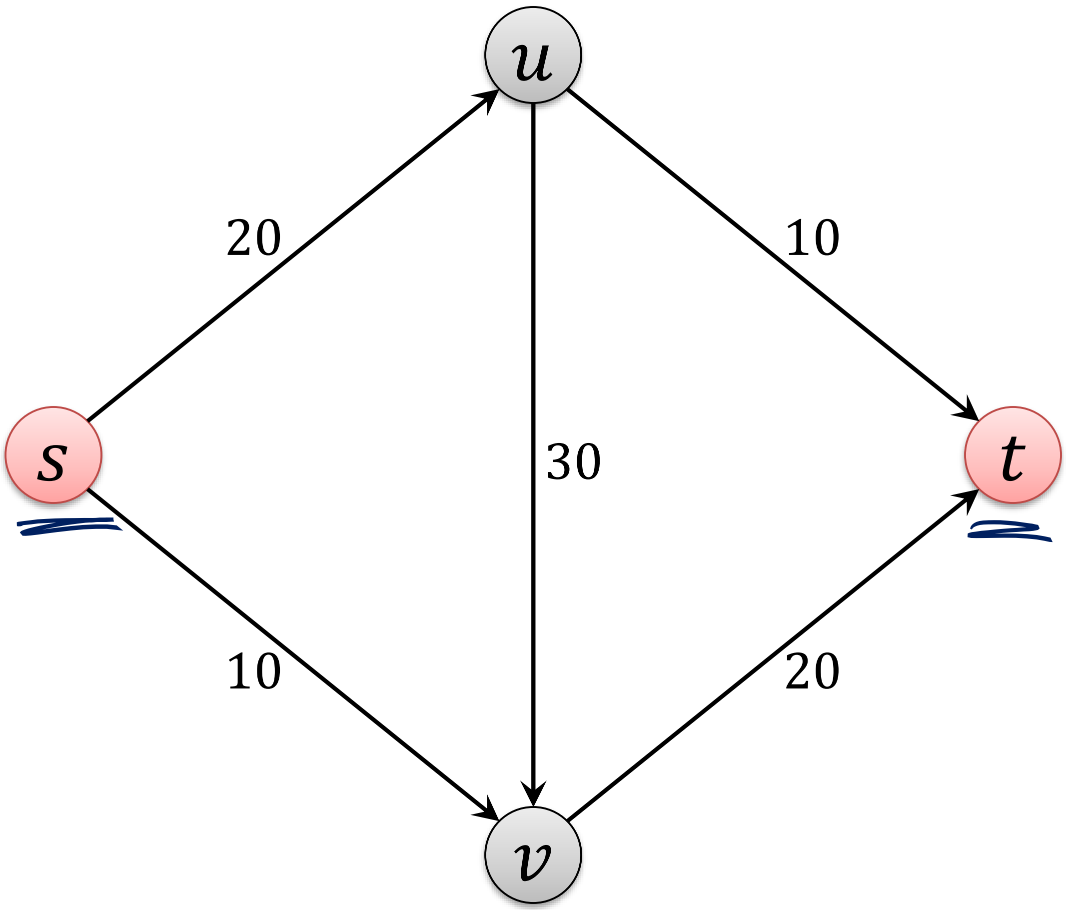
Chapter 6
Graph Algorithms

Algorithm Theory
WS 2018/19

Fabian Kuhn

Example: Flow Network



Network Flow: Definition

Flow: function $\boldsymbol{f}: \boldsymbol{E} \to \mathbb{R}_{\geq 0}$

- $f(e)$ is the amount of flow carried by edge e

Capacity Constraints:

- For each edge $e \in E$, $f(e) \leq c_e$

Flow Conservation:

- For each node $v \in V \setminus \{s, t\}$,

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

Flow Value:

$$|f| \coloneqq \sum_{e \text{ out of } s} f\big((s,u)\big) = \sum_{e \text{ into } t} f\big((v,t)\big)$$

The Maximum-Flow Problem

Maximum Flow:

Given a flow network, find a flow of maximum possible value

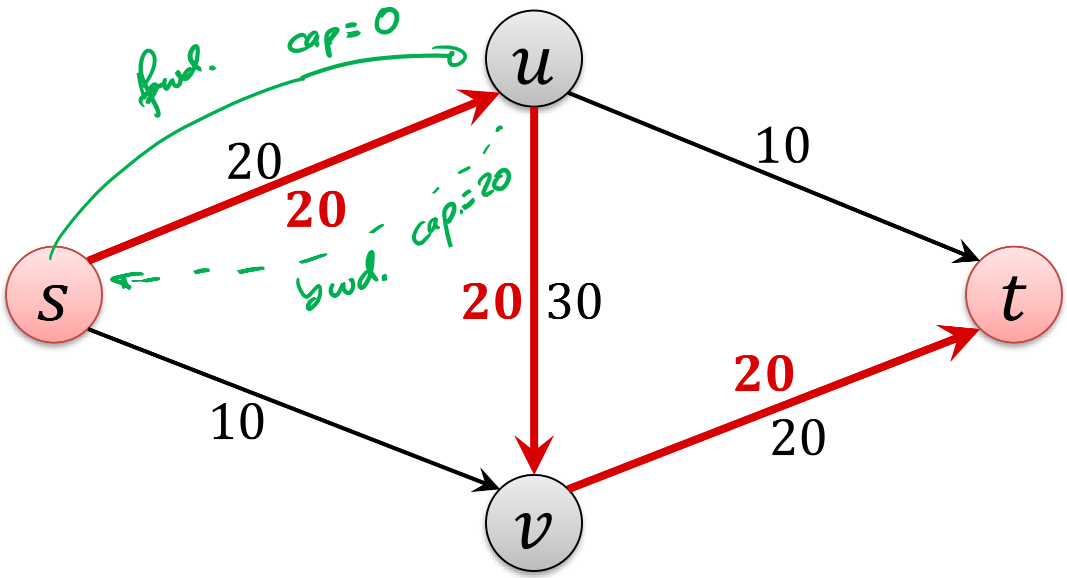
- Classical graph optimization problem
- Many applications (also beyond the obvious ones)
- Requires new algorithmic techniques

Residual Graph

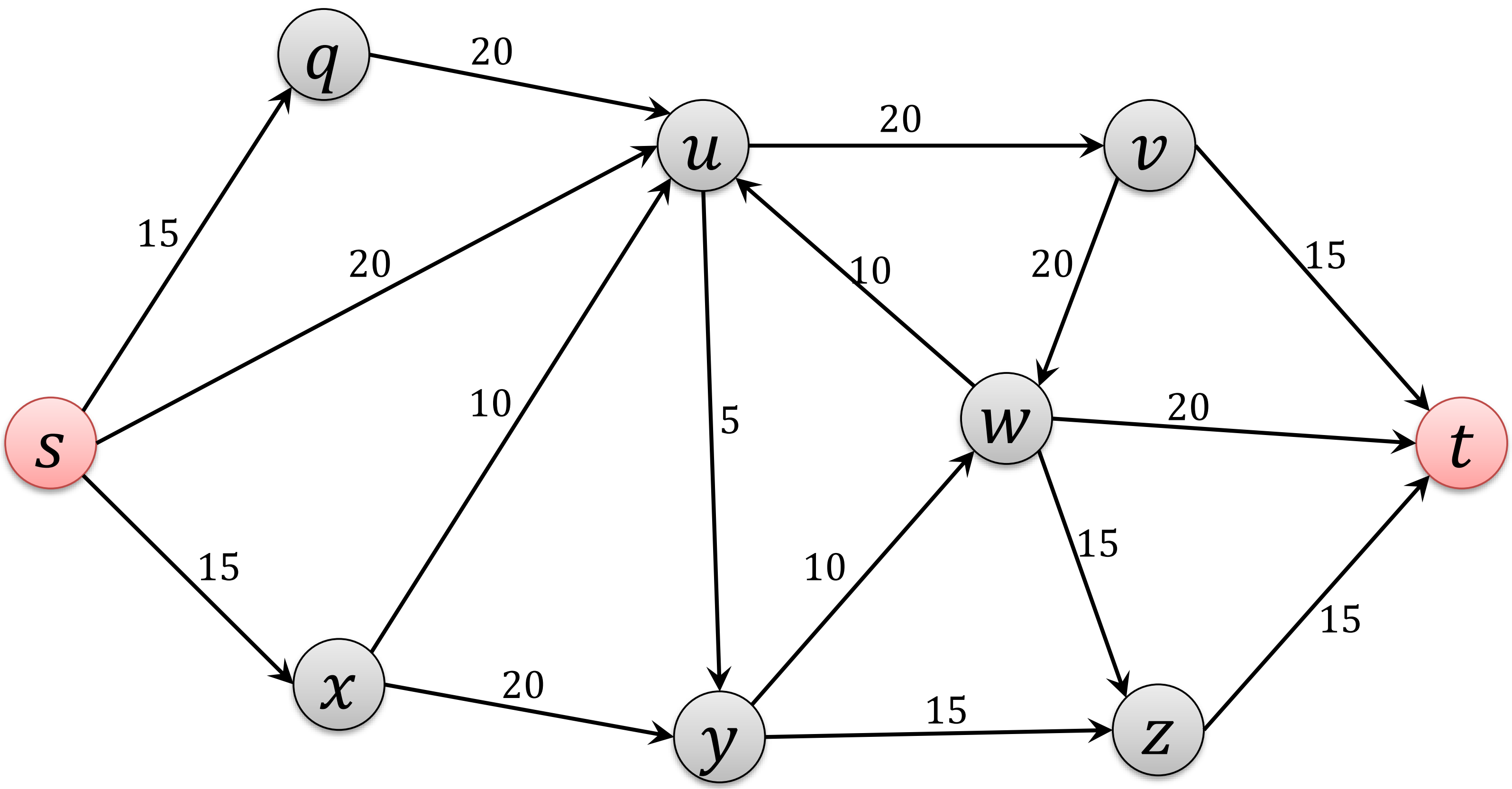
Given a flow network $G = (V, E)$ with capacities c_e (for $e \in E$)

