}$ (forests of G)

Basic properties: $\emptyset \in \mathcal{F}$ + hereditary property

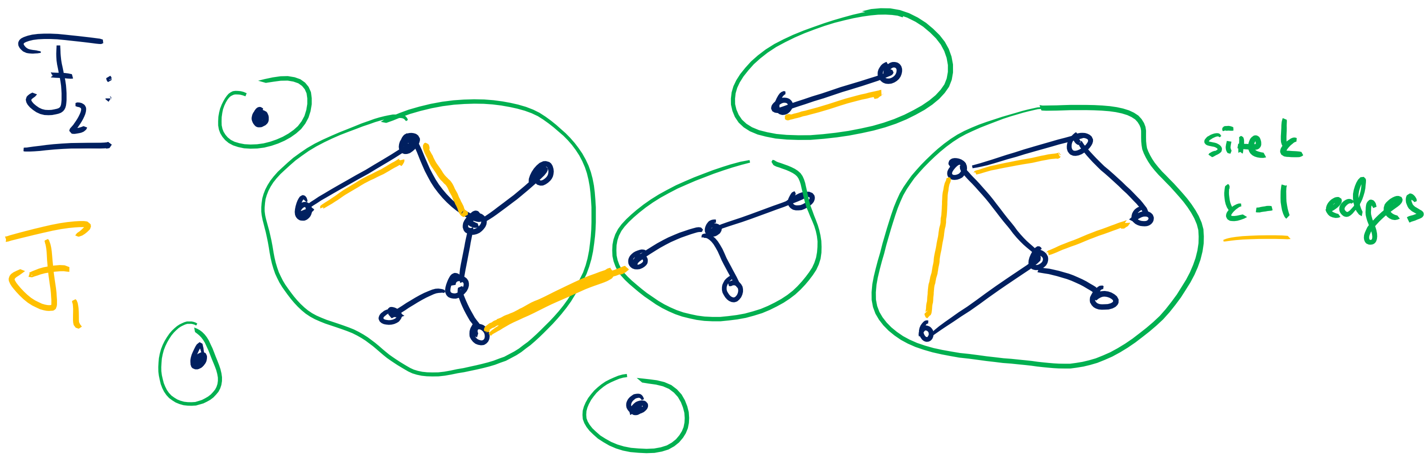
- Empty graph has no cycles, removing edges doesn't create cycles

Independent set exchange property:

- Given $\mathcal{F}_1, \mathcal{F}_2$ s.t. $|\mathcal{F}_1| > |\mathcal{F}_2|$

$\exists e \in \mathcal{F}_1$ s.t. $\mathcal{F}_2 \cup \{e\}$ is a forest

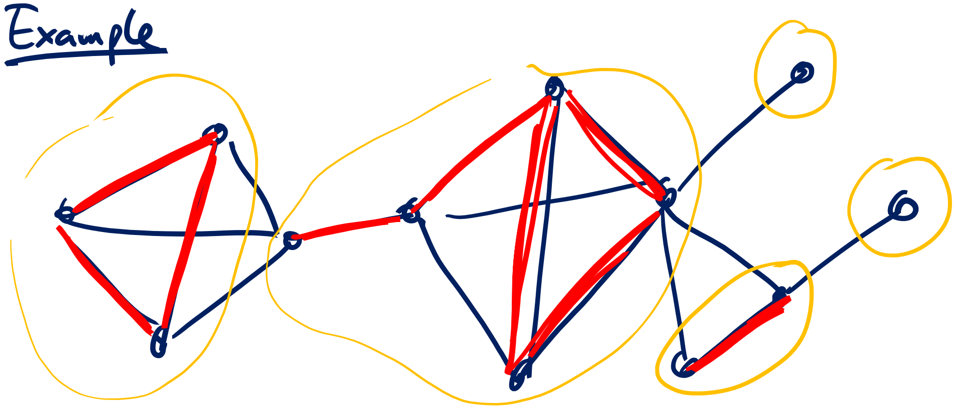
- $\mathcal{F}_1$ needs to have an edge e connecting two components of $\mathcal{F}_2$
  - Because it can only have $|\mathcal{F}_2|$ edges connecting nodes inside components



