Chapter 7
Randomization

Algorithm Theory
WS 2018/19

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Minimum Cut

Reminder: Given a graph $G = (V, E)$, a cut is a partition (A, B) of V such that $V = A \cup B$, $A \cap B = \emptyset$, $A, B \neq \emptyset$

Size of the cut (A, B): # of edges crossing the cut

- For weighted graphs, total edge weight crossing the cut

edge connectivity

Goal: Find a cut of minimal size (i.e., of size $\lambda(G)$)

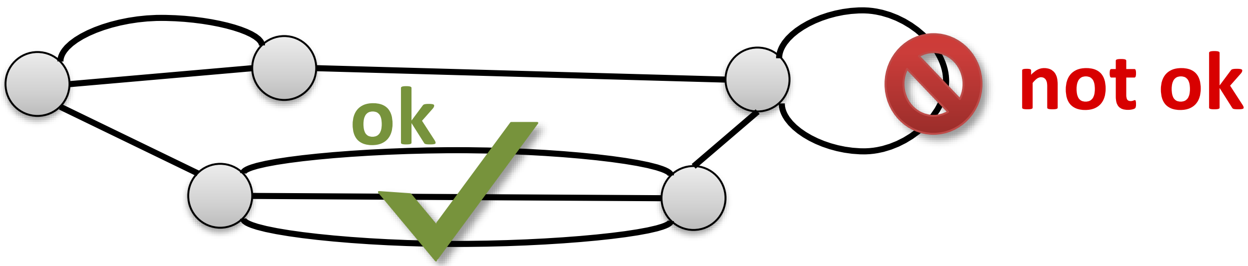
Maximum-flow based algorithm:

- Fix s, compute min s-t-cut for all $t \neq s$
- $O\big(m \cdot \lambda(G)\big) = O(mn)$ per s-t cut
- Gives an $O\big(mn\lambda(G)\big) = O(mn^2)$-algorithm

Best-known deterministic algorithm: $O(mn + n^2 \log n)$

Edge Contractions

- In the following, we consider multi-graphs that can have multiple edges (but no self-loops)



Contracting edge $\{u, v\}$:

- Replace nodes u, v by new node w
- For all edges $\{u, x\}$ and $\{v, x\}$, add an edge $\{w, x\}$
- Remove self-loops created at node w

Randomized Contraction Algorithm

Algorithm:

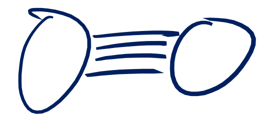
min degr. $\geq k$

#edges $\geq \frac{n \cdot k}{2}$

while there are >2 nodes **do**

contract a uniformly random edge

return cut induced by the last two remaining nodes



(cut defined by the original node sets represented by the last 2 nodes)

Theorem: The probability that the algorithm outputs a (specific) minimum cut (A, B) is at least $2/n(n-1) = 2/\binom{n}{2}$.

- Consider a fixed min cut (A, B), assume (A, B) has size k
- If no edge crossing (A, B) is contracted before, the probability to contract an edge crossing (A, B) in step i is at most

$n-(i-1) = n-i+1$

$$\frac{k}{\boxed{\frac{(n+1-i)k}{2}}} = \frac{2}{n+1-i}.$$

Getting The Min Cut

Theorem: The probability that the algorithm outputs a (specific) minimum cut (A, B) is at least $2/n(n-1) = 2/\binom{n}{2}$.

Proof:

- Event $\mathcal{E}_i$: edge contracted in step i is **not** crossing (A, B)

$$\mathbb{P}(\mathcal{E}_{i+1} | \mathcal{E}_1 \cap \dots \cap \mathcal{E}_i) \geq 1 - \frac{2}{n-i} = \frac{n-i-2}{n-i}$$

- Probability that cut (A, B) survives to the end:

$$\mathbb{P}(\mathcal{E}_1 \cap \dots \cap \mathcal{E}_{n-2}) = \prod_{i=0}^{n-3} \mathbb{P}(\mathcal{E}_{i+1} | \mathcal{E}_1 \cap \dots \cap \mathcal{E}_i)$$

$$= \frac{\cancel{n-2}}{n} \cdot \frac{\cancel{n-3}}{n-1} \cdot \frac{\cancel{n-4}}{\cancel{n-2}} \cdot \frac{\cancel{n-5}}{\cancel{n-3}} \cdot \dots \cdot \frac{2}{\cancel{3}} \cdot \frac{1}{\cancel{2}} = \frac{2}{n(n-1)}$$

Number of Minimum Cuts ← cut of size k

- Given a graph G, how many minimum cuts can there be?