For a flow f on G, define directed graph $G_f = (V_f, E_f)$ as follows:

- Node set $V_f = V$
- For each edge $e = (u, v)$ in E, there are two edges in E_f:
  - forward edge $e = (u, v)$ with residual capacity $c_e - f(e)$
  - backward edge $e' = (v, u)$ with residual capacity $f(e)$



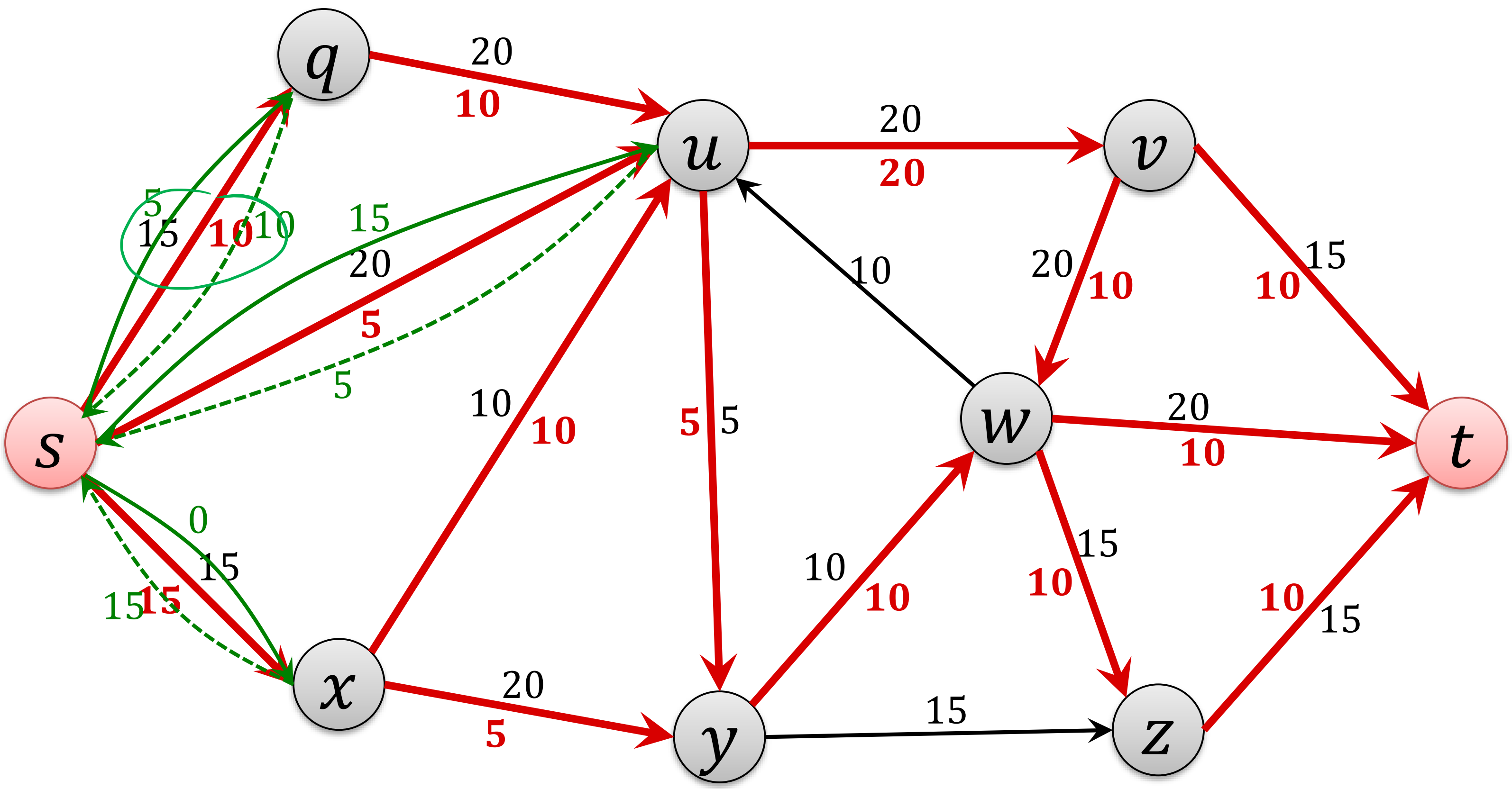
Residual Graph: Example



Residual Graph: Example

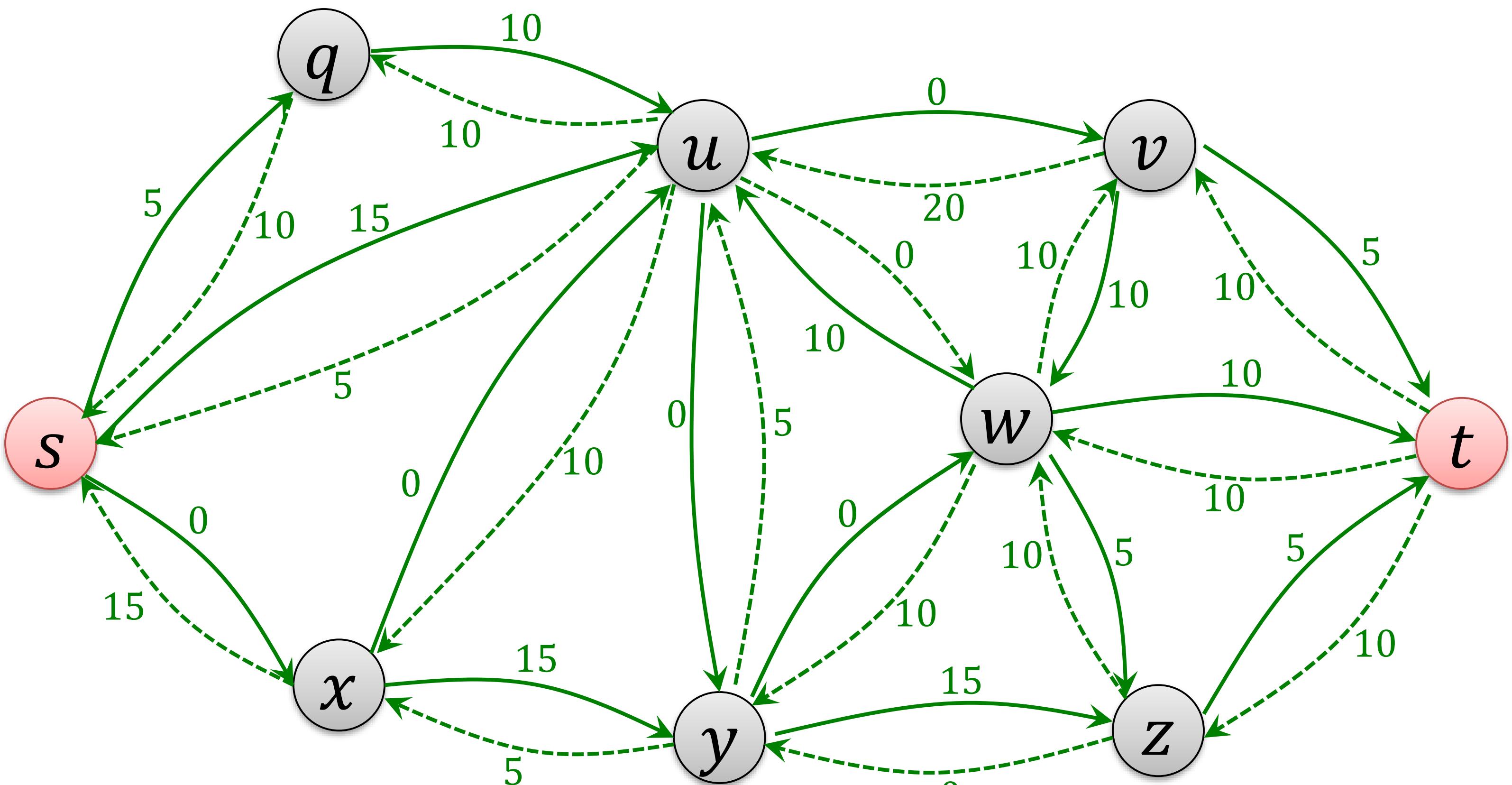
|f|=30

Flow f



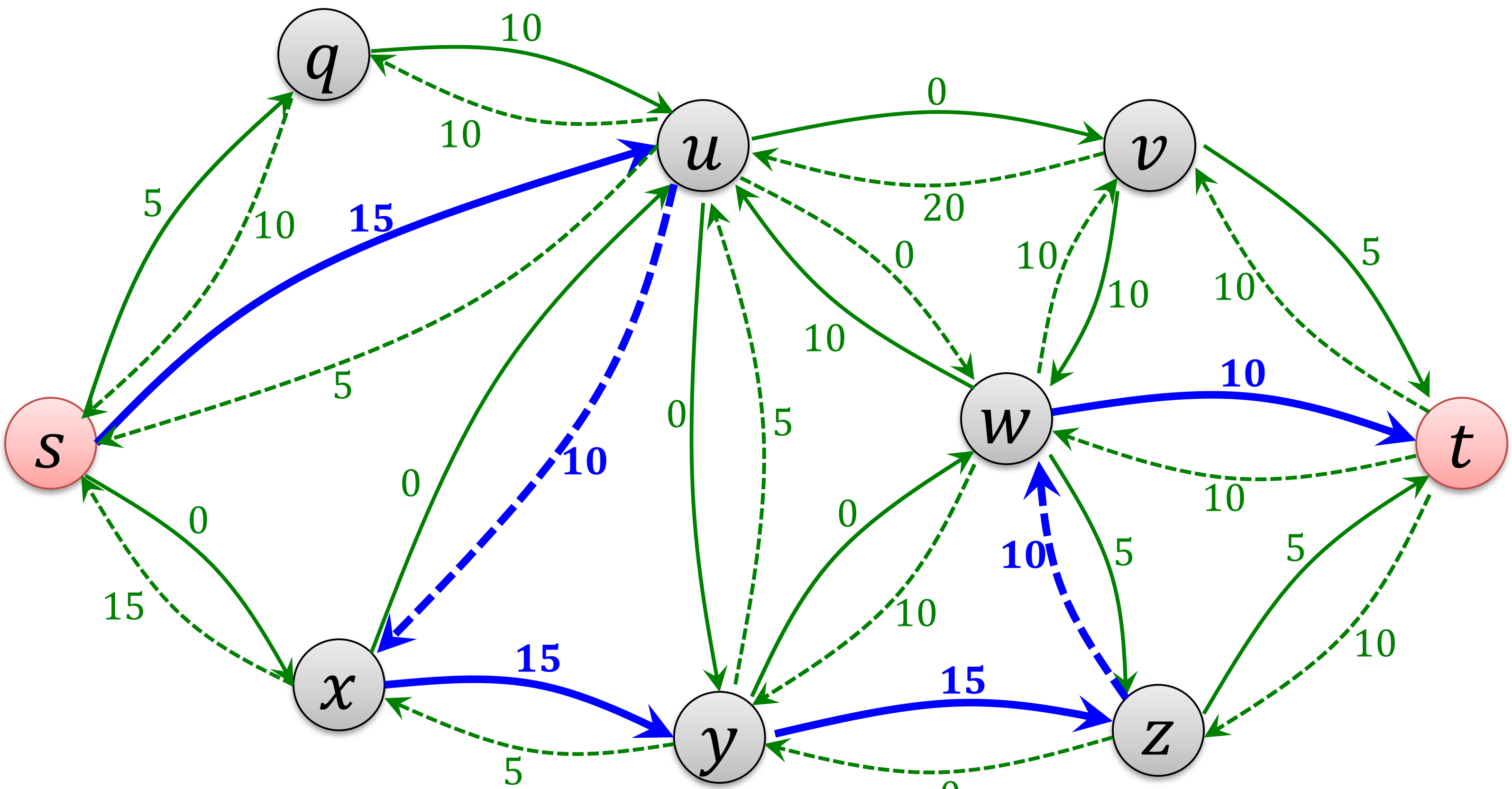
Residual Graph: Example

Residual Graph G_f



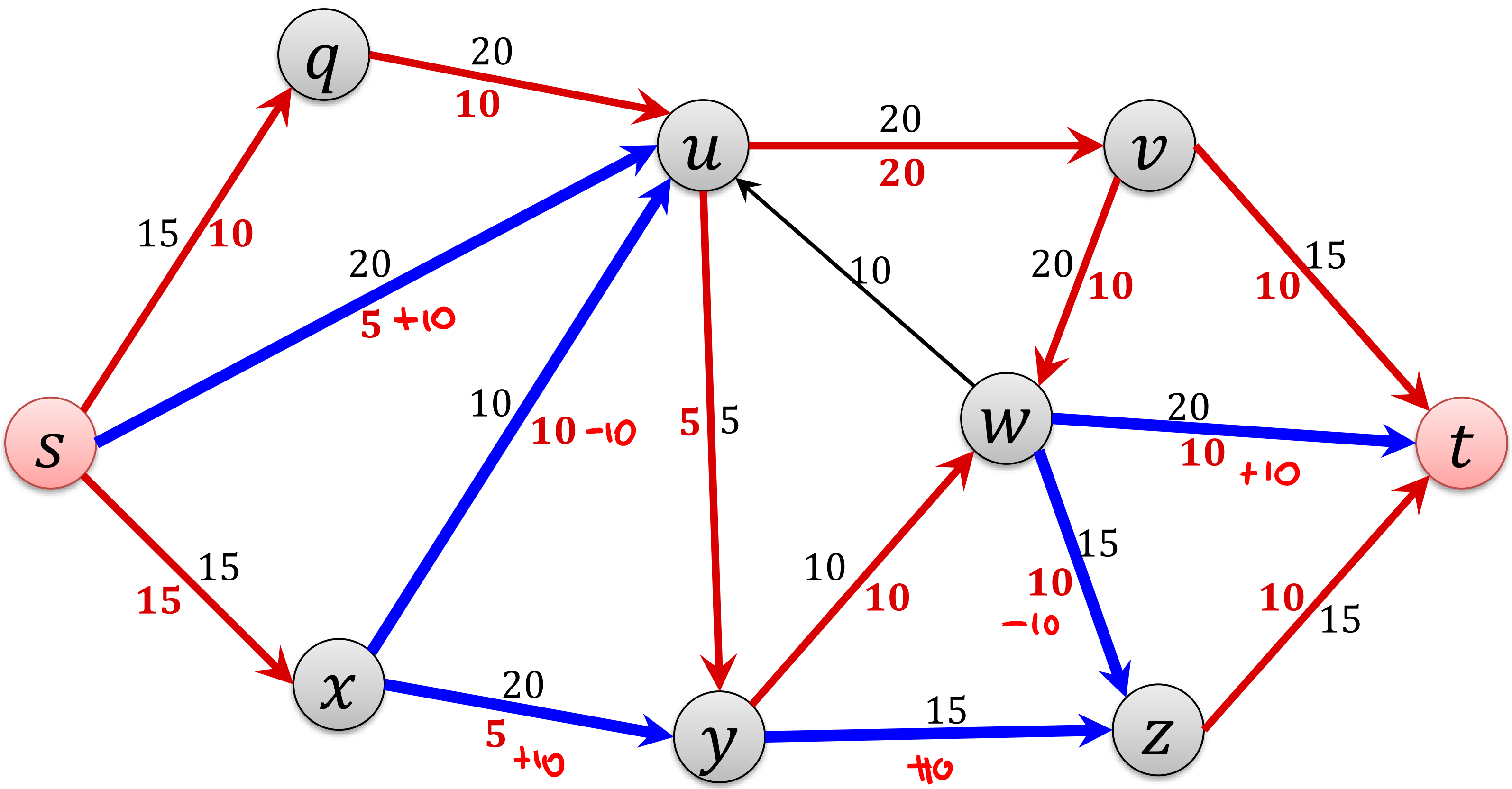
Augmenting Path

Residual Graph G_f



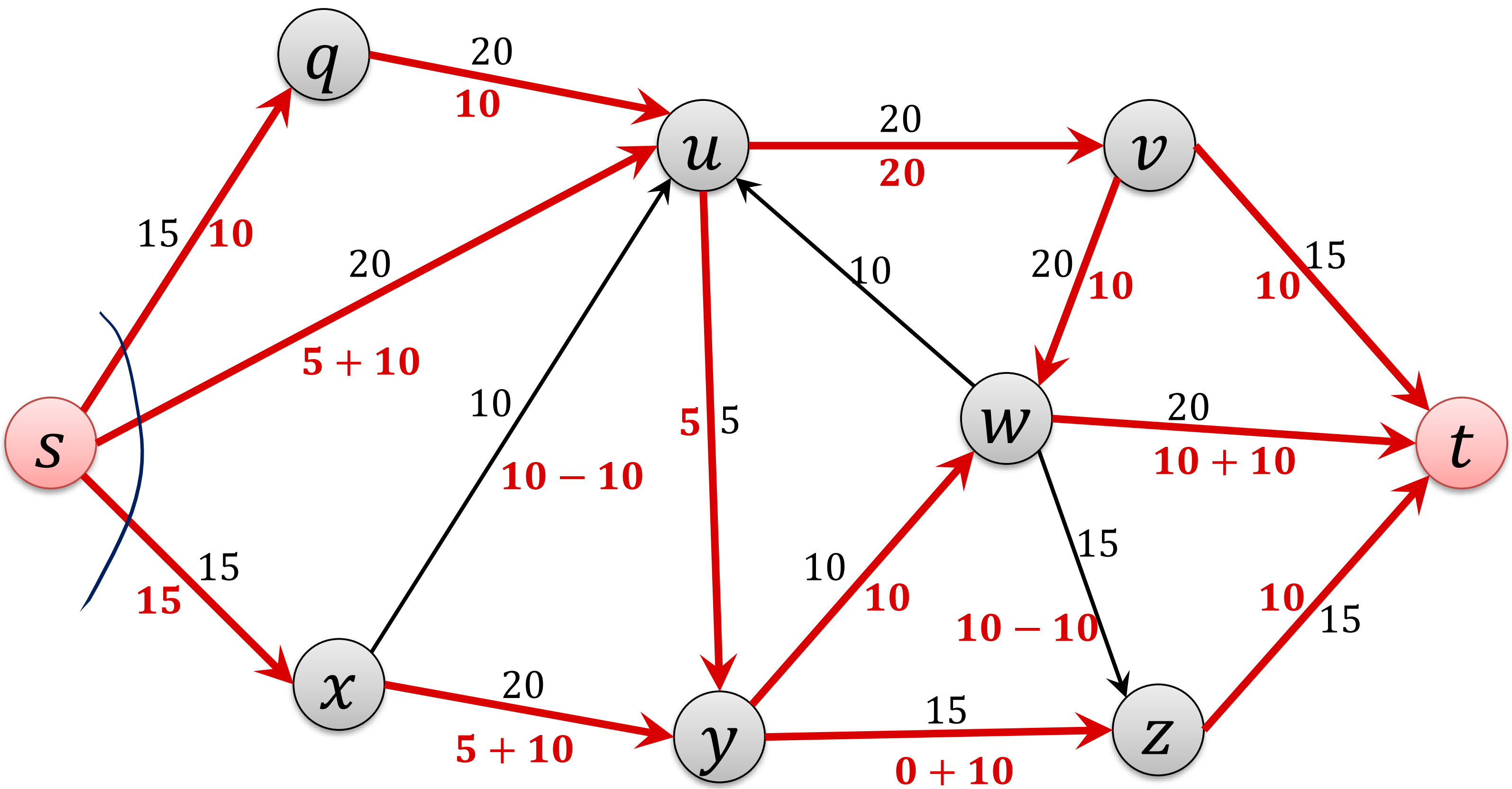