Bicircular Matroid

$G=(V,E)$, matroid $(E,\mathcal{B})$ $\quad S\subseteq E:$ $S\in\mathcal{B}$ iff all comp. of (V,S) have ≤ 1 cycle

Example



Claim: $(E,\mathcal{B})$ is a matroid

Proof: Need to show that $(E,\mathcal{B})$ satisfies properties 1, 2, and 3

Prop. 1: $\emptyset\in\mathcal{B}$ $\quad (V,\emptyset)$ has no cycles ✓

Prop. 2: $A\in\mathcal{B}$, $A'\subseteq A \Rightarrow A'\in\mathcal{B}$ ✓

Exch. prop. 3: edge sets $A\in\mathcal{B}$, $C\in\mathcal{B}$ $\qquad (V,A) \qquad (V,C)$

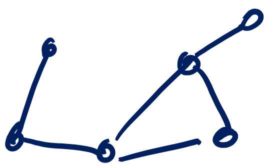
$|C|>|A| \Rightarrow \exists e\in C\setminus A$ $\qquad$ every comp. has ≤ 1 cycle

s.t. $A\cup\{e\}\in\mathcal{B}$

Bicircular Matroid

Components with ≤ 1 cycle

comp. has $k \geq 1$ nodes



$\Longrightarrow$ #edges

no cycle: $k-1$ edges

1 cycle: k edges

(V,S) : $S \subseteq \mathcal{B}$

$|S| \leq n$

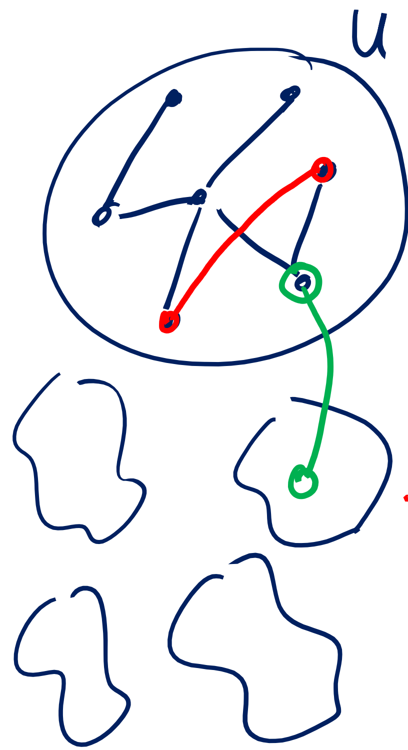
$|S| = n \Longleftrightarrow$ all comp. have exactly one cycle

(V,A), (V,C) $\quad (A, C \in \mathcal{B})$

$|A| < |C| \Longrightarrow |A| \leq n-1$

(V,A)

↳ there is a component $U \subseteq V$ with no cycle



$|U|-1$ edges in A

show: can add an edge $e \in C \setminus A$

$\boxed{\text{case 1:}}$ C has an edge conn. U to $V \setminus U$

$\boxed{\text{case 2:}}$ $C \setminus A$ contains an edge conn. 2 nodes in U

case 3: consider graph defined by $V \setminus U$

Greedoid

- Matroids can be generalized even more

- Relax hereditary property:

  Replace $\quad A' \subseteq A \subseteq I \quad \Longrightarrow \quad A' \in I$

  by $\quad \emptyset \neq A \subseteq I \quad \Longrightarrow \quad \exists a \in A,\ \text{s.t. } A \setminus \{a\} \in I$

- Augmentation property holds as before

- Under certain conditions on the weights, greedy is optimal for computing the max. weight $A \in I$ of a greedoid.
  - Additional conditions automatically satisfied by hereditary property

- More general than matroids