- Or alternatively: If G has edge connectivity $\underline{k \geq 1}$, how many ways are there to remove k edges to disconnect G?

- Note that the total number of cuts is large.

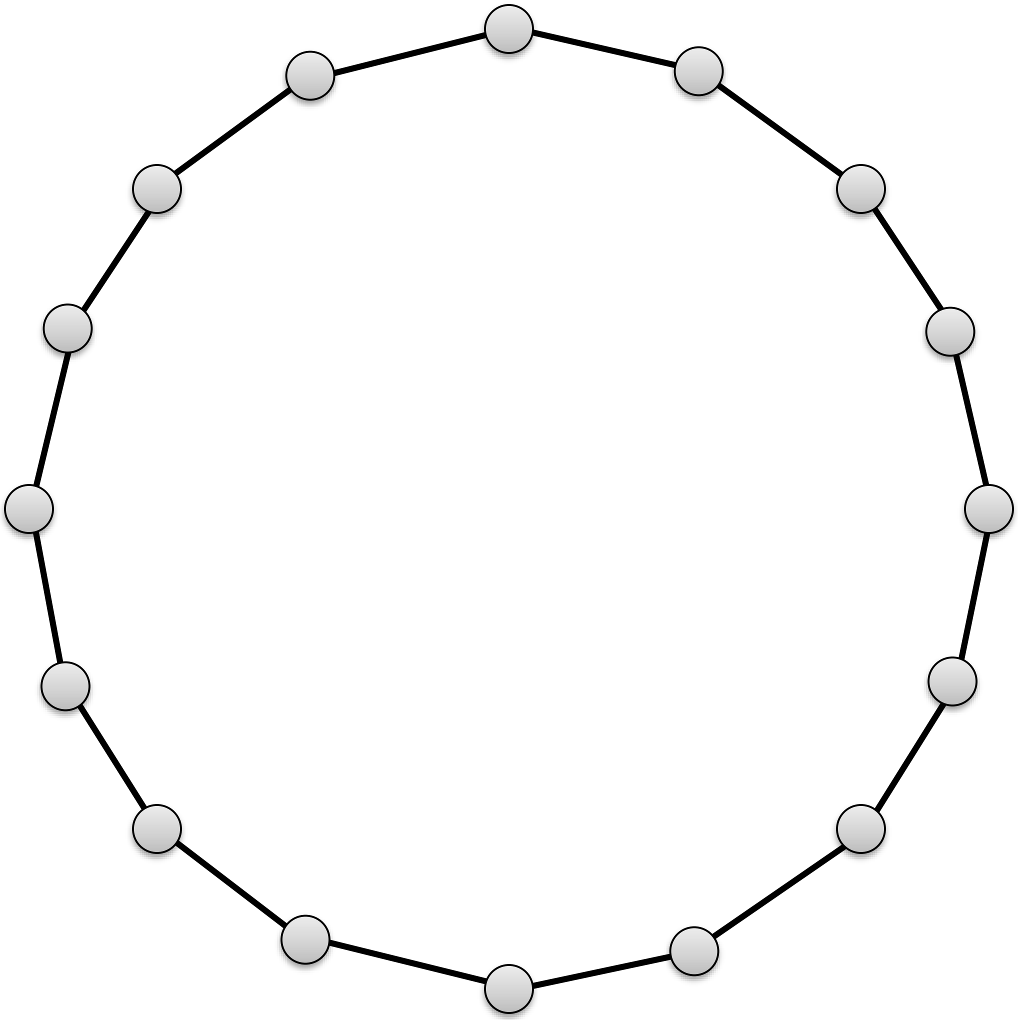
n nodes cut: partition of the n nodes into 2 non-empty parts

how many cuts are there? $\underline{2^{n-1} - 2}$

$(A, V \setminus A)$ (A, B) (B, A)

Number of Minimum Cuts

Example: Ring with n nodes



- Minimum cut size: 2
- Every two edges induce a min cut
- Number of edge pairs:

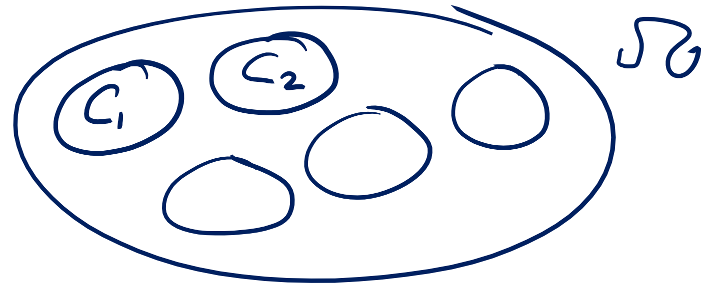
$$\binom{n}{2}$$

- Are there graphs with more min cuts?