Augmenting Path

Augmenting Path



Augmenting Path

New Flow of value 40



Augmenting Path

Definition:
An augmenting path P is a (simple) s-t-path on the residual graph G_f on which each edge has residual capacity ≥ 0.

$\underbrace{\text{bottleneck}(P, f)}_{>0}$: minimum residual capacity on any edge of the augmenting path P

Augment flow f to get flow f':

- For every forward edge (u, v) on P:

$$f'\big((u,v)\big) := f\big((u,v)\big) + \text{bottleneck}(P, f)$$

- For every backward edge (u, v) on P:

$$f'\big((v,u)\big) := f\big((v,u)\big) - \text{bottleneck}(P, f)$$

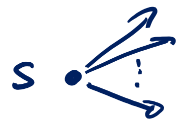
Augmented Flow

Lemma: Given a flow f and an augmenting path P, the resulting augmented flow f' is legal and its value is

$$|f'| = |f| + \underbrace{\textbf{bottleneck}(P, f)}_{b}.$$

Proof:

$|f'| = |f| + b$ ✓ :



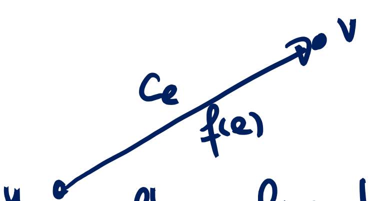
P leaves s on a forward edge

f' is legal

$\forall e \in E$: $\quad 0 \le f'(e) \le c_e \qquad$ (I)

$\forall v \in V \setminus \{s,t\}$: $\quad f'^{\text{in}}(v) = f'^{\text{out}}(v) \qquad$ (II)

(I): fwd. edge

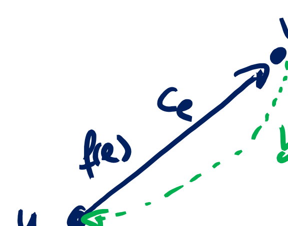


$0 \le f'(e) = f(e) + b \le c_e$

cap. of fwd. edge: $c_e - f(e) \ge b$

backw. edge

$b \le f(e)$



bwd. edge cap. = f(e)

$c_e \ge f'(e) = f(e) - b \ge 0$

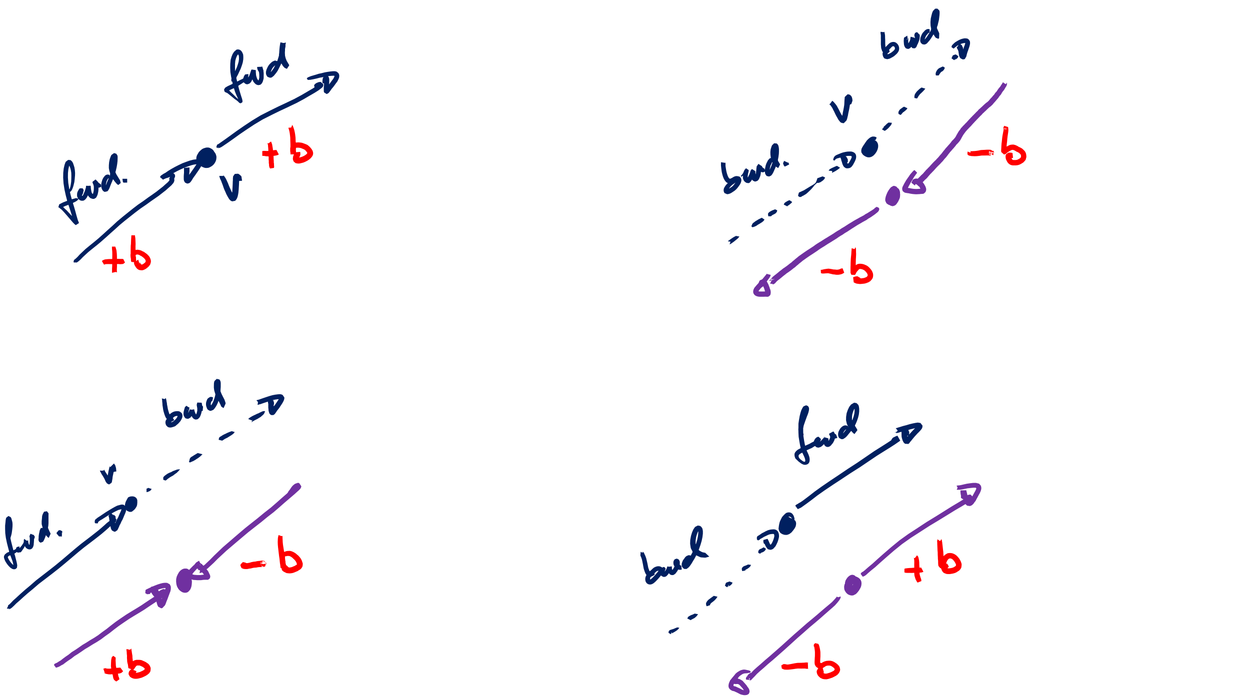
Augmented Flow

Lemma: Given a flow f and an augmenting path P, the resulting augmented flow f' is legal and its value is

$$|f'| = |f| + \textbf{bottleneck}(P, f).$$

Proof:

flow conservation (consider some $v \neq s,t$ on P)



□

Ford-Fulkerson Algorithm

- Improve flow using an augmenting path as long as possible:

1. Initially, $f(e) = 0$ for all edges $e \in E$, $G_f = G$
2. **while** there is an augmenting s-t-path P in G_f **do**
3. Let P be an augmenting s-t-path in G_f;
4. $f' \coloneqq \mathrm{augment}(f, P)$;
5. update f to be f';
6. update the residual graph G_f
7. **end**;

Ford-Fulkerson Running Time

Theorem: If all edge capacities are integers, the Ford-Fulkerson algorithm terminates after at most C iterations, where

$$C = \text{"max flow value"} \leq \sum_{e \text{ out of } s} c_e .$$

Proof:

At all times, for all $e \in E$, $f(e)$ is an integer

$\hookrightarrow$ res. cap. of G_f are integers

initially: $f(e) = 0$

in one iter.: augm. P : res. cap. are integers

bottleneck$(P, f) > 0$ (also bottleneck(P, f) is an int.)

$\Longrightarrow$ bottleneck$(P, f) \geq 1$

$\longrightarrow$ new flow values are int.

$\longrightarrow |f'| \geq |f| + 1$

$\Longrightarrow \leq C$ iterations

Ford-Fulkerson Running Time

Theorem: If all edge capacities are integers, the Ford-Fulkerson algorithm can be implemented to run in $O(mC)$ time.

m: #edges

Proof:

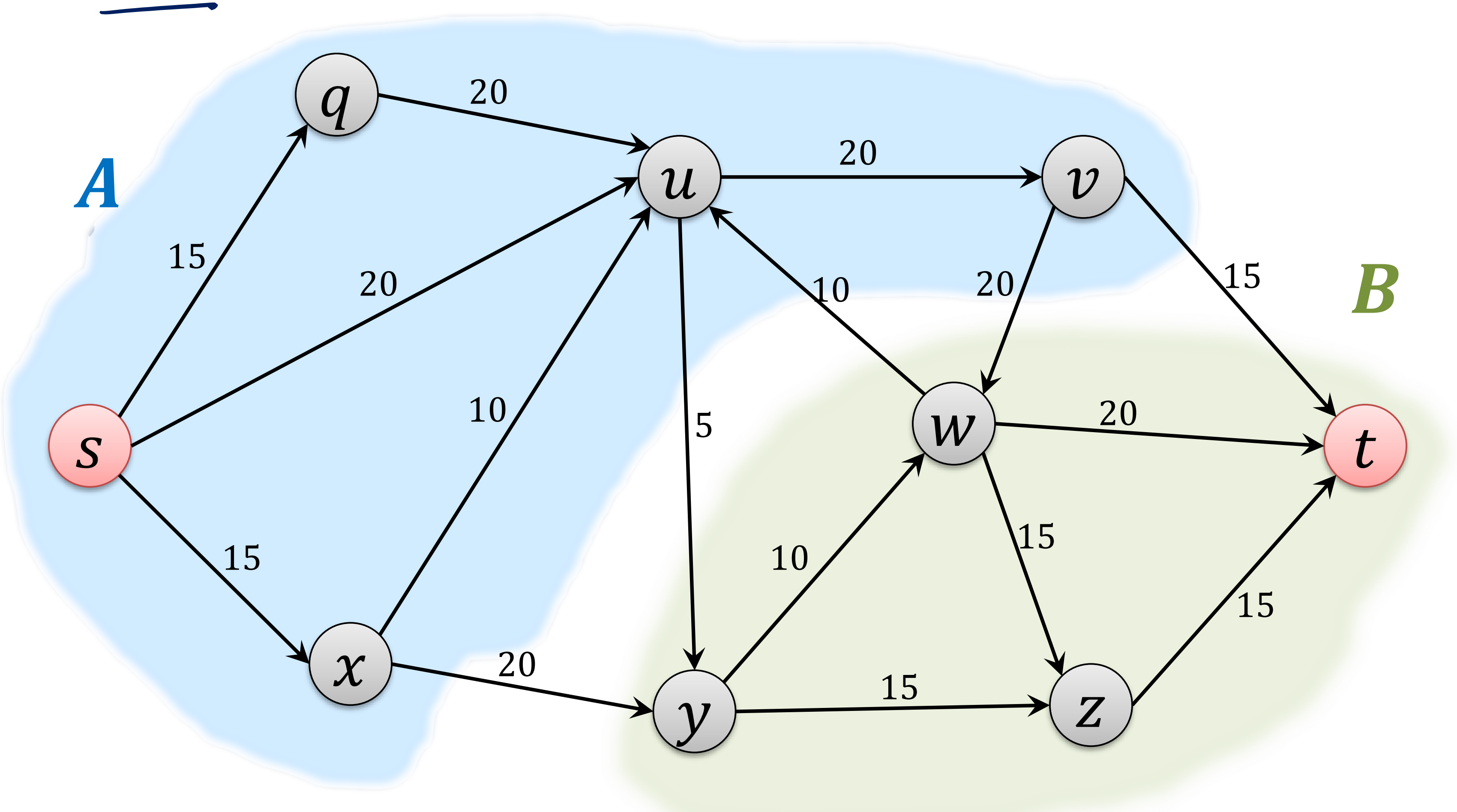
Claim: One iteration can be computed in $O(m)$ time

1. compute / update residual graph
   - first iter.: $O(m)$
   - later iter.: $O(n)$
2. find augm. path / conclude there is no augm. path
   - s-t path in G_f with res. cap. > 0
   - graph traversal (DFS / BFS): $O(m)$ time
3. update flow values : $O(n)$

s-t Cuts

$B = V \setminus A$

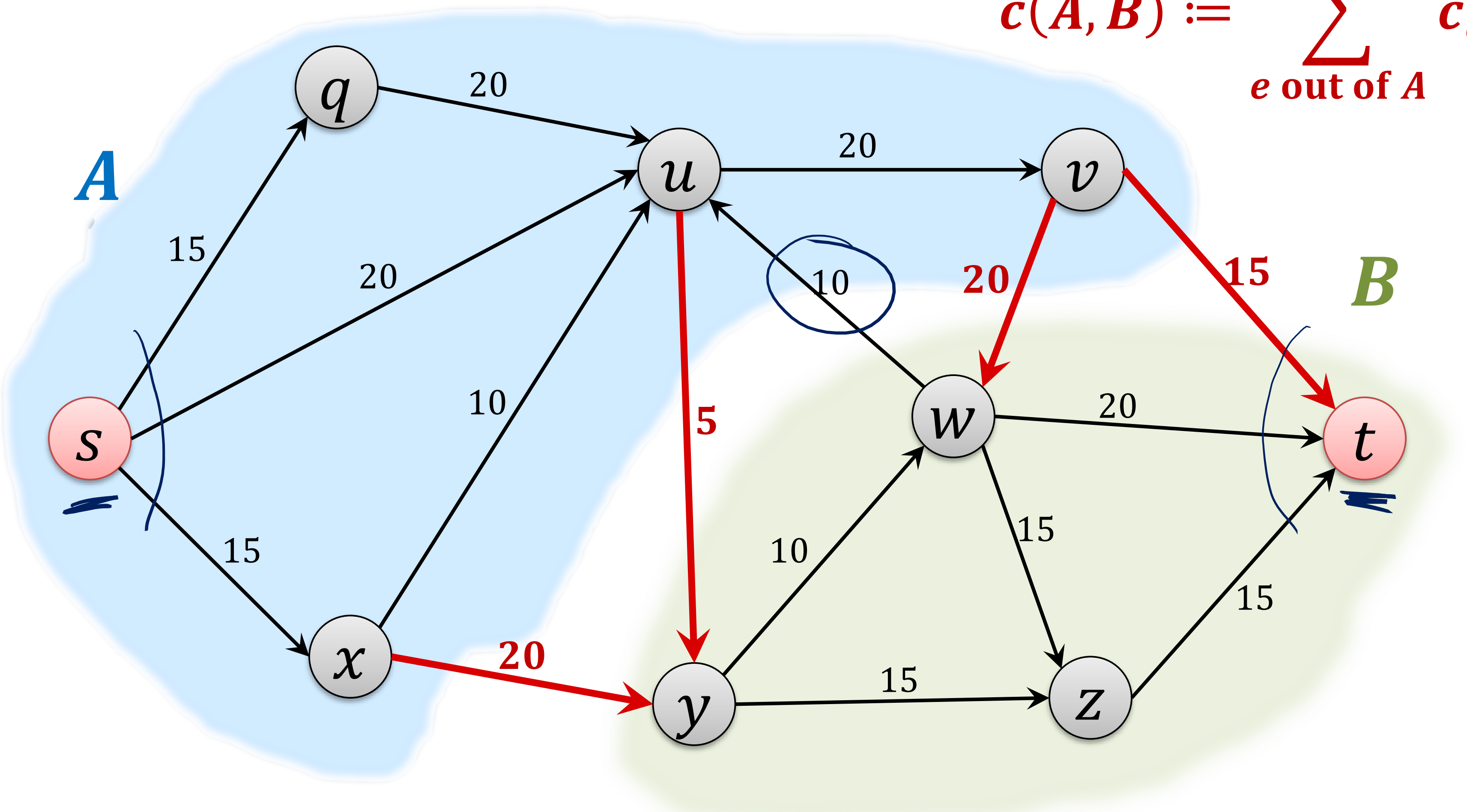
Definition:
An s-t cut is a partition (A, B) of the vertex set such that $s \in A$ and $t \in B$



Cut Capacity

Definition:
The capacity $c(A, B)$ of an s-t-cut (A, B) is defined as

$$\boldsymbol{c(A,B) \coloneqq \sum_{e \text{ out of } A} c_e} .$$



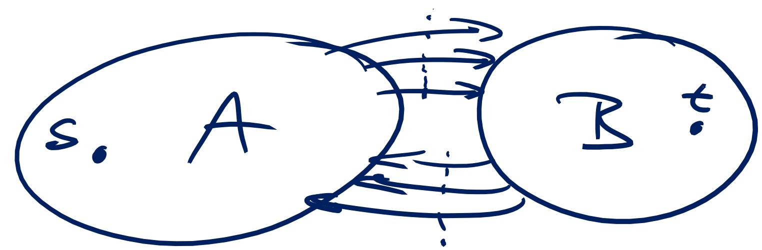
Cuts and Flow Value

Lemma: Let f be any s-t flow, and (A, B) any s-t cut. Then,

$$|f| = f^{\text{out}}(A) - f^{\text{in}}(A).$$

Proof:

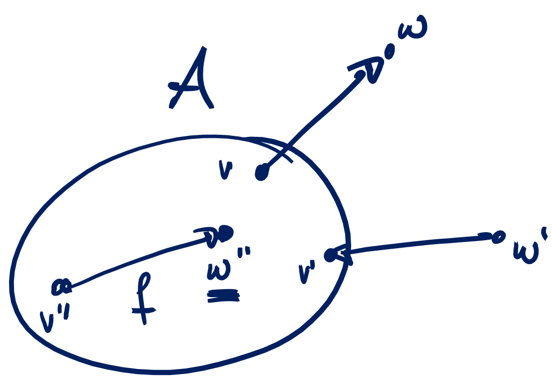
$$|f| = f^{\text{out}}(s) \quad \left(= f^{\text{in}}(t)\right)$$



$$|f| = f^{\text{out}}(s) - \underbrace{f^{\text{in}}(s)}_{=0}$$

$$= \sum_{v \in A} \big(\underbrace{f^{\text{out}}(v) - f^{\text{in}}(v)}_{= 0 \text{ except for } s} \big)$$

$$= f^{\text{out}}(A) - f^{\text{in}}(A)$$



Cuts and Flow Value

Lemma: Let f be any s-t flow, and (A, B) any s-t cut. Then,

$$|\boldsymbol{f}| = \boldsymbol{f}^{\text{out}}(\boldsymbol{A}) - \boldsymbol{f}^{\text{in}}(\boldsymbol{A}).$$

Lemma: Let f be any s-t flow, and (A, B) any s-t cut. Then,

$$|\boldsymbol{f}| = \underline{\boldsymbol{f}^{\text{in}}(\boldsymbol{B})} - \underline{\boldsymbol{f}^{\text{out}}(\boldsymbol{B})}.$$

Proof:

$$f^{out}(A) = f^{in}(B)$$

$$f^{in}(A) = f^{out}(B)$$

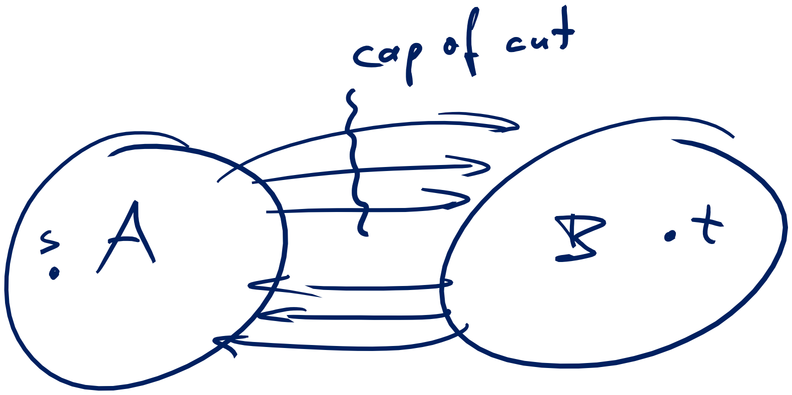
Upper Bound on Flow Value

Lemma:

Let f be any s-t flow and (A, B) any s-t cut. Then $\boldsymbol{|f| \leq c(A, B)}$.

Proof:

$$|f| = f^{out}(A) - f^{in}(A) \leq c(A,B)$$



$$f^{out}(A) \leq c(A,B)$$

$$f^{in}(A) \geq 0$$

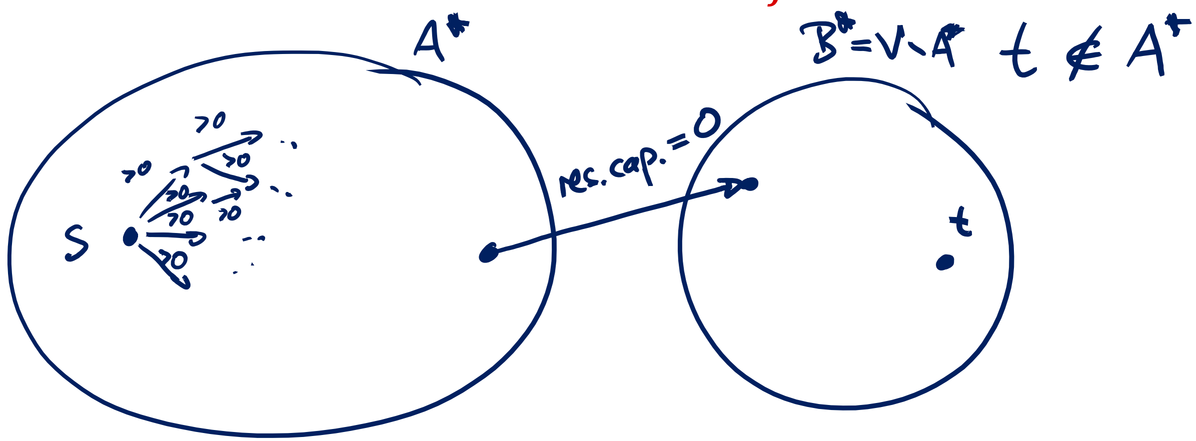
Ford-Fulkerson Gives Optimal Solution

Lemma: If f is an s-t flow such that there is no augmenting path in G_f, then there is an s-t cut (A^*, B^*) in G for which

$$|\boldsymbol{f}| = \boldsymbol{c}(\boldsymbol{A}^*, \boldsymbol{B}^*).$$

Proof:

- Define $\boldsymbol{A}^*$: set of nodes that can be reached from s on a path with positive residual capacities in G_f:



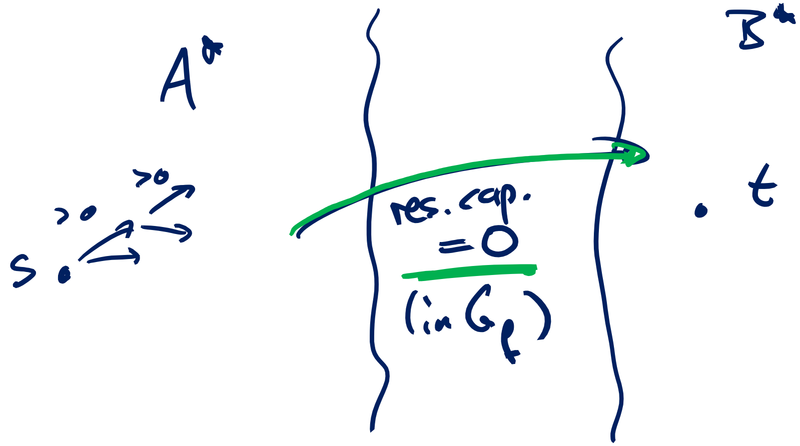
- For $B^* = V \setminus A^*$, (A^*, B^*) is an s-t cut
  - By definition $s \in A^*$ and $t \notin A^*$

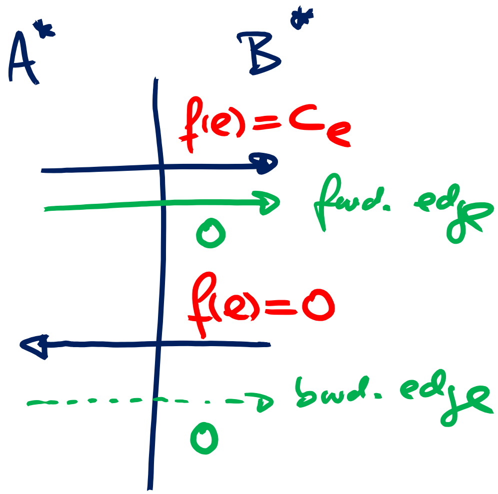
Ford-Fulkerson Gives Optimal Solution

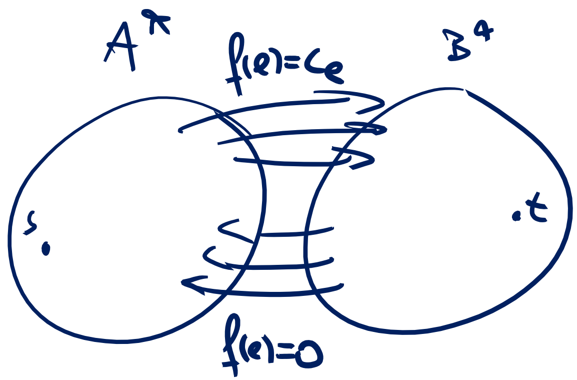
Lemma: If f is an s-t flow such that there is no augmenting path in G_f, then there is an s-t cut (A^*, B^*) in G for which

$$|f| = c(A^*, B^*).$$

Proof:







$$f^{out}(A^*) = c(A^*, B^*)$$

$$f^{in}(A^*) = 0$$

$$\underline{|f| = c(A^*, B^*)}$$

Ford-Fulkerson Gives Optimal Solution

Lemma: If f is an s-t flow such that there is no augmenting path in G_f, then there is an s-t cut (A^*, B^*) in G for which

$$|\boldsymbol{f}| = \boldsymbol{c}(\boldsymbol{A}^*, \boldsymbol{B}^*).$$

Proof:

Ford-Fulkerson Gives Optimal Solution

Theorem: The flow returned by the Ford-Fulkerson algorithm is a maximum flow.

Proof:

FF gives a flow f^* and a s-t cut (A^*, B^*)

s.t. $\underline{|f^*| = c(A^*, B^*)}$

we have shown that for all flows f

$\underline{|f| \leq c(A^*, B^*)}$

Min-Cut Algorithm

Ford-Fulkerson also gives a min-cut algorithm:

Theorem: Given a flow f of maximum value, we can compute an s-t cut of minimum capacity in $O(m)$ time.

Proof:

f maximum $\rightarrow$ no augm. path

can find cut (A^*, B^*) s.t. $|f| = c(A^*, B^*)$

↳ as before by using BFS/DFS

(A^*, B^*) is an s-t cut of min. cap.

because for every other s-t cut (A, B)

we know that $|f| \leq c(A, B)$

$c(A^*, B^*) \leq c(A, B)$

Max-Flow Min-Cut Theorem

Theorem: **(Max-Flow Min-Cut Theorem)**

In every flow network, the maximum value of an s-t flow is equal to the minimum capacity of an s-t cut.

Proof:

FF gives f^* & $c(A^*, B^*)$

s.t. f^* max. flow

$c(A^*, B^*)$ min. s-t cut

$|f^*| = c(A^*, B^*)$

Integer Capacities

Theorem: **(Integer-Valued Flows)**

If all capacities in the flow network are integers, then there is a maximum flow f for which the flow $f(e)$ of every edge e is an integer.

Proof:

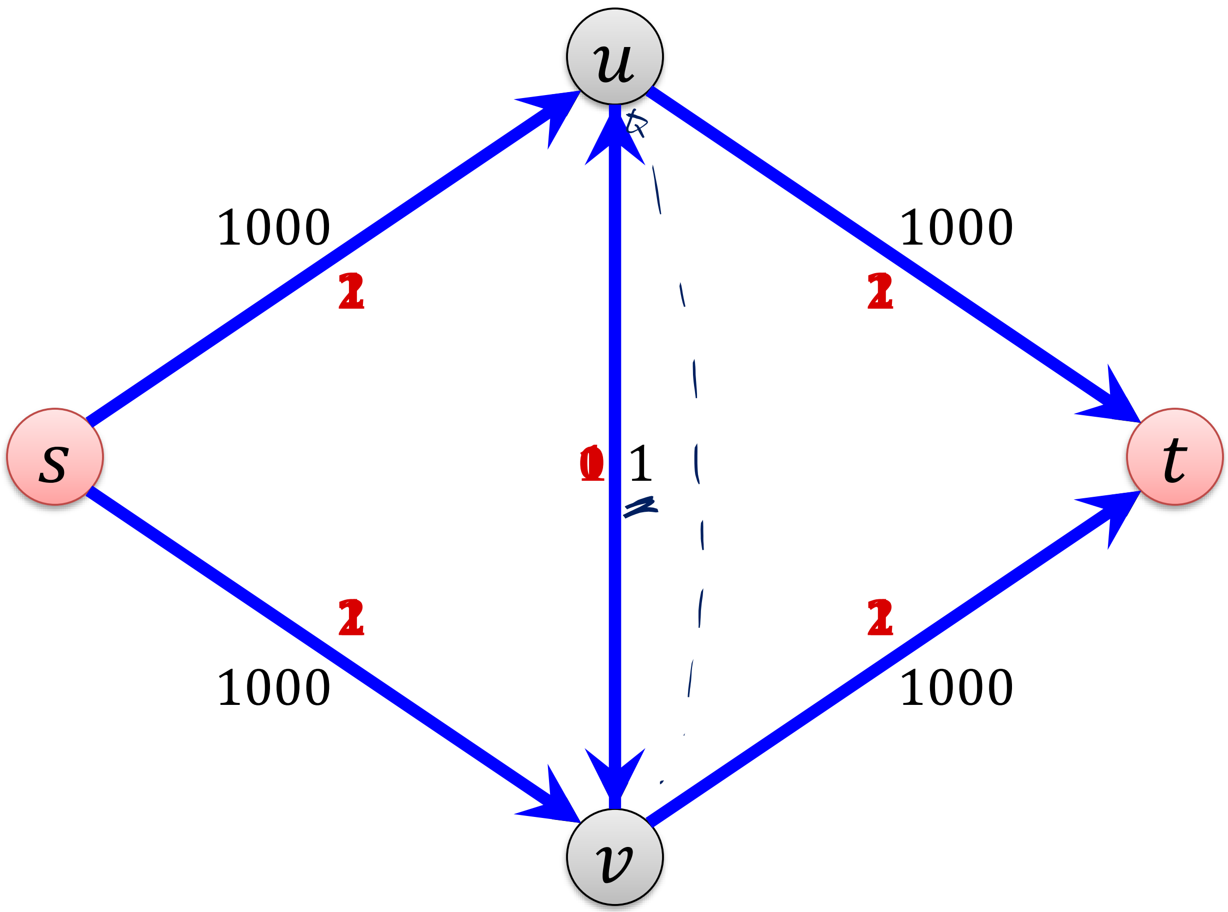
FF gives an opt. int flow

Non-Integer Capacities

What if capacities are not integers?

- rational capacities:
  - can be turned into integers by multiplying them with large enough integer
  - algorithm still works correctly
- real (non-rational) capacities:
  - not clear whether the algorithm always terminates
- even for integer capacities, time can linearly depend on the value of the maximum flow

Slow Execution



- Number of iterations: 2000 (value of max. flow)

Improved Algorithm

Idea: Find the best augmenting path in each step

- best: path P with maximum $\text{bottleneck}(P, f)$

- Best path might be rather expensive to find
  → find almost best path

- **Scaling parameter Δ:** $\Delta = 2^k$
  (initially, $\Delta =$ "$\max c_e$ rounded down to next power of 2")

- As long as there is an augmenting path that improves the flow by at least Δ, augment using such a path

- If there is no such path: $\Delta \coloneqq {}^{\Delta}/_{2}$

Scaling Parameter Analysis

Lemma: If all capacities are integers, number of different scaling parameters used is $\leq 1 + \lfloor \log_2 C \rfloor$.

$C \geq \underbrace{\max_e c_e}_{c_{max}}$

initially

$\Delta = 2^{\lfloor \log_2 c_{max} \rfloor}$: #scaling param. : $\lfloor \log_2 c_{max} \rfloor + 1$

- **Δ-scaling phase:** Time during which scaling parameter is Δ

run time:

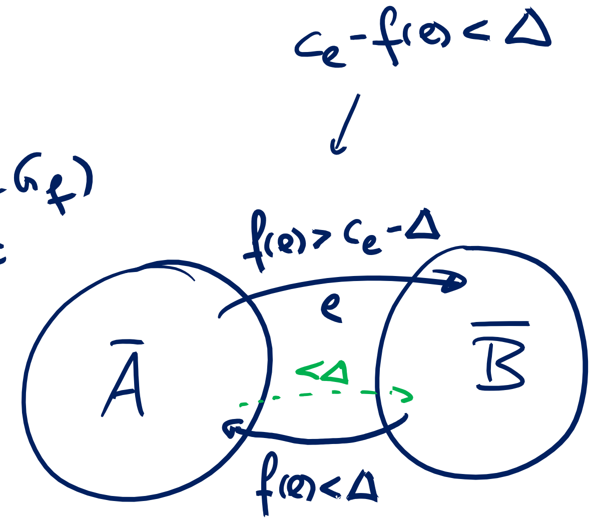
$\underbrace{\#\text{scaling phases}}_{O(\log C)} \cdot \underbrace{\#\text{iter. per phase}}_{??} \cdot O(m)$

Length of a Scaling Phase

Lemma: If f is the flow at the end of the Δ-scaling phase, the maximum flow in the network has value at most $|f| + m\Delta$.

$|f^*| < |f| + m \cdot \Delta$

define cut $(\bar{A}, \bar{B})$



$|f^*| \leq c(\bar{A}, \bar{B}) < |f| + m\Delta$

$|f| = f^{out}(\bar{A}) - f^{in}(A)$

$> c(\bar{A}, \bar{B}) - m_1 \cdot \Delta - m_2 \cdot \Delta$

m_1: # edges from $\bar{A}$ to $\bar{B}$

m_2: # edges from $\bar{B}$ to $\bar{A}$

$\geq c(\bar{A}, \bar{B}) - m\Delta$

$(m_1 + m_2 \leq m)$

Length of a Scaling Phase

Lemma: The number of augmentation in each scaling phase is at most $2m$.

at the beginning of the Δ-scaling phase

$\hookrightarrow$ at the end of the 2Δ-scaling phase

$\Longrightarrow |f^*| < |f| + 2m\Delta$ (prev. lemma)

each augm. improves flow by $\geq \Delta$

$\Longrightarrow \leq 2m$ augm. in Δ-scaling phase

Running time: $O(\log C) \cdot O(m) \cdot O(m) = O(m^2 \cdot \log C)$

Running Time: Scaling Max Flow Alg.

Theorem: The number of augmentations of the algorithm with scaling parameter and integer capacities is at most $O(m \log C)$. The algorithm can be implemented in time $O(m^2 \log C)$.

Strongly Polynomial Algorithm

- Time of regular Ford-Fulkerson algorithm with integer capacities:

$$O(mC)$$

- Time of algorithm with scaling parameter:

C_{max}

$$O(m^2 \log C)$$

- $O(\log C)$ is polynomial in the size of the input, but not in n
- Can we get an algorithm that runs in time polynomial in n?
- Always picking a shortest augmenting path leads to running time

$$O(m^2 n)$$

  – also works for arbitrary real-valued weights

Other Algorithms

- There are many other algorithms to solve the maximum flow problem, for example:

- **Preflow-push algorithm:**
  - Maintains a preflow (∀ nodes: inflow ≥ outflow)
  - Alg. guarantees: As soon as we have a flow, it is optimal
  - Detailed discussion in 2012/13 lecture
  - Running time of basic algorithm: $O(m \cdot n^2)$
  - Doing steps in the "right" order: $O(n^3)$

- **Current best known complexity: $\boldsymbol{O(m \cdot n)}$**
  - For graphs with $m \geq n^{1+\epsilon}$ (for every constant $\epsilon > 0$) [King,Rao,Tarjan 1992/1994]
  - For sparse graphs with $m \leq n^{16/15-\delta}$ [Orlin, 2013]
  - approximate max flow in undirected graphs in time $O(m \cdot n^{o(1)})$ nec.