Number of Min Cuts

Theorem: The number of minimum cuts of a graph is at most $\binom{n}{2}$.

Proof:



- Assume there are s min cuts
- For $i \in \{1, \dots, s\}$, define event $\mathcal{C}_i$:

  $\mathcal{C}_i := \{\text{basic contraction algorithm returns min cut } i\}$

- We know that for $i \in \{1, \dots, s\}$: $\mathbb{P}(\mathcal{C}_i) \geq 1/\binom{n}{2} = \frac{2}{n(n-1)}$
- Events $\mathcal{C}_1, \dots, \mathcal{C}_s$ are disjoint:

$$1 \geq \mathbb{P}\left(\bigcup_{i=1}^{s} \mathcal{C}_i\right) = \sum_{i=1}^{s} \mathbb{P}(\mathcal{C}_i) \geq \frac{s}{\binom{n}{2}}$$

$$\frac{s}{\binom{n}{2}} \leq 1$$

$$s \leq \binom{n}{2}$$

Counting Larger Cuts

- In the following, assume that min cut has size $\lambda = \lambda(G)$
- How many cuts of size $\leq k = \alpha \cdot \lambda$ can a graph have?
- Consider a specific cut (A, B) of size $\leq k$
- As before, during the contraction algorithm:
  - min cut size $\geq \lambda$
  - number of edges $\geq \lambda \cdot \#\text{nodes}/2$
  - cut (A, B) remains as long as none of its edges gets contracted
- Prob. that an edge crossing (A, B) is chosen in i^{th} contraction

$$\leq \frac{k}{\#\text{edges}} \leq \frac{2k}{\lambda \cdot \#\text{nodes}} = \frac{2\alpha}{n - i + 1}$$

For simplicity, in the following, assume that 2α is an integer

Counting Larger Cuts

Lemma: If $2\alpha \in \mathbb{N}$, the probability that cut (A, B) of size $\alpha \cdot \lambda$ survives the first $n - 2\alpha$ edge contractions is at least

$$\frac{(2\alpha)!}{n(n-1)\cdot \ldots \cdot (n-2\alpha+1)} \geq \frac{2^{2\alpha-1}}{n^{2\alpha}}.$$

Proof:

- As before, event $\mathcal{E}_i$: cut (A, B) survives contraction i

$$\mathbb{P}(\mathcal{E}_1) \cdot \mathbb{P}(\mathcal{E}_2 \mid \mathcal{E}_1) \cdot \ldots$$

$$= \frac{n-2\alpha}{n} \cdot \frac{n-2\alpha-1}{n-1} \cdot \frac{n-2\alpha-2}{n-2} \cdot \ldots \cdot \frac{2}{2\alpha+2} \cdot \frac{1}{2\alpha+1}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2\alpha}{n(n-1)(n-2)\ldots(n-2\alpha+1)} = \frac{(2\alpha)!}{n(n+1)\ldots(n-2\alpha+1)} = \frac{1}{\binom{n}{2\alpha}}$$

Number of Cuts

Theorem: If $2\alpha \in \mathbb{N}$, the number of edge cuts of size at most $\alpha \cdot \lambda(G)$ in an n-node graph G is at most $n^{2\alpha}$.

Proof:

$$\mathbb{P}(\text{cut } (A,B) \text{ survives } n-2\alpha \text{ contr.}) \geq \frac{2^{2\alpha-1}}{n^{2\alpha}}$$

2α nodes : $\#\text{cuts} \leq 2^{2\alpha-1}$

return a random remaining cut

$$\mathbb{P}(\text{return } (A,B)) \geq \frac{2^{2\alpha-1}}{n^{2\alpha}} \cdot \frac{1}{2^{2\alpha-1}} = \frac{1}{n^{2\alpha}}$$

$\Rightarrow \leq n^{2\alpha}$ cuts of size $\leq \alpha \cdot \lambda$

Remark: The bound also holds for general $\alpha \geq 1$.

Application: Resilience To Edge Failures

- Consider a connected network (graph) G with n nodes
- Assume that each link (edge) of G fails independently with probability p
- How large can p be such that the remaining graph is still connected with probability $1 - \varepsilon$?

- In the exercises, you will show that if $p \leq 1 - \frac{c \cdot \ln(n)}{\lambda}$ for a sufficiently large constant $c > 0$, the graph remains connected with high probability.
  - Idea: graph connected $\Leftrightarrow$ there is an edge over each possible cut
  - Analyze the probability that a cut of a specific size survives
  - Do a union bound over all possible cut sizes and all cuts of